Ray tracing and stability analysis of parametric systems

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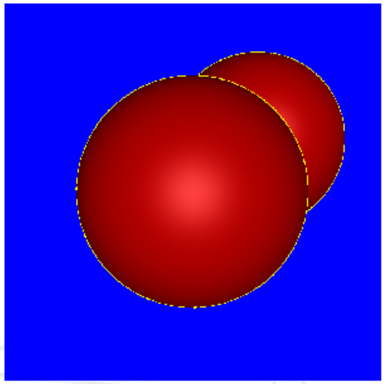
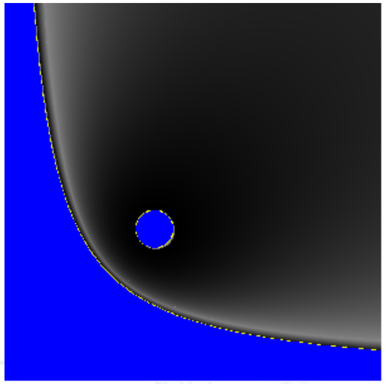
> Plan

Introduction

Introduction

- Goal : Show similarities between 2 problems apparently different : ray tracing and parametric stability analysis





- Use of interval analysis

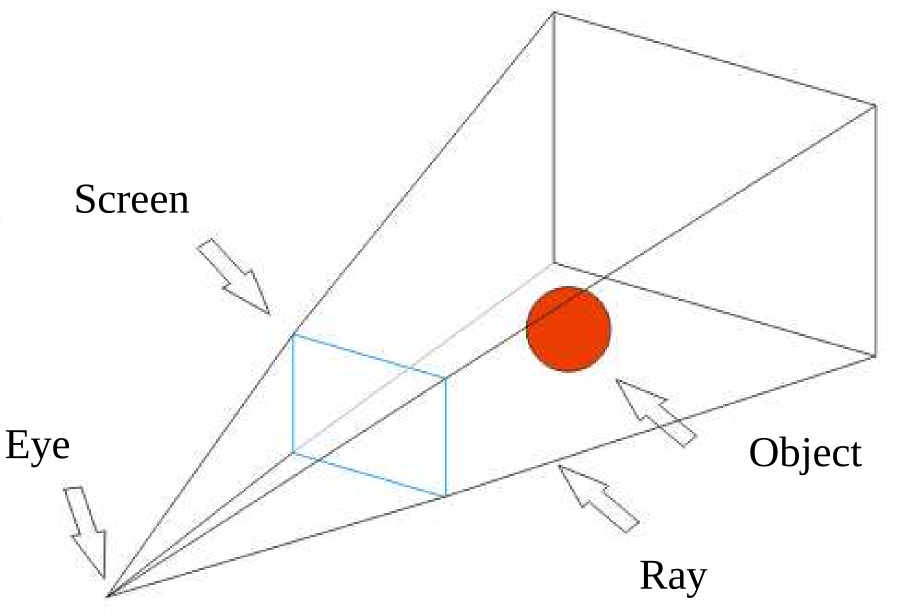
Ray tracing

Ray tracing

- Description
  - Ray tracing, ray casting
  - 3D scene display
  - Method : build the reverse light path starting from the screen to the object

Ray tracing

- Hypothesis
  - Objects are defined by implicit functions
  - The eye is at the origin of a coordinate space R(O,i,j,k) and the screen is at z=1
  - The screen is not in the object



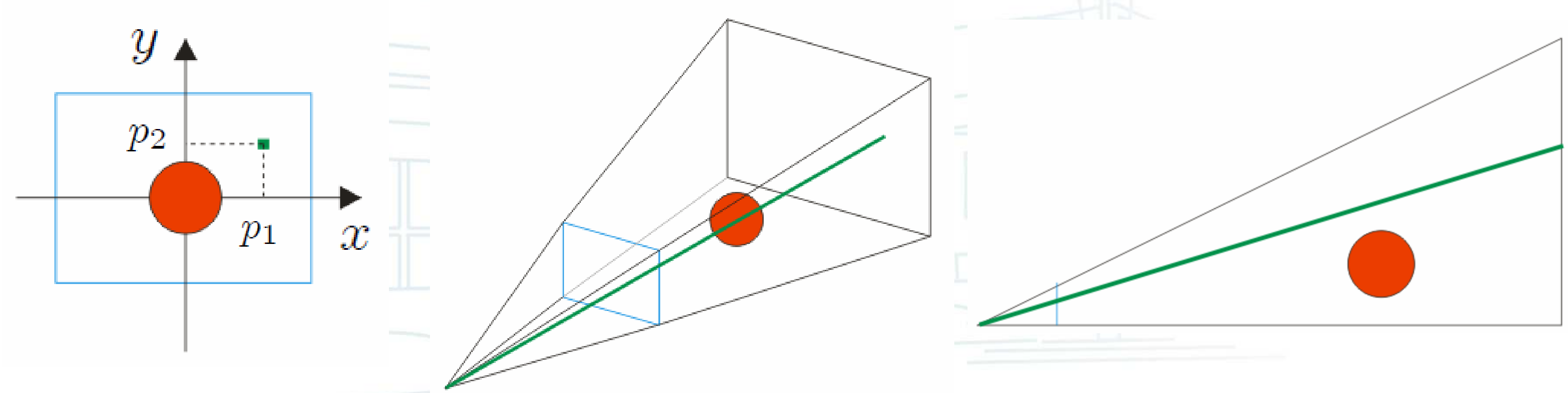
Ray tracing

- Problem description
  - A ray assiociated with the pixel

$$\mathbf{p} = (p_1, p_2) \in [\mathbf{p}]$$

satisfies

$$x = p_1.d$$
$$y = p_2.d$$
$$z = d$$

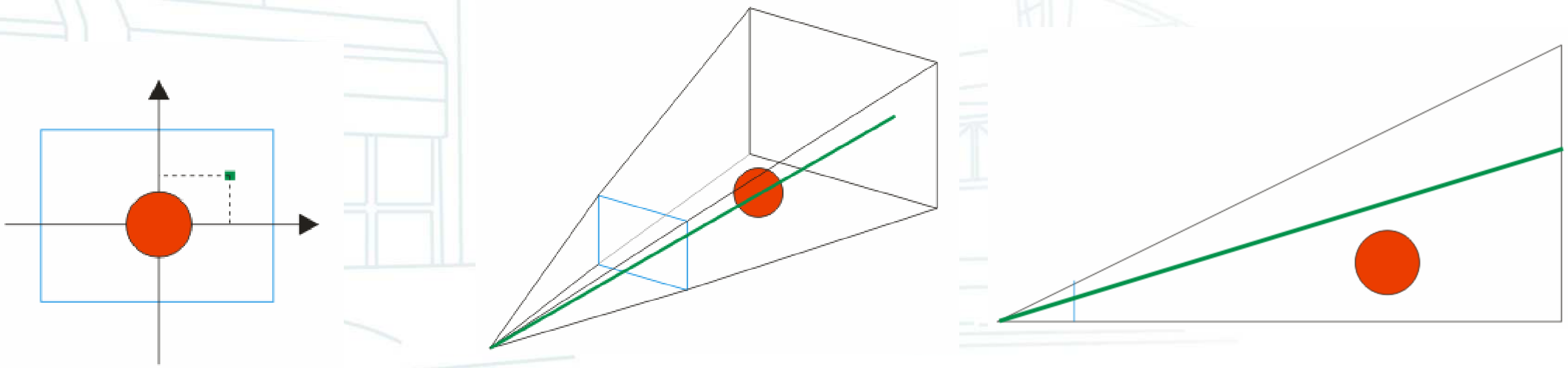


Ray tracing

- Problem description
  - The point (x, y, z) is in the object if

$$f(x, y, z) \leq 0$$

  - A pixel displays a point of the object if the associated ray intersects the object



Ray tracing

The ray associated with **p** intersects the object if

$$\exists d \geq 0, g(\mathbf{p}, d) \leq 0$$

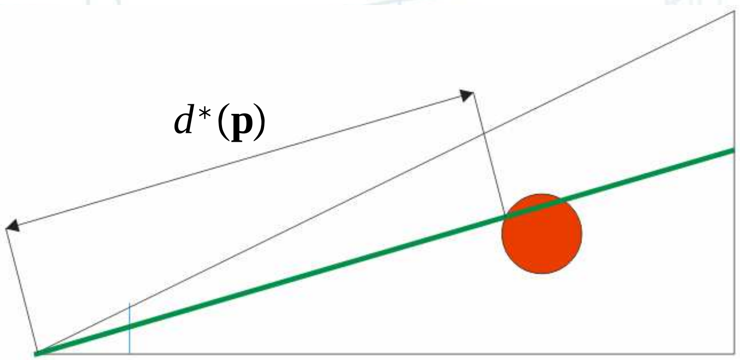
with

$$g(\mathbf{p}, d) = f(p_1 . d, p_2 . d, d)$$

Ray tracing

- Light effects handling
  - Realism => illumination model
  - Phong : needs the distance from the eye to the object
  - We need to compute for each pixel $\mathbf{p}$:

$$d^*(\mathbf{p}) = \min_{\substack{d \geq 0 \\ g(\mathbf{p},d) \leq 0}} d$$



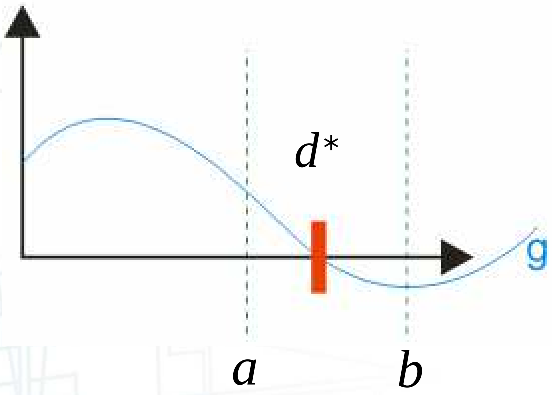
Ray tracing

- Computation of d^*
  – If

$$\begin{cases} g([0,a]) \subset [0,\infty[\\ g(b) < 0 \end{cases}$$

Then

$$d^* \in [a,b]$$



Moreover, if

$$g'([a,b]) \subset]-\infty,0]$$

We can use a dichotomy to get d^*

Ray tracing

- Computation of d^*
  - Interval computations are used to find $[a,b]$
  - A dichotomy finds d^*

Ray tracing

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- Parametric version
  - $g(\mathbf{p}, d)$ now depends on $\mathbf{p} \in [\mathbf{p}]$
  - If

$$\begin{cases} g([\mathbf{p}],[0,a]) \subset [0,\infty[\\ g([\mathbf{p}],b) \subset]-\infty,0] \end{cases}$$

Then

$$d^*([\mathbf{p}]) \subset [a,b]$$

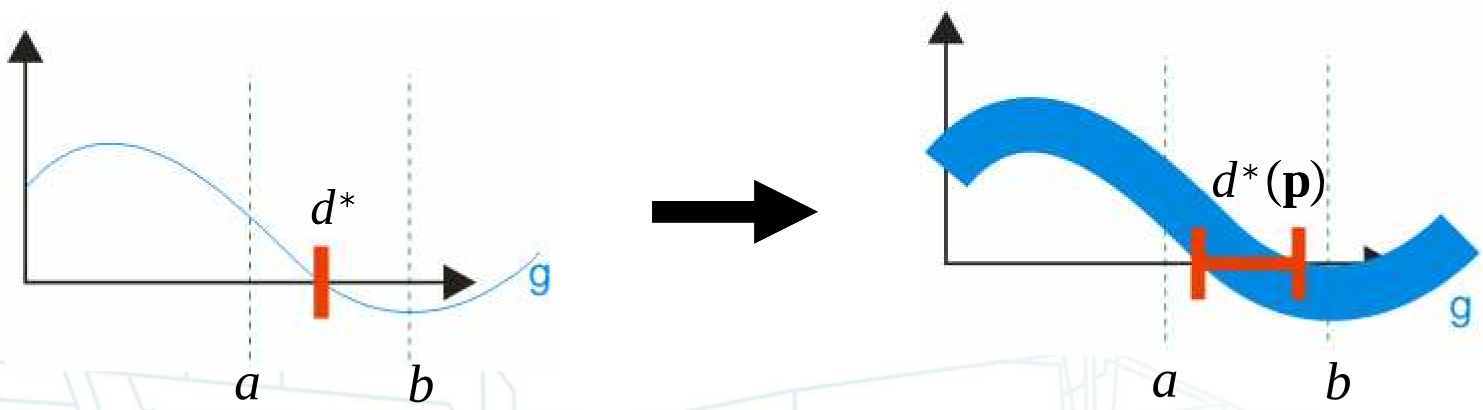
Moreover if

$$\frac{\partial g}{\partial d}([\mathbf{p}],[a,b]) \subset]-\infty,0]$$

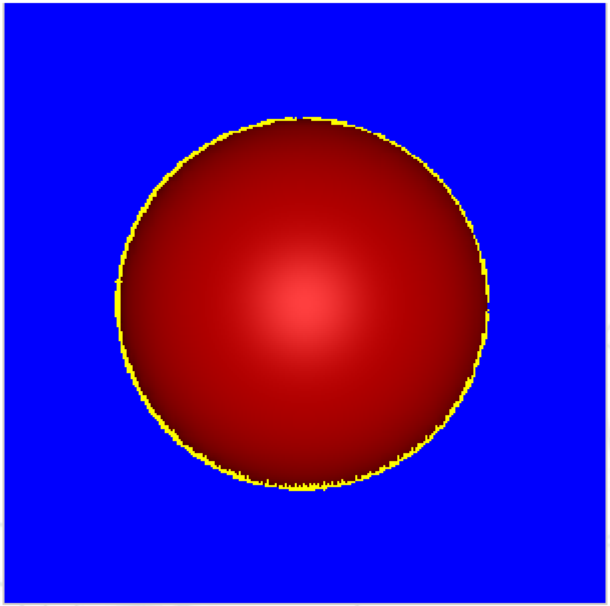
We can use a dichotomy to get d^* for each $\mathbf{p}$

Ray tracing

- From d^* to $d^*(\mathbf{p})$



Ray tracing



Stability analysis of a parametric system

Stability analysis of a parametric system

- Stability

$$P(s,\mathbf{p}) \text{ stable} \Leftrightarrow \text{all its roots have a real part} \leq 0$$

$$\overset{\text{(Routh)}}{\Leftrightarrow} r(\mathbf{p}) \leq 0$$

where r is retrieved from the Routh table

Stability analysis of a parametric system

- δ stability

$$P(s,\mathbf{p}) \text{ is } \delta \text{ stable} \Leftrightarrow \text{all its roots have a real part} \leq \delta$$

(Routh)

$$\Leftrightarrow r(\mathbf{p},\delta) \leq 0$$

Stability analysis of a parametric system

- Example : Ackermann

$$
\begin{aligned}
P(s,\mathbf{p}) &= s^3 + (p_1 + p_2 + 2)s^2 + (p_1 + p_2 + 2)s \\
&\quad +2p_1 p_2 + 6p_1 + 6p_2 + 2.25.
\end{aligned}
$$

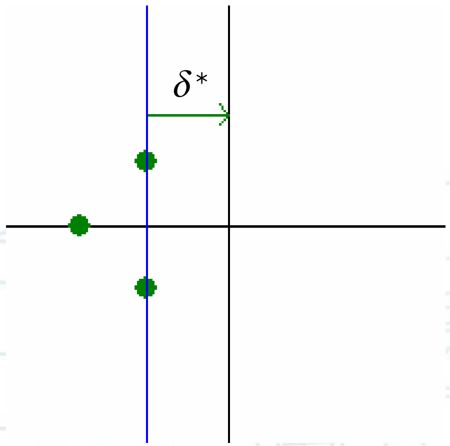
is δ stable if

$$
r(\mathbf{p},\delta) = \min\begin{pmatrix} p_1 + p_2 + 2 - 3\delta \\ (p_1 - 1)^2 + (p_2 - 1)^2 - 0.25 - 2\delta((p_1 + p_2 + 2)(p_1 + p_2 + 3 - 4\delta) + 4\delta^2) \\ 2(p_1 + 3)(p_2 + 3) - 15.75 - \delta((p_1 + p_2 + 2)(1 + \delta) - \delta^2) \end{pmatrix} \leq 0
$$

Stability analysis of a parametric system

- Stability degree

$$\delta^*(\mathbf{p}) = \min_{\substack{\delta \geq 0 \\ r(\mathbf{p},\delta) \leq 0}} \delta$$

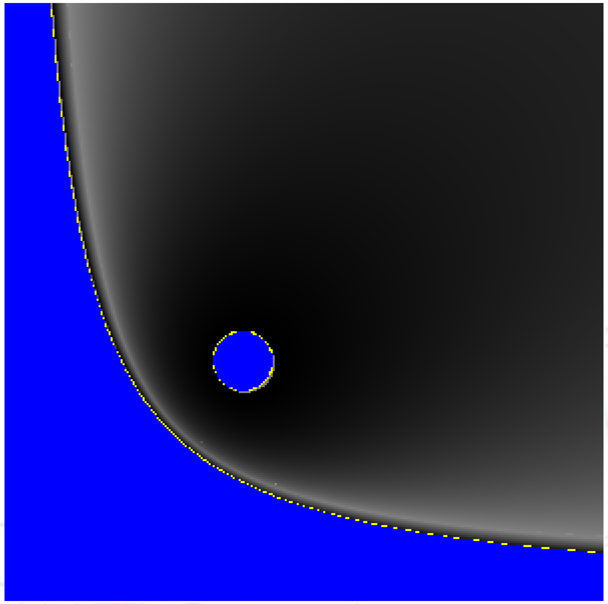


Stability analysis of a parametric system

- Similarities with ray tracing

Ray tracing		Stability degree
d	$\leftrightarrow$	δ
g	$\leftrightarrow$	r
$d^*(\mathbf{p}) = \min\limits_{\substack{d\geq 0 \\ g(\mathbf{p},d)\leq 0}} d$	$\leftrightarrow$	$\delta^*(\mathbf{p}) = \min\limits_{\substack{\delta\geq 0 \\ r(\mathbf{p},\delta)\leq 0}} \delta$

Stability analysis of a parametric system



Conclusion

Conclusion

- Ray tracing and stability degree drawing of a linear system are similar problems

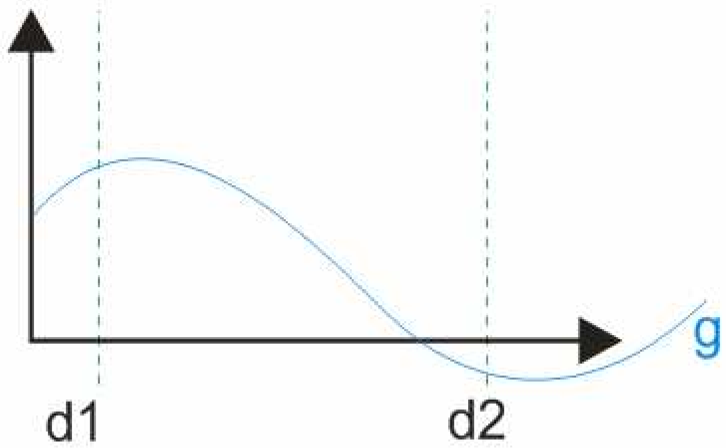
- A common algorithm based on intervals and dichotomy has been proposed

References

- L. Jaulin. *Solution globale et garantie de problèmes ensemblistes; Application à l'estimation non linéaire et à la commande robuste.* PhD thesis, Université Paris XI Orsay, 1994.
- S. Bazeille. *Vision sous-marine monoculaire pour la reconnaissance d'objets.* PhD thesis, Université de Bretagne Occidentale, 2008.
- J. Flórez. Improvements in the ray tracing of implicit surfaces based on interval arithmetic. PhD thesis, Universitat de Girona, 2008.
- L. Jaulin, M. Kieffer, O. Didrit et E. Walter, *Applied interval analysis*, Springer-Verlag, London, Great Britain, 2001.
- L. Jaulin, E. Walter, O. Lévêque et D. Meizel, "Set inversion for chi-algorithms, with application to guaranteed robot localization", *Math. Comput Simulation*, 52, pp. 197-210, 2000.
- J. Ackermann, "Does it suffice to check a subset of multilinear parameters in robustness analysis?", *IEEE Transactions on Automatic Control*, 37(4), pp. 487-488, 1992.
- J. Ackermann, H. Hu et D. Kaesbauer, "Robustness analysis: a case study", *IEEE Transactions on Automatic Control*, 35(3), pp. 352-356, 1990.

Ray tracing

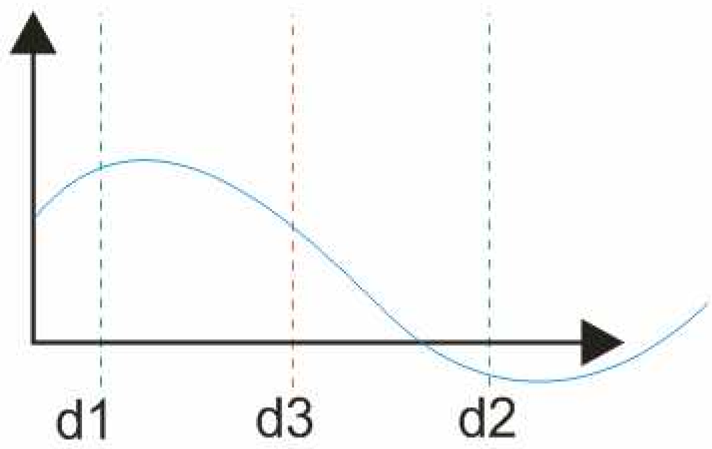
$$\mathcal{L} = \{[d_1, d_2]\} \quad \longrightarrow \quad \begin{array}{l} d = [d_1, d_2] \\ \mathcal{L} = \{\} \end{array}$$



$$[g]\,([d]) \not\subset [0, \infty[$$
$$[g']\,([d]) \not\subset]-\infty, 0]$$

Ray tracing

$$\mathcal{L} = \{[d_1, d_3], [d_3, d_2]\} \longrightarrow \begin{array}{l} d = [d_1, d_3] \\ \mathcal{L} = \{[d_3, d_2]\} \end{array}$$

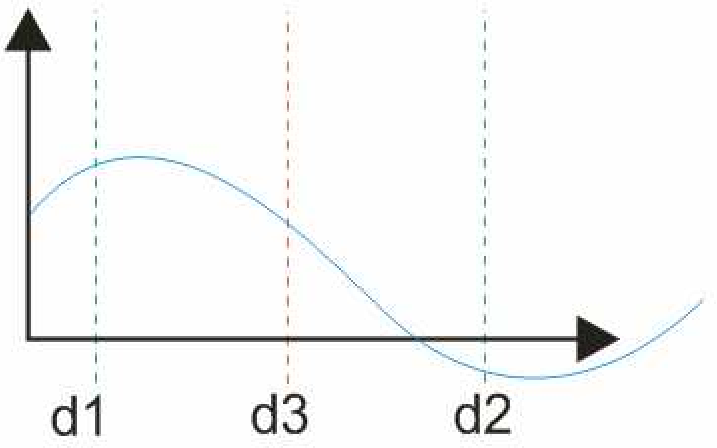


$$[g]([d]) \subset [0, \infty[$$

=> a = d3

Ray tracing

$$\mathcal{L} = \{[d_3, d_2]\}$$ → $$d = [d_3, d_2]$$ $$\mathcal{L} = \{\}$$



$$[g]\left([d]\right) \not\subset [0, \infty[$$

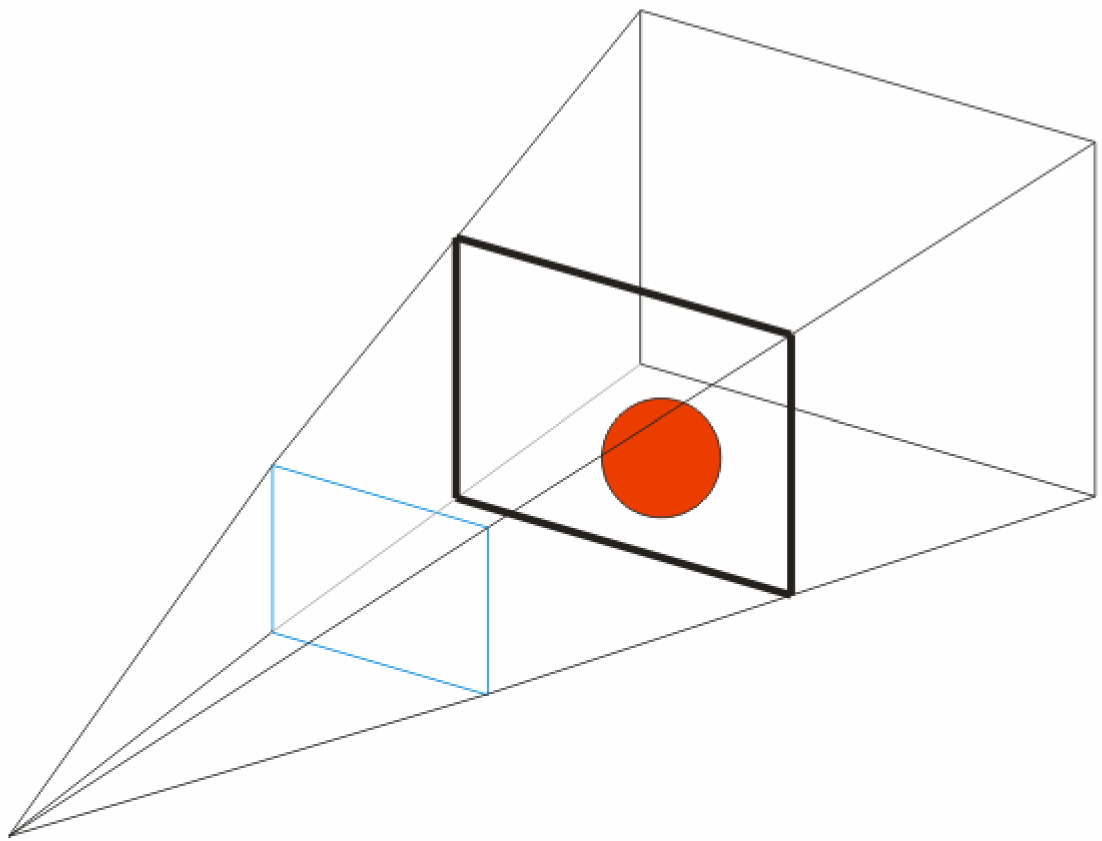
$$[g']\left([d]\right) \subset]-\infty, 0]$$

$$[g]\left([d^+]\right) < 0$$

=> b = d2

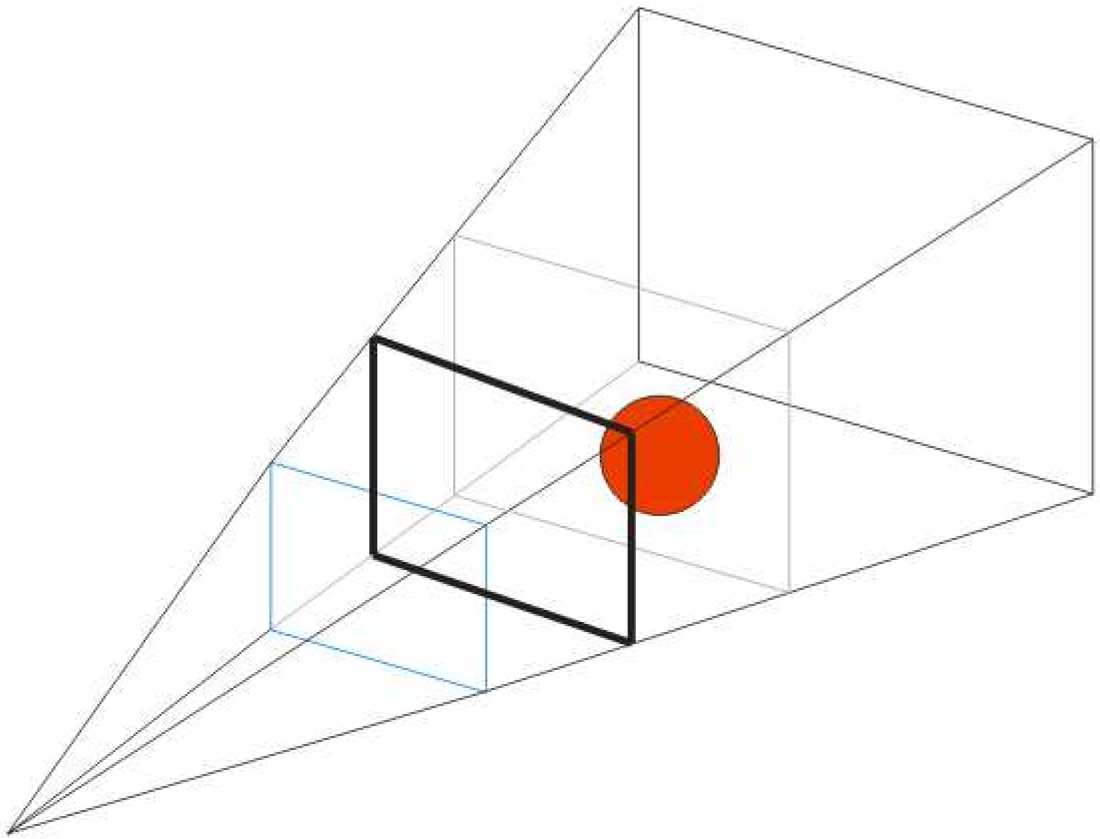
Ray tracing

- Division in d



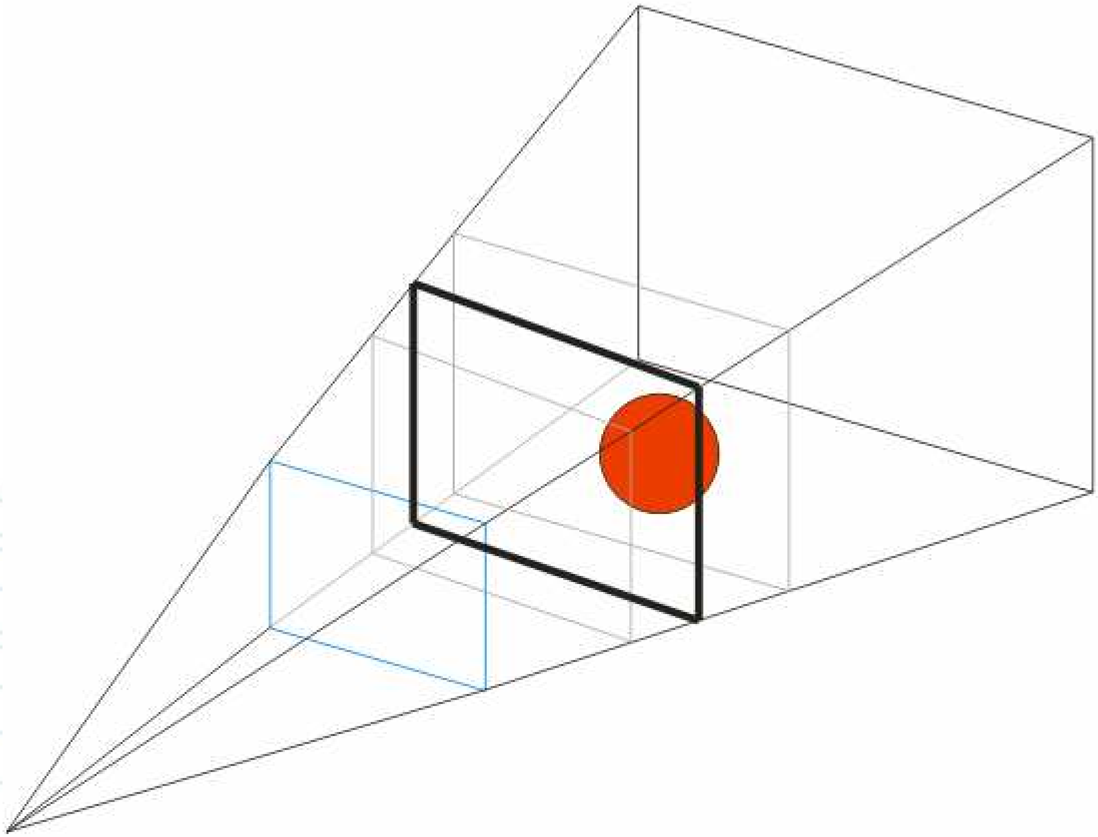
Ray tracing

- Division in d



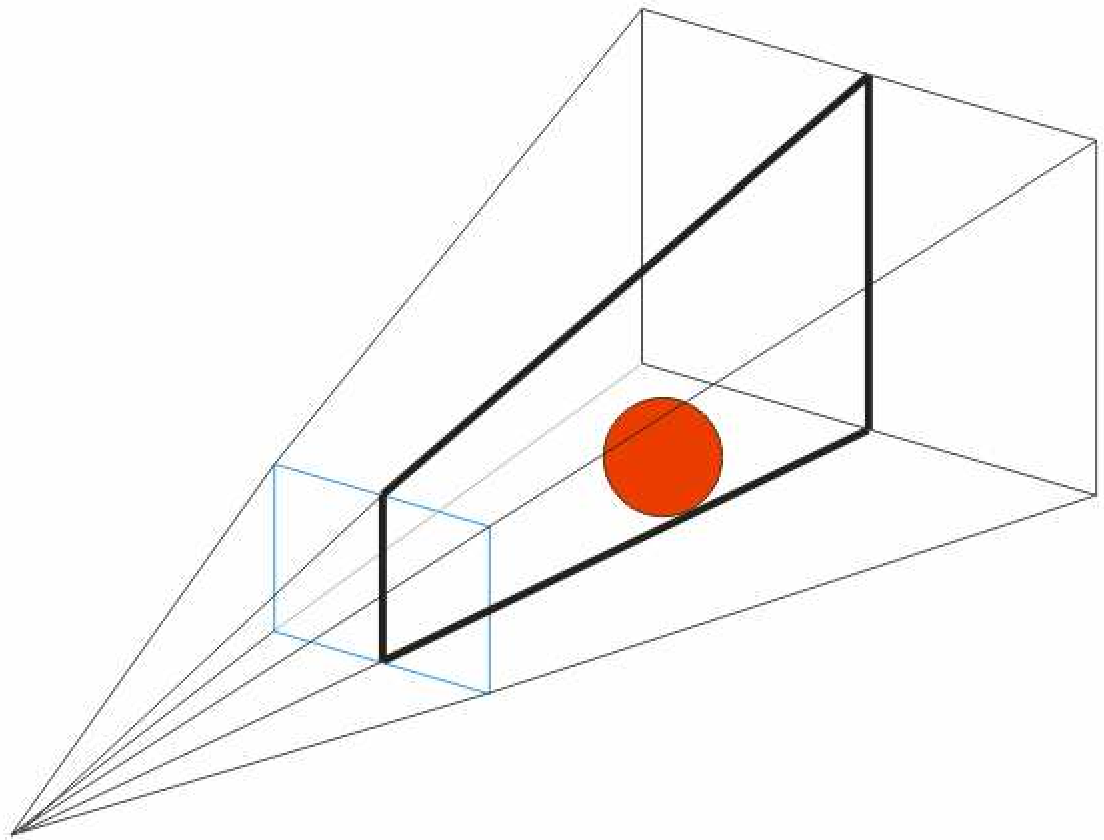
Ray tracing

- Division in d



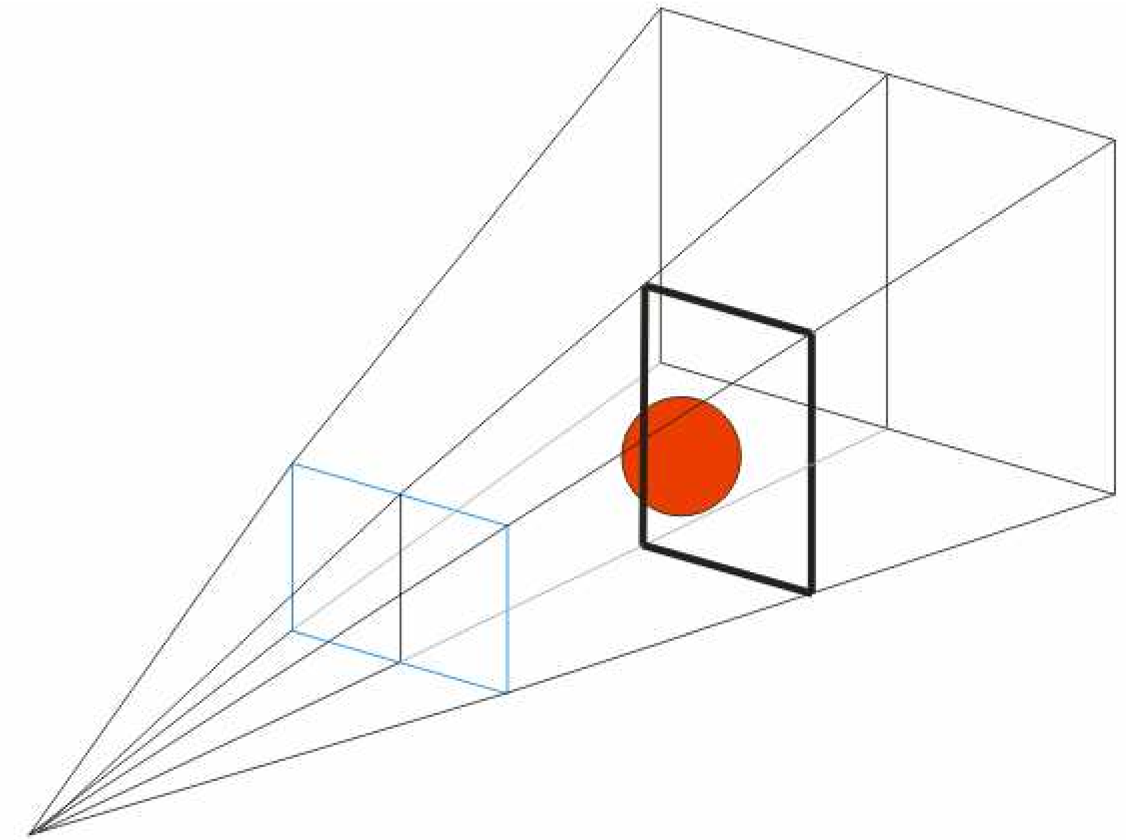
Ray tracing

- Division in **p**



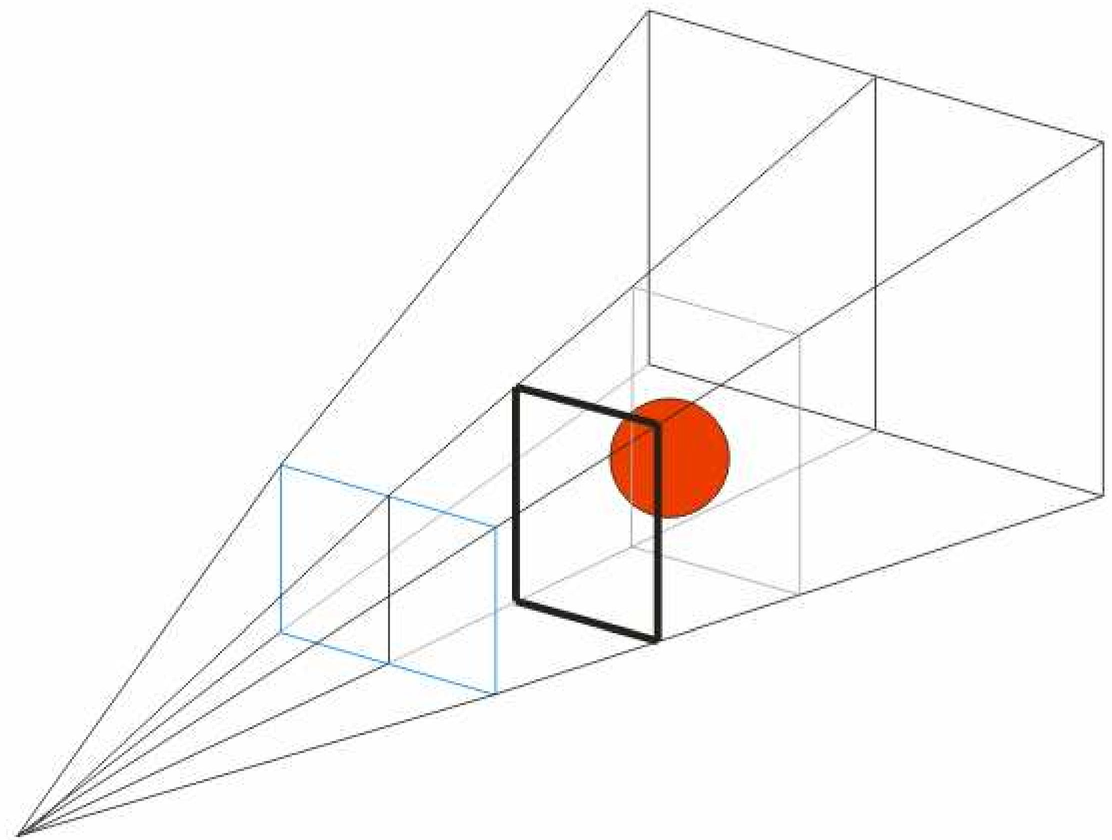
Ray tracing

- Division in **p**



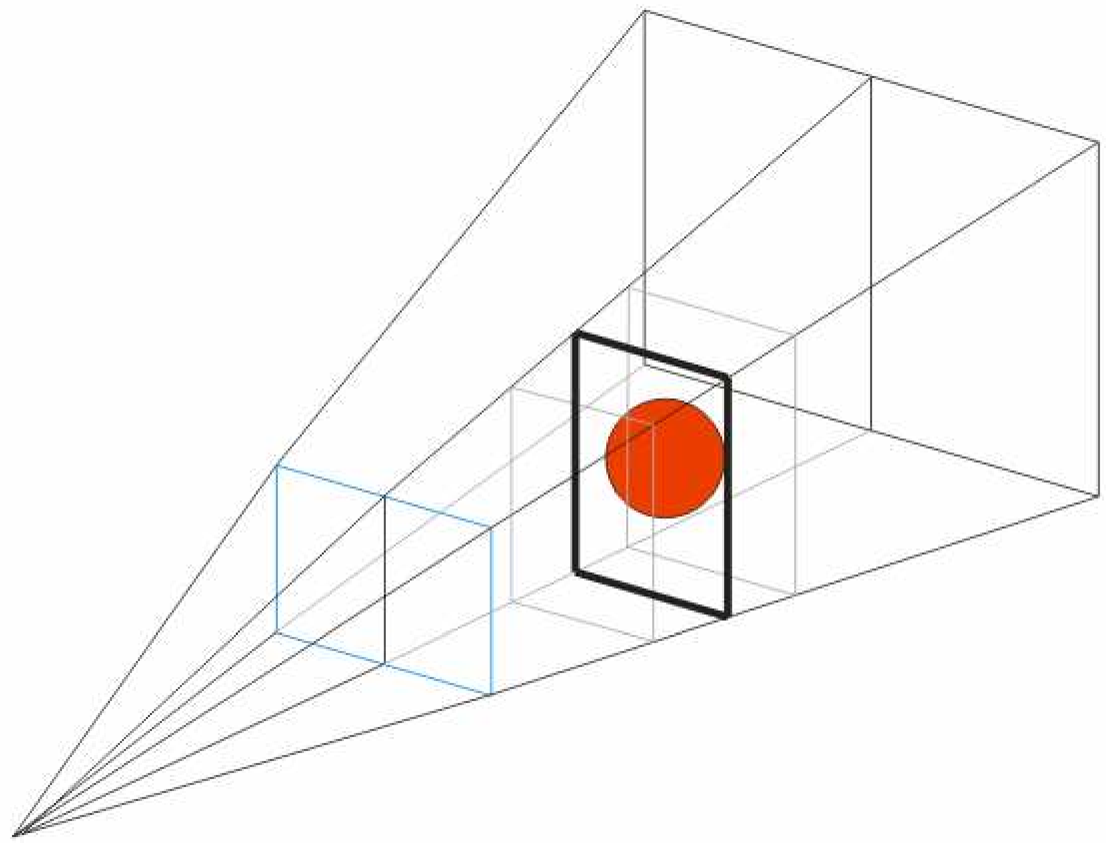
Ray tracing

- Division in **p**



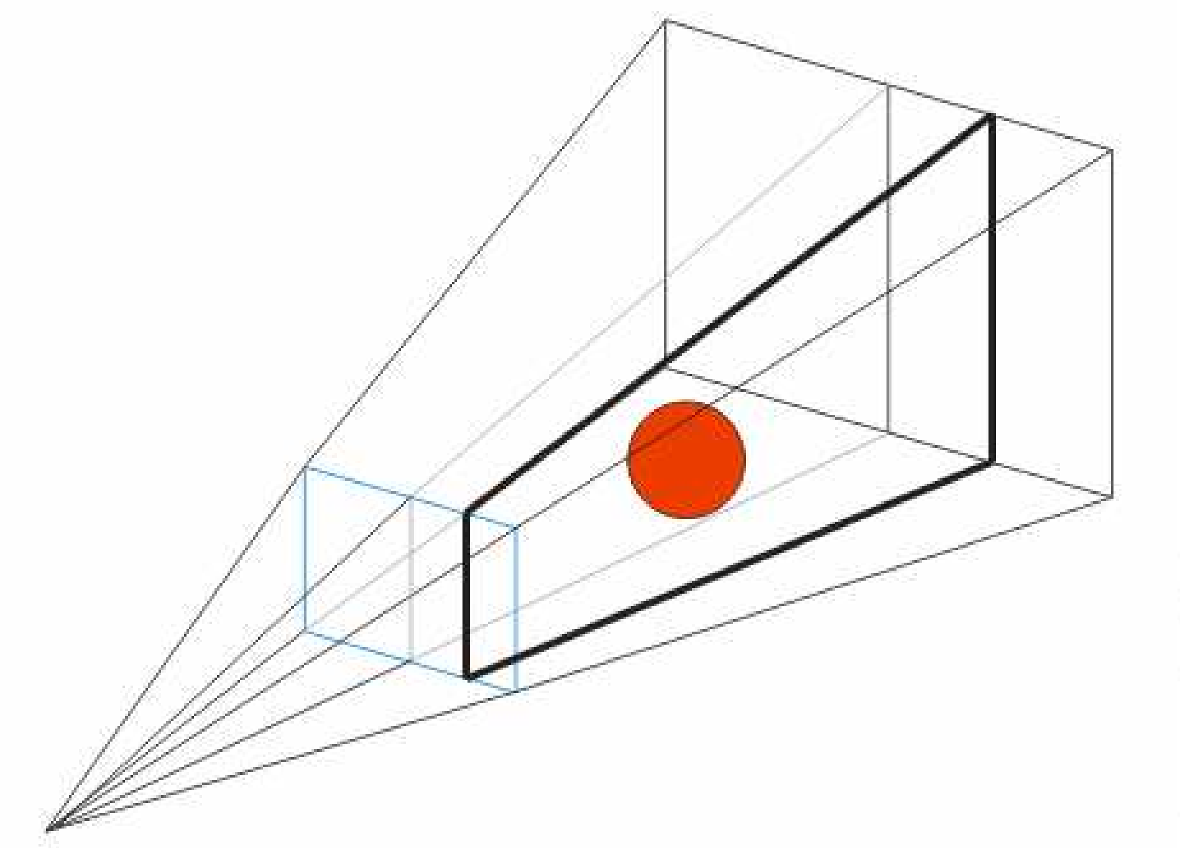
Ray tracing

- Division in **p**



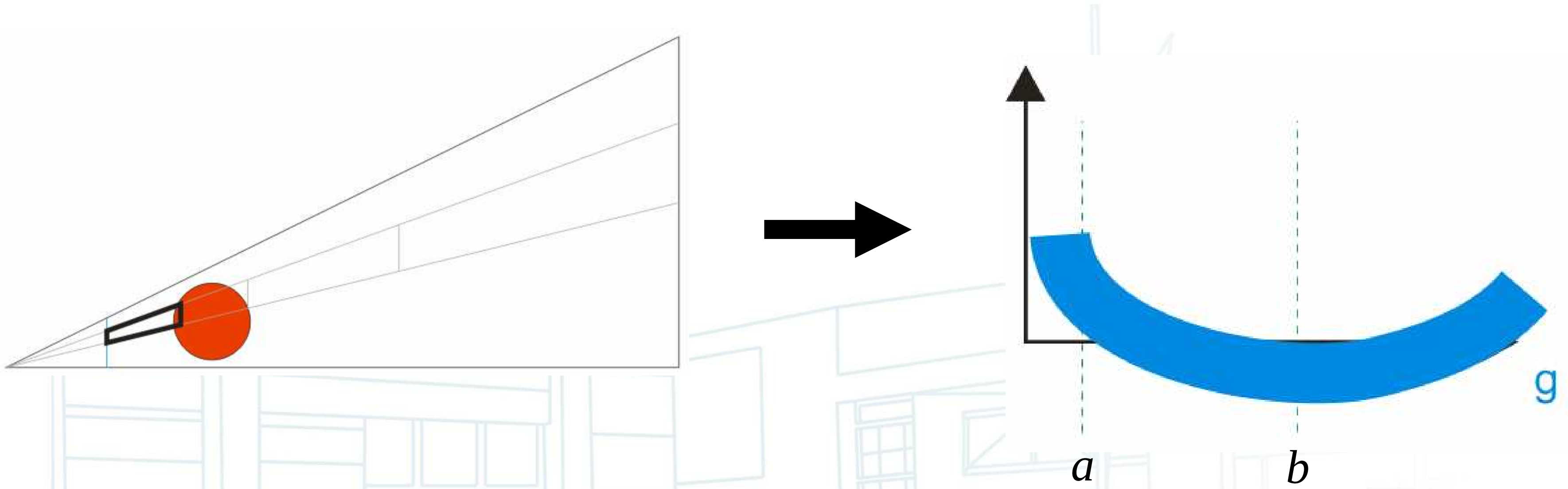
Ray tracing

- Division in **p**

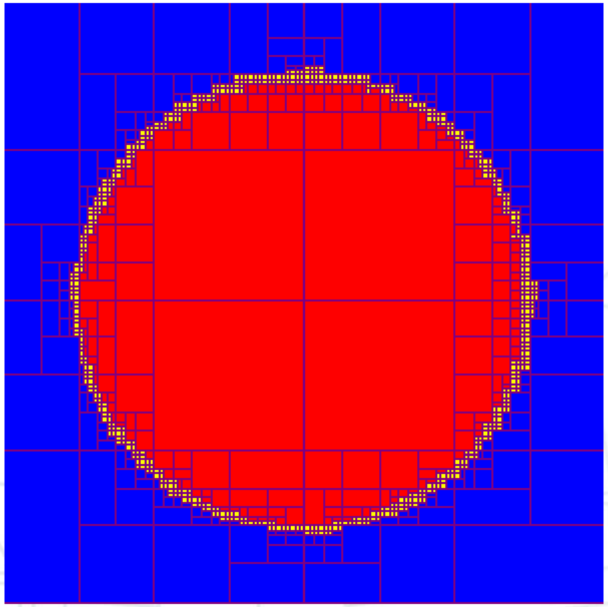


Ray tracing

- Division in **p** and in d



Ray tracing

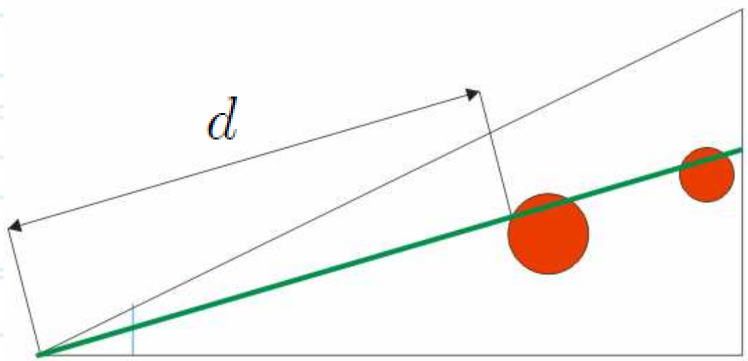


Ray tracing

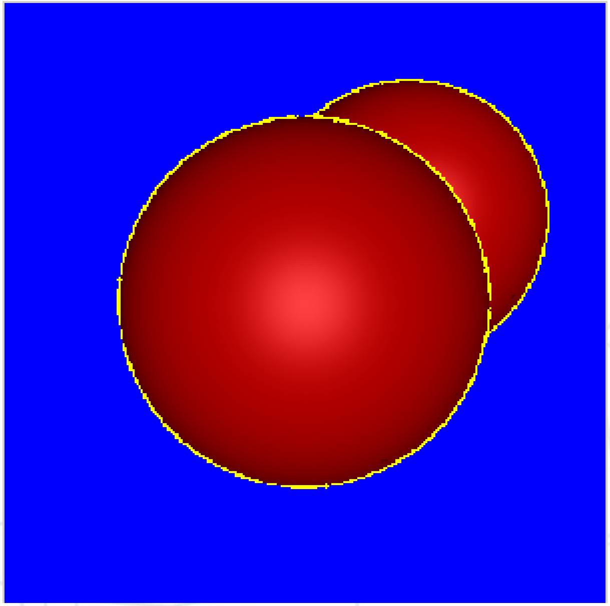
- Several objects display handling
  - We apply the previous algorithm for the function :

$$g_{\min} : (\mathbf{p}, d) \to \min_i g_i(\mathbf{p}, d)$$

  - Indeed, we have to consider only the first object crossed by the ray



Ray tracing



Stability analysis of a parametric system

– Stability degree of an invariant linear system of caracteristic polynomial P(s) :

$$\delta^* = \min_{P(s-\delta)\ \text{unstable}} \delta.$$

– We consider an invariant linear system parametized with a vector of parameter **p** :

$$\begin{aligned} P(s,\mathbf{p}) &= s^3 + (p_1+p_2+2)s^2 + (p_1+p_2+2)s \\ &\quad +2p_1p_2 + 6p_1 + 6p_2 + 2.25. \end{aligned}$$

Stability analysis of a parametric system

– The stability degree becomes :

$$\delta^*(\mathbf{p}) = \min_{P(s-\delta,\mathbf{p})\ \text{unstable}} \delta.$$

– With

$$P(s-\delta,\mathbf{p}) = s^3 + b_2 s^2 + b_1 s + b_0$$

the polynomial is stable if (Routh) :

$$\begin{pmatrix} b_2 \\ b_1 b_2 - b_0 \\ b_0 \end{pmatrix} \geq 0$$

Stability analysis of a parametric system

– If we note

$$r_{\min}(\mathbf{p},\delta) = \min\left(b_2, b_1 b_2 - b_0, b_0\right)$$

We get

$$P(s-\delta,\mathbf{p}) \text{ unstable } \Leftrightarrow r_{\min}(\mathbf{p},\delta) \leq 0$$

– Therefore, the stability degree is :

$$\delta^*(\mathbf{p}) = \min_{\substack{\delta \geq 0 \\ r_{\min}(\mathbf{p},\delta) \leq 0}} \delta$$