Global Optimization of H_∞ problems: Application to robust control synthesis under structural constraints

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Abstract. In this paper, a new technique to compute a synthesis structured Robust Control Law is developed. This technique is based on global optimization methods using a Branch-and-Bound algorithm. The original problem is reformulated as a min/max problem with non-convex constraint. Our approach uses interval arithmetic to compute bounds and accelerate the convergence.

1 Context

Controlling an autonomous vehicle or a robot requires the synthesis of control laws for steering and guiding. To generate efficient control laws, a lot of specifications, constraints and requirements have been translated into norm constraints and then into a constraint feasibility problem. This problem has been solved, sometimes with relaxations, using numerical methods based on LMI (Linear Matrix Inequalities) or SDP (Semi Definite Program) [2, 3]. The main limitation of these approaches is the complexity of the controller for implementation in an embedded system. However, if a physical structure is imposed on the control law in order to make the implementation easier, the synthesis of this robust control law is much more complex. And this complexity has been identified as a key issue for several years. A efficient first approach based on local non-smooth optimization was given by Apkarian and Noll [1].

In this talk, we will present a new approach based on **global optimization** in order to generate **robust control laws**.

2 H_∞ control synthesis under structural constraints

We illustrate our approach with an example of the control of a periodic second order system G with a PID controller K subjected to two frequency constraints on the error e and on the command u of the closed-loop system, see Figure 1. The objective is to find $\boldsymbol{k} = (k_p, k_i, k_d)$ to stabilize the closed-loop system while minimizing the H_∞ norm of the controlled system to ensure robustness.

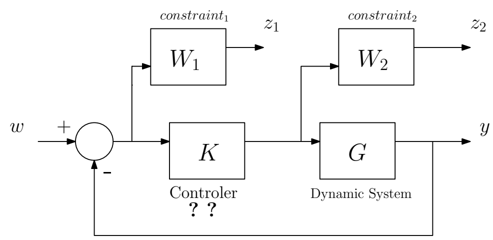

Fig. 1. 2-blocks H_∞ problem

The H_∞ norm of a dynamic system P is defined as follow:

$$||P||_\infty = \sup_\omega(\sigma_{\max}(P(j\omega))),$$

with $\sigma_{\max}$ the greatest singular value of the transfert function P and j the imaginary unit.

In our particular case, the closed-loop system can be interpreted as two SISO systems (Single In Single Out). The H_∞ norm of a SISO system is the maximum of the absolute value of the transfer function. Indeed, to minimize the H_∞ norm of our example, we need to solve the following min/max problem:

$$\begin{cases} \min\limits_{\boldsymbol{k}} \max \left(\sup\limits_\omega \left| \dfrac{W_1(j\omega)}{1+G(j\omega)K(j\omega)} \right|, \sup\limits_\omega \left| \dfrac{W_2(j\omega)K(j\omega)}{1+G(j\omega)K(j\omega)} \right| \right), \\ \\ s.t. \qquad \text{The closed-loop system must be stable.} \end{cases} \tag{1}$$

The stability constraint of a closed-loop is well-known: the roots of denominator part of the transfer function $\frac{1}{1+G(s)K(s)}$ must have a non-positive real part [4]. Using Routh-Hurwitz stability criterion [5], this constraint can be reformulated as a set of non-convex constraints.

Proposition 1. *Let us consider a polynomial $Q(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$. The real parts of its roots are negative if the entries in the first column of the following table are positive:*

$v_{1,1} = a_n$	$v_{1,2} = a_{n-2}$	$v_{1,3} = a_{n-4}$	$v_{1,4} = a_{n-6}$	$\cdots$
$v_{2,1} = a_{n-1}$	$v_{2,2} = a_{n-3}$	$v_{2,3} = a_{n-5}$	$v_{2,4} = a_{n-7}$	$\cdots$
$v_{3,1} = \frac{-1}{v_{2,1}} \begin{vmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{vmatrix}$	$v_{3,2} = \frac{-1}{v_{2,1}} \begin{vmatrix} v_{1,1} & v_{1,3} \\ v_{2,1} & v_{2,3} \end{vmatrix}$	$v_{3,3} = \frac{-1}{v_{2,1}} \begin{vmatrix} v_{1,1} & v_{1,4} \\ v_{2,1} & v_{2,4} \end{vmatrix}$	$\cdots$	$\cdots$
$v_{4,1} = \frac{-1}{v_{3,1}} \begin{vmatrix} v_{2,1} & v_{2,2} \\ v_{3,1} & v_{3,2} \end{vmatrix}$	$v_{4,2} = \frac{-1}{v_{3,1}} \begin{vmatrix} v_{2,1} & v_{2,3} \\ v_{3,1} & v_{3,3} \end{vmatrix}$	$\cdots$	$\cdots$	$\cdots$
$v_{5,1} = \frac{-1}{v_{4,1}} \begin{vmatrix} v_{3,1} & v_{3,2} \\ v_{4,1} & v_{4,2} \end{vmatrix}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

Indeed, applying Proposition 1 with $Q(s) = 1 + G(s)K(s)$, the H_∞ control synthesis under structural constraint is reformulated as a min/max problem with non-convex constraints.

3 Global optimization of min/max problems

In order to solve Problem (1), our approach is based on an Branch-and-Bound technique [7]. At each iteration, the domain under study is bisected to improve the computation of bounds. Boxes are eliminated if and only if it is certified that no point in the box can produce a better solution than the current best one, or that at least one constraint cannot be satisfied by any point in such a box.

The non-convex contraint can be handled with constraint programming techniques. In our approach, we use the ACID algorithm [8] which reduces the width of the boxes and so accelerates the convergence of the branch-and-bound.

The key point of our approach concerns the computation of the bounds of the objective function. In our example, the objective function can be reformulated as the following expression, with $x = (k_p,\, k_i,\, k_d)$:

$$f(x) = \sup_{\omega \in [\omega_{\min}, \omega_{\max}]} g(x, \omega). \tag{2}$$

At each iteration, Algorithm 1 is used to compute a lower bound of this function over a box $[\boldsymbol{x}]$. This algorithm is also a branch-and-bound algorithm based on Interval Arithmetic. But, for not wasting time, we limit the maximum number of iterations for computing faster lower bounds. Each element $([\boldsymbol{\omega}], ub_{\boldsymbol{\omega}})$ stored in $\mathcal{L}$ is composed of: (i) $[\boldsymbol{\omega}]$ a sub-interval of $[\boldsymbol{\omega}_{\min}, \boldsymbol{\omega}_{\max}]$ and (ii) $ub_{\boldsymbol{\omega}}$ an upper bound of g over $[\boldsymbol{x}] \times [\boldsymbol{\omega}]$.

Algorithm 1 Computation of bounds of f over a box $[\boldsymbol{x}]$

Require: g: the function under study (see Equation 2); $\boldsymbol{x}$: a initial box; $\mathcal{L}$: the list of boxes; $nbIter$: the maximal number of iterations.

Initialization: $(lb_{out}, ub_{out}) = (-\infty, \infty)$.
for nb := 1 **to** nbIter **do**
  Extract an element $(\boldsymbol{\omega}, ub_{\boldsymbol{\omega}})$ from $\mathcal{L}$.
  Bisect $\boldsymbol{\omega}$ into two sub-boxes $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$.
  for i:=1 **to** 2 **do**
    Compute $lb_{\boldsymbol{\omega}_i}$ and $ub_{\boldsymbol{\omega}_i}$ a lower and an upper bound of $g(x, \boldsymbol{\omega})$ over $[\boldsymbol{x}] \times [\boldsymbol{\omega}_i]$ using Interval Arithmetic techniques [6].
    if $lb_{\boldsymbol{\omega}_i} > lb_{out}$ **then**
      $lb_{out} := lb_{\boldsymbol{\omega}_i}$, {Update the best lower bound}
      Remove from $\mathcal{L}$ all the element j such as $ub_{\boldsymbol{\omega}_j} < lb_{out}$,
    end if
    if $ub_{\boldsymbol{\omega}_i} > lb_{out}$ **then**
      Add $(\boldsymbol{\omega}_i, ub_{\boldsymbol{\omega}_i})$ in $\mathcal{L}$,
    end if
  end for
end for
$ub_{out} := \max_{(\boldsymbol{\omega}_i, ub_{\boldsymbol{\omega}_i}) \in \mathcal{L}} ub_{\boldsymbol{\omega}_i}$
return (lb_{out}, ub_{out}): a lower and an upper bound of f over $\boldsymbol{x}$.

Thanks to Interval Analysis, at the end of Algorithm 1, we can ensure that the value of the maximum of f over $[\boldsymbol{x}]$ is included in $[lb_{out}, ub_{out}]$.

4 Application

In our example, we consider a second-order system and weighting functions W_1 and W_2 penalizing the error signal and control signal respectively:

$$G(s) = \frac{1}{s^2 + 1.4s + 1}, \; K(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{1 + s}.$$

$$W_1(s) = \frac{s + 100}{100s + 1}, \qquad W_2(s) = \frac{10s + 1}{s + 10}.$$

We want to find k_p, k_i and k_d the coefficients of the structured controller K such that the closed-loop system respects the constraints:

$$\max\left(|| \frac{W_1(j\omega)}{1 + G(j\omega)K(j\omega)} ||_\infty, || \frac{W_2(j\omega)K(j\omega)}{1 + G(j\omega)K(j\omega)} ||_\infty \right) \leq 1$$

The control is bounded in $[-2, 2]$, and we limit the interval of ω to $[10^{-2}, 10^2]$.

Our algorithm gives the following result:

$$\max\left(\sup_\omega \left| \frac{W_1(j\omega)}{1 + G(j\omega)K(j\omega)} \right|, \sup_\omega \left| \frac{W_2(j\omega)K(j\omega)}{1 + G(j\omega)K(j\omega)} \right| \right) = 2.1414$$

$$\text{with } k_p = -0.0425, \; k_i = 0.4619, \; k_d = 0.2566$$

Unfortunately, the value of the solution of the min/max problem is greater than 1. So, the constraints are not respected as shown on Figure 2 (solid lines are above dotted lines of the same color at some frequencies).

In this example, the main advantage of our global optimization approach is that unlike classical method based on non-smooth optimization, we can certify that no robust solution of our problem exists.

References

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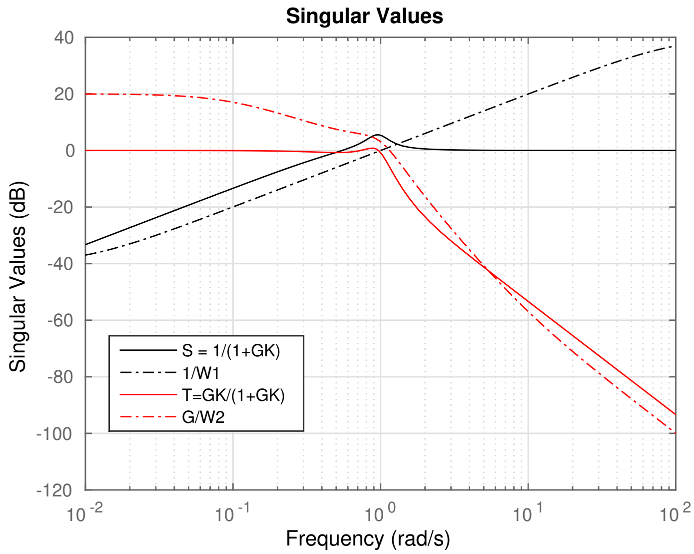

Fig. 2. Weighting functions and singular values of the solution

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