Ray tracing and stability analysis of parametric systems
Plan

1. Introduction
2. Ray tracing
3. Stability analysis of a parametric system
4. Conclusion
Introduction
Goal: Show similarities between 2 problems apparently different: ray tracing and parametric stability analysis

Use of interval analysis
Ray tracing
Ray tracing

- Description
  - Ray tracing, ray casting
  - 3D scene display
  - Method: build the reverse light path starting from the screen to the object
Ray tracing

- **Hypothesis**
  - Objects are defined by implicit functions
  - The eye is at the origin of a coordinate space R(O,i,j,k) and the screen is at z=1
  - The screen is not in the object
Ray tracing

- Problem description
  - A ray associated with the pixel
    \[ p = (p_1, p_2) \in [p] \]

satisfies

\[
\begin{align*}
x &= p_1 d \\
y &= p_2 d \\
z &= d
\end{align*}
\]
Ray tracing

- **Problem description**
  - The point \((x, y, z)\) is in the object if
    \[
    f(x, y, z) \leq 0
    \]
  - A pixel displays a point of the object if the associated ray intersects the object
Ray tracing

The ray associated with $\mathbf{p}$ intersects the object if

$$\exists d \geq 0, g(\mathbf{p}, d) \leq 0$$

with

$$g(\mathbf{p}, d) = f(p_1.d, p_2.d, d)$$
Ray tracing

- Light effects handling
  - Realism => illumination model
  - Phong: needs the distance from the eye to the object
  - We need to compute for each pixel $p$:

$$d^*(p) = \min_{d \geq 0} \begin{cases} d & g(p,d) \leq 0 \end{cases}$$
**Ray tracing**

- **Computation of** $d^*$
  - If
    \[
    \begin{cases}
    g([0,a]) \subset [0,\infty] \\
    g(b) < 0
    \end{cases}
    \]
  Then
  \[d^* \in [a,b]\]

Moreover, if
\[g'( [a,b]) \subset ] -\infty, 0]\]
We can use a dichotomy to get $d^*$
Ray tracing

- Computation of $d^*$
  - Interval computations are used to find $[a, b]$
  - A dichotomy finds $d^*$
Ray tracing

- Parametric version
  - $g(p, d)$ now depends on $p \in [p]$
  - If
    $$\begin{cases} g([p], [0, a]) \subset [0, \infty[ \\ g([p], b) \subset ] - \infty, 0] \end{cases}$$

  Then
  $$d^*(p) \subset [a, b]$$

  Moreover if
  $$\frac{\partial g}{\partial a}([p], [a, b]) \subset ] - \infty, 0]$$

  We can use a dichotomy to get $d^*$ for each $p$
Ray tracing

- From $d^*$ to $d^*(p)$
Ray tracing
Stability analysis of a parametric system
Stability analysis of a parametric system

- **Stability**

\[ P(s, p) \text{ stable} \iff \text{all its roots have a real part } \leq 0 \]

(Routh)

\[ \iff r(p) \leq 0 \]

where \( r \) is retrieved from the Routh table
Stability analysis of a parametric system

- $\delta$ stability

$P(s, \mathbf{p})$ is $\delta$ stable $\iff$ all its roots have a real part $\leq \delta$

(Routh)

$\iff r(\mathbf{p}, \delta) \leq 0$
Stability analysis of a parametric system

- Example: Ackermann

\[ P(s, p) = s^3 + (p_1 + p_2 + 2)s^2 + (p_1 + p_2 + 2)s + 2p_1p_2 + 6p_1 + 6p_2 + 2.25. \]

is \( \delta \) stable if

\[
 r(p, \delta) = \min \left( \frac{p_1 + p_2 + 2 - 3\delta}{(p_1 - 1)^2 + (p_2 - 1)^2 - 0.25 - 2\delta((p_1 + p_2 + 2)(p_1 + p_2 + 3 - 4\delta) + 4\delta^2)}, \frac{2(p_1 + 3)(p_2 + 3) - 15.75 - \delta((p_1 + p_2 + 2)(1 + \delta) - \delta^2)}{2(p_1 + 3)(p_2 + 3) - 15.75 - \delta((p_1 + p_2 + 2)(1 + \delta) - \delta^2)} \right) \leq 0
\]
Stability analysis of a parametric system

- Stability degree

\[ \delta^*(p) = \min_{\delta \geq 0} \delta \]

\[ r(p, \delta) \leq 0 \]
Stability analysis of a parametric system

- Similarities with ray tracing

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<td>$\delta$</td>
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<td>$g$</td>
<td>$r$</td>
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$d^*(p) = \min_{d \geq 0} d \iff \delta^*(p) = \min_{\delta \geq 0} \delta$
Stability analysis of a parametric system
Conclusion

- Ray tracing and stability degree drawing of a linear system are similar problems.

- A common algorithm based on intervals and dichotomy has been proposed.
References

Ray tracing

\[ \mathcal{L} = \{ [d_1, d_2] \} \]

\[ d = [d_1, d_2] \]

\[ \mathcal{L} = \{ \} \]

\[ [g] ([d]) \not\in [0, \infty[ \]

\[ [g'] ([d]) \not\in (-\infty, 0] \]
Ray tracing

\[ \mathcal{L} = \{[d_1, d_3], [d_3, d_2]\} \]

\[ d = [d_1, d_3] \]

\[ \mathcal{L} = \{[d_3, d_2]\} \]

\[ [g] ([d]) \subset [0, \infty[ \]

\[ \Rightarrow a = d_3 \]
Ray tracing

\[ \mathcal{L} = \{[d_3, d_2]\} \quad \rightarrow \quad d = [d_3, d_2] \]

\[ \mathcal{L} = \{\} \]

\[ [g]\, ([d]) \not\in [0, \infty[ \]

\[ [g']\, ([d]) \subset ] - \infty, 0] \]

\[ [g]\, ([d^+]) < 0 \]

\[ \Rightarrow b = d_2 \]
Ray tracing

- Division in d
Ray tracing

- Division in d
Ray tracing

- Division in d
Ray tracing

- Division in $p$
Ray tracing

- Division in $p$
Division in $\mathbf{p}$
Ray tracing

- Division in p
Ray tracing

- Division in $p$
Ray tracing

- Division in \( p \) and in \( d \)
Ray tracing
Ray tracing

- Several objects display handling
  - We apply the previous algorithm for the function:

  \[ g_{\text{min}} : (p, d) \to \min_i g_i(p, d) \]

  - Indeed, we have to consider only the first object crossed by the ray.
Ray tracing
Stability analysis of a parametric system

– Stability degree of an invariant linear system of characteristic polynomial $P(s)$:

$$
\delta^* = \min_{P(s-\delta) \text{ unstable}} \delta.
$$

– We consider an invariant linear system parametrized with a vector of parameter $p$:

$$
P(s, p) = s^3 + (p_1 + p_2 + 2)s^2 + (p_1 + p_2 + 2)s + 2p_1p_2 + 6p_1 + 6p_2 + 2.25.
$$
Stability analysis of a parametric system

– The stability degree becomes:

\[ \delta^*(p) = \min_{P(s-\delta,p) \text{ unstable}} \delta. \]

– With

\[ P(s - \delta, p) = s^3 + b_2 s^2 + b_1 s + b_0 \]

the polynomial is stable if (Routh):

\[
\begin{pmatrix}
  b_2 \\
  b_1 b_2 - b_0 \\
  b_0
\end{pmatrix} \geq 0
\]
Stability analysis of a parametric system

– If we note

\[ r_{\text{min}}(p, \delta) = \min (b_2, b_1 b_2 - b_0, b_0) \]

We get

\[ P(s - \delta, p) \text{ unstable} \iff r_{\text{min}}(p, \delta) \leq 0 \]

– Therefore, the stability degree is:

\[ \delta^*(p) = \min_{\delta \geq 0} \delta \quad \text{subject to} \quad r_{\text{min}}(p, \delta) \leq 0 \]