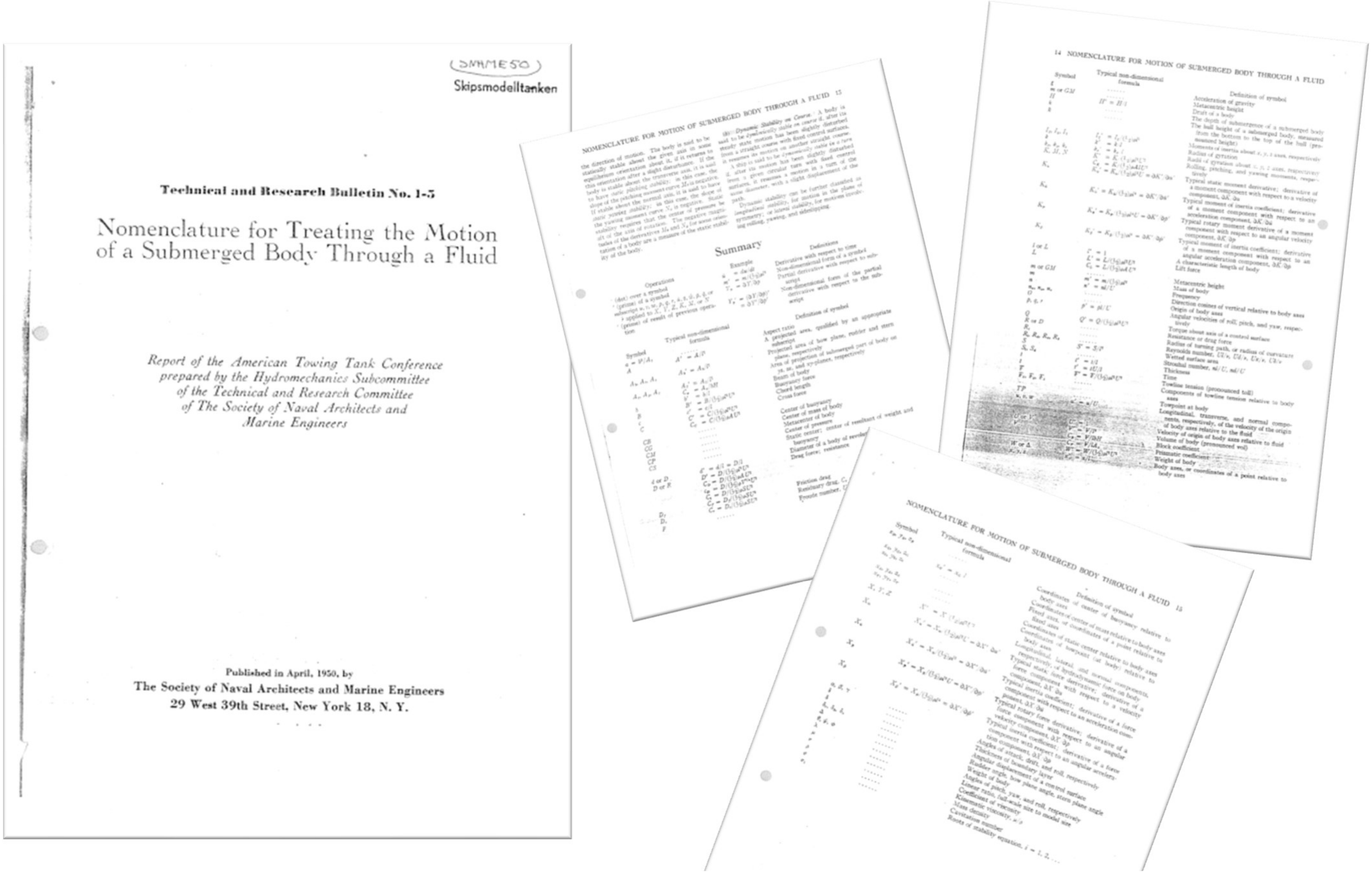
# II - Models

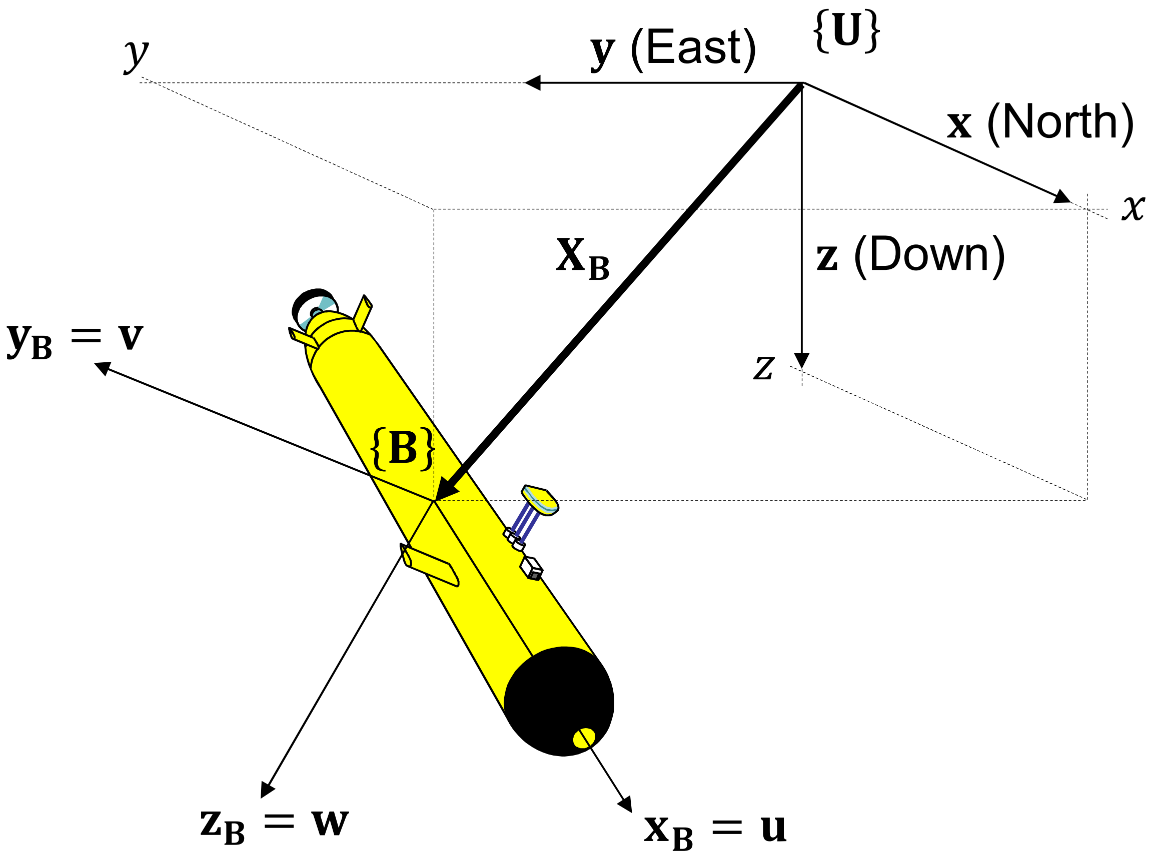
# Notation


SNAME50
Skipsmodelltanken
Technical and Research Bulletin No. 1-5
Nomenclature for Treating the Motion of a Submerged Body Through a Fluid
Report of the American Towing Tank Conference prepared by the Hydromechanics Subcommittee of the Technical and Research Committee of The Society of Naval Architects and Marine Engineers
Published in April, 1950, by
The Society of Naval Architects and Marine Engineers
29 West 39th Street, New York 18, N. Y.
NOMENCLATURE FOR MOTION OF SUBMERGED BODY THROUGH A FLUID
Summary

# II – Models
# a) Representation Formalism

# Representation Formalism

- Position



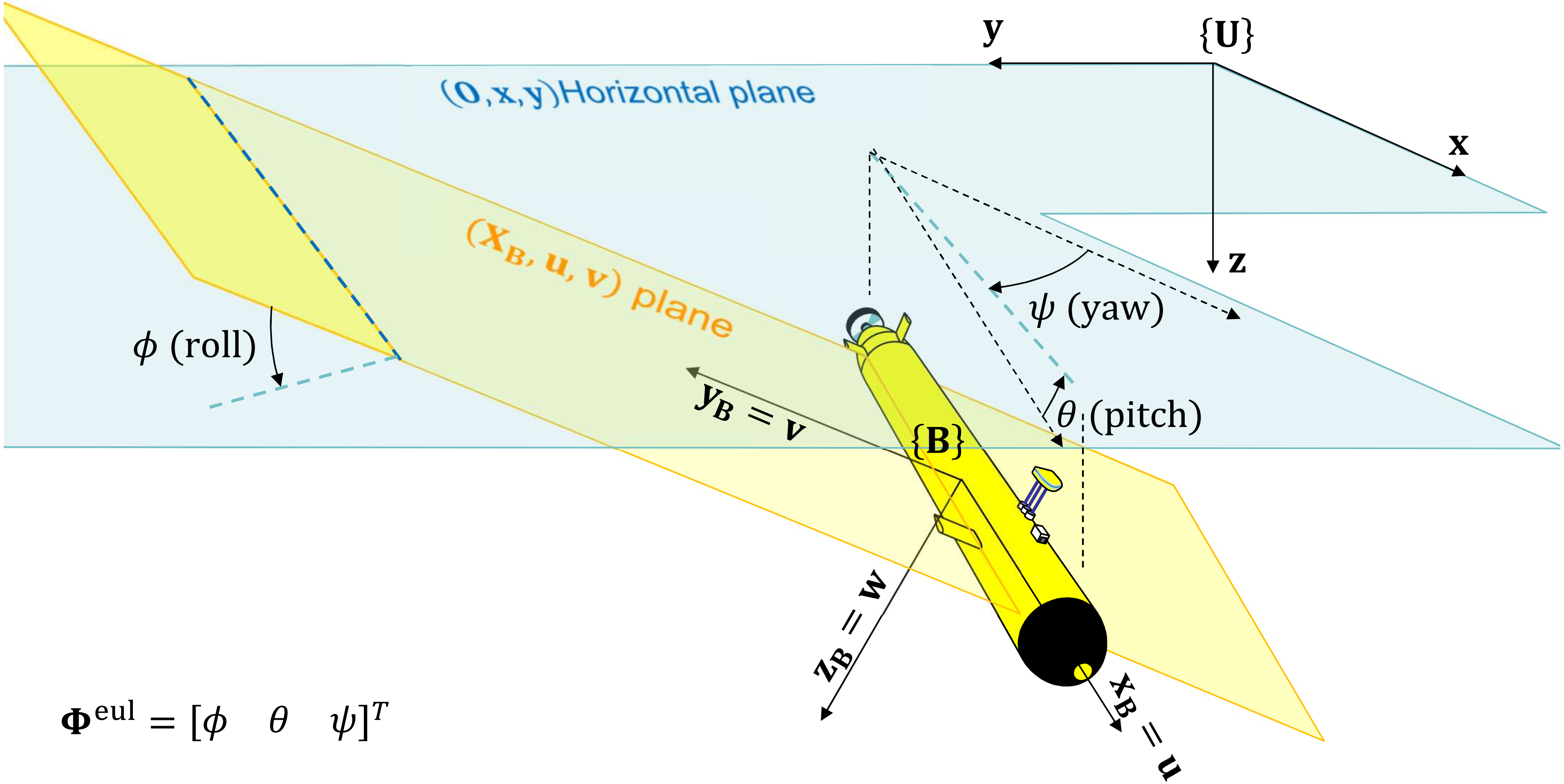

$\{\mathbf{U}\}: (\mathbf{O}, \mathbf{x}, \mathbf{y}, \mathbf{z})$

$\{\mathbf{B}\}: (\mathbf{X_B}, \mathbf{u}, \mathbf{v}, \mathbf{w})$

$\mathbf{X} = [x \quad y \quad z]^T$

# Representation Formalism

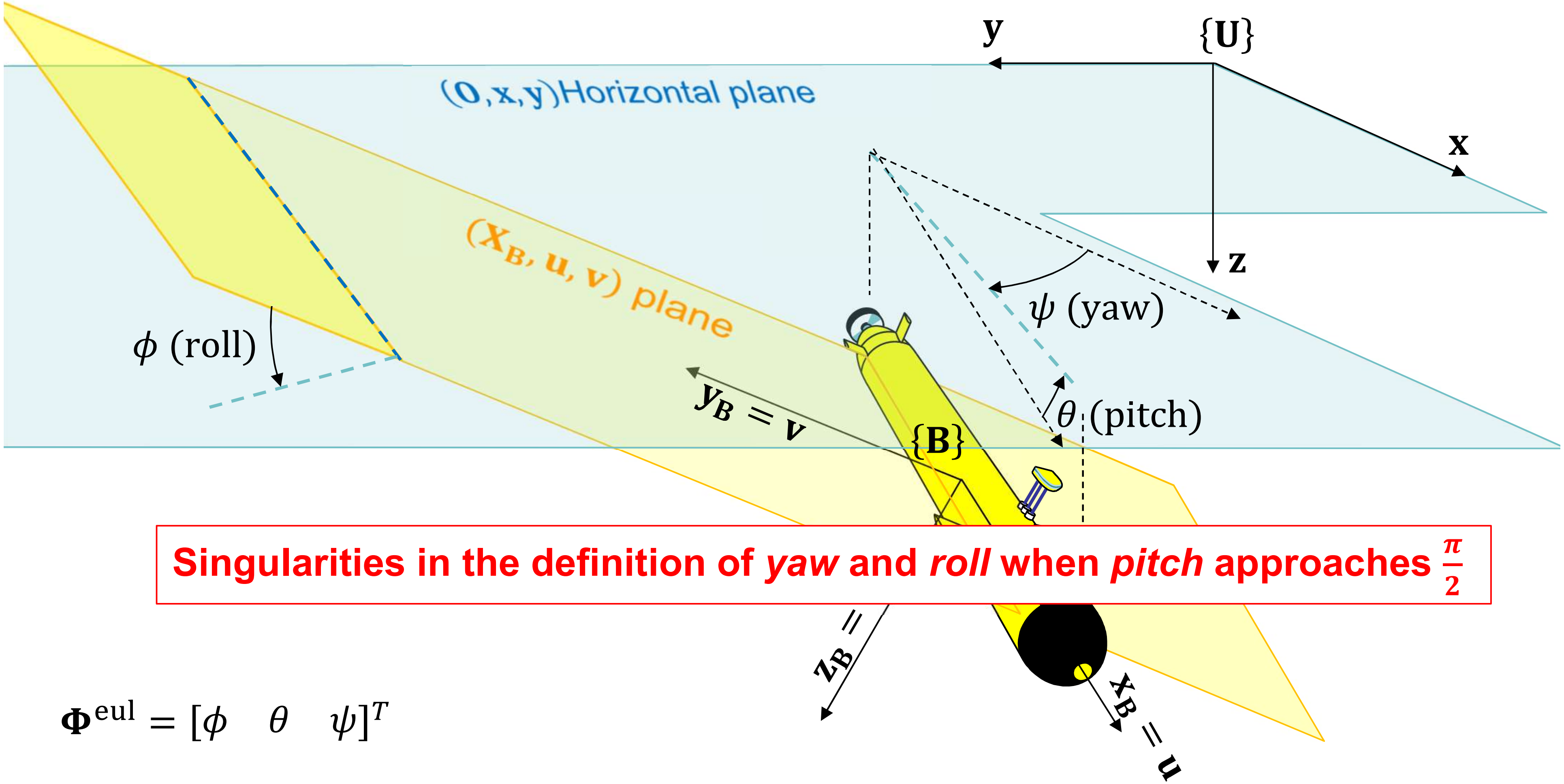
- Attitude, Orientation : Euler angles




$$\mathbf{\Phi}^{\text{eul}} = [\phi \quad \theta \quad \psi]^T$$

# Representation Formalism

- Attitude, Orientation : Euler angles




**Singularities in the definition of *yaw* and *roll* when *pitch* approaches $\frac{\pi}{2}$**

$$\mathbf{\Phi}^{\text{eul}} = [\phi \quad \theta \quad \psi]^T$$

# Representation Formalism

- Attitude, Orientation : Euler angles

$$\mathbf{\Phi}^{\text{eul}} = [\phi \quad \theta \quad \psi]^T$$

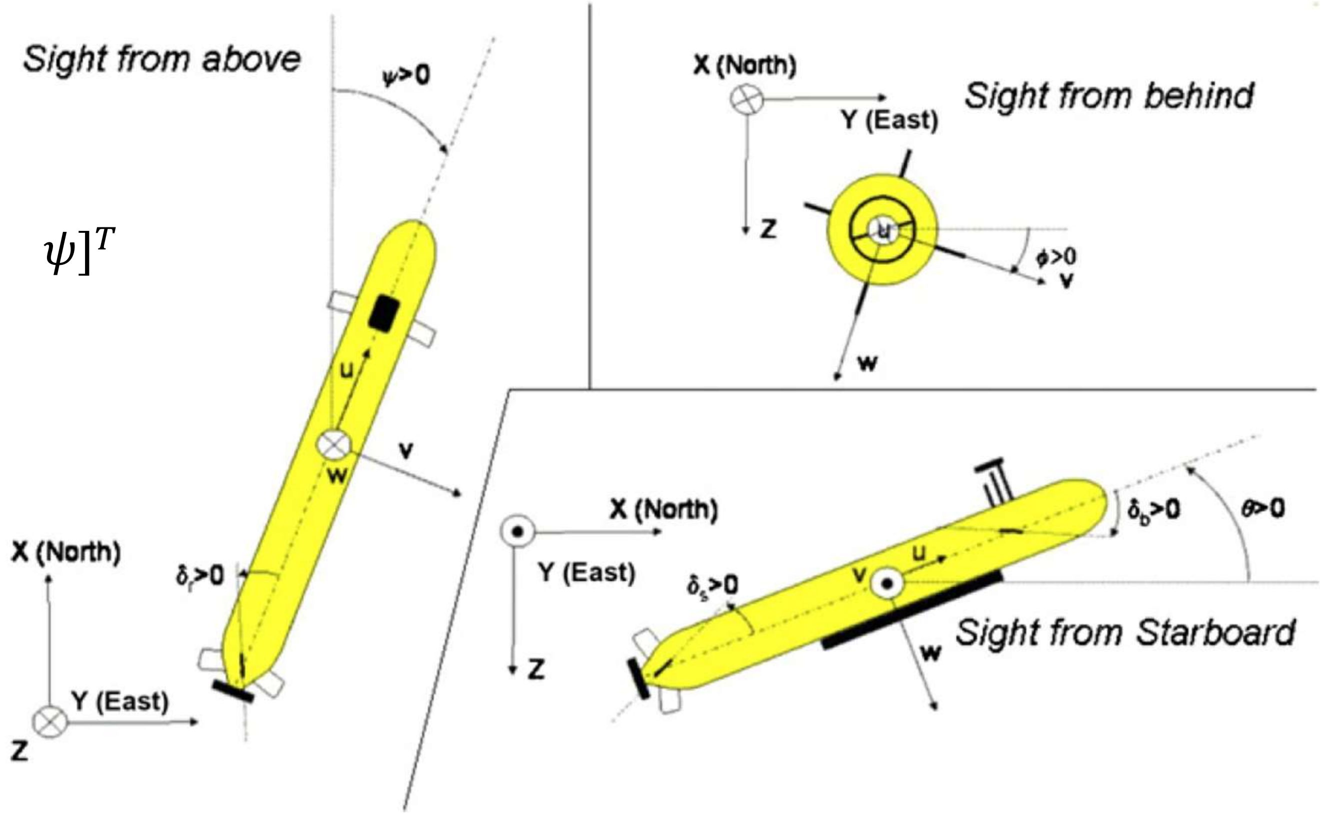
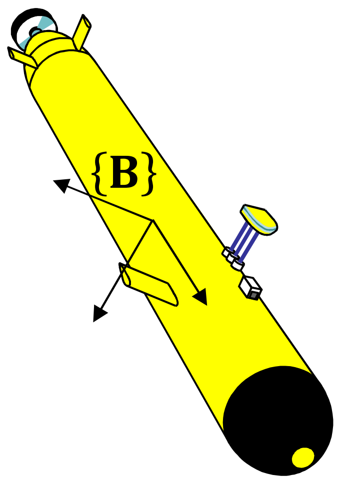
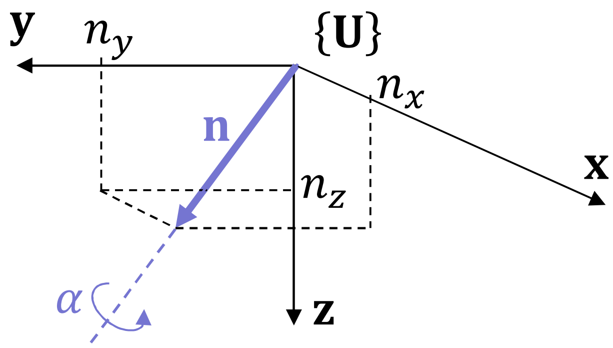



**Fig. 2.** Body frame and control surfaces angles definition.

# Representation Formalism

- Attitude, Orientation : Rotation Matrix




$$\mathbf{n} = [n_x \quad n_y \quad n_z]^T, \|\mathbf{n}\| = 1$$

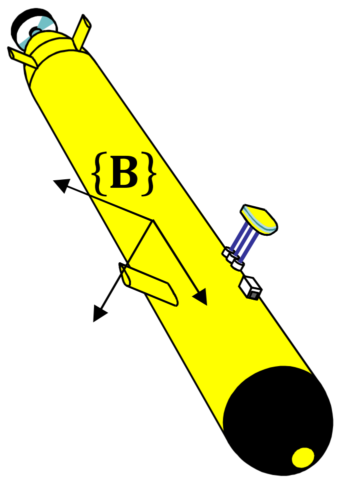
$$\mathbf{R} = \mathbf{P} + \cos\alpha \cdot (\mathbf{I} - \mathbf{P}) + \sin\alpha \cdot \mathbf{Q}$$

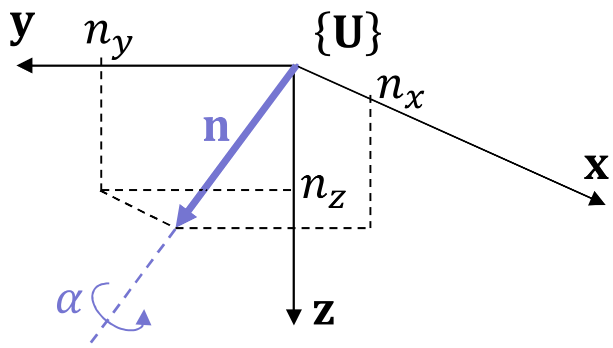
with : $\mathbf{P} = \mathbf{n} \cdot \mathbf{n}^T, \boldsymbol{Q} = \wedge(\mathbf{n})$

$$\mathbf{Q} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$

# Representation Formalism

- Attitude, Orientation : Quaternion




$$\mathbf{n} = [n_x \quad n_y \quad n_z]^T, \|\mathbf{n}\| = 1$$

$$\mathbf{Q}_{\boldsymbol{B}} = \left[ cos\left(\frac{\alpha}{2}\right) \quad \mathbf{n}^T \cdot sin\left(\frac{\alpha}{2}\right) \right]^T$$

$$\mathbf{Q}_{\boldsymbol{B}} = [a \quad b \quad c \quad d]^T$$

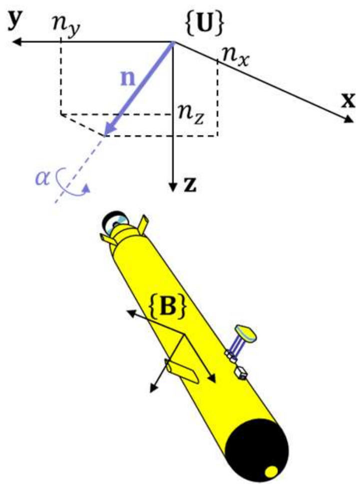
# Representation Formalism

- Attitude, Orientation : Rot mat. vs Quaternion

$$\mathbf{R} = \mathbf{P} + \cos\alpha \cdot (\mathbf{I} - \mathbf{P}) + \sin\alpha \cdot \mathbf{Q}$$

with : $\mathbf{P} = \mathbf{n} \cdot \mathbf{n}^T, \boldsymbol{Q} = \wedge(\mathbf{n})$

$$\mathbf{Q} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$




$$\mathbf{Q}_{\boldsymbol{B}} = \left[ cos\left(\frac{\alpha}{2}\right) \quad \mathbf{n}^T \cdot sin\left(\frac{\alpha}{2}\right) \right]^T$$

$$\mathbf{Q}_{\boldsymbol{B}} = [a \quad b \quad c \quad d]^T$$

$$\mathbf{R} = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2 \cdot b \cdot c - 2 \cdot a \cdot d & 2 \cdot a \cdot c + 2 \cdot b \cdot d \\ 2 \cdot a \cdot d + 2 \cdot b \cdot c & a^2 - b^2 + c^2 - d^2 & 2 \cdot c \cdot d - 2 \cdot a \cdot b \\ 2 \cdot b \cdot d - 2 \cdot a \cdot c & 2 \cdot a \cdot b - 2 \cdot c \cdot d & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

$$r = \pm\frac{1}{2} \cdot \sqrt{1 + \mathrm{Tr}(\mathbf{R})}; \begin{cases} a = r & c = \frac{1}{4 \cdot r} \cdot (R_{13} - R_{31}) \\ b = \frac{1}{4 \cdot r} \cdot (R_{32} - R_{23}) & d = \frac{1}{4 \cdot r} \cdot (R_{21} - R_{12}) \end{cases}$$

# Quaternion, basic relations

- Any rotation of an angle $\alpha$ around a unitary vector $\mathbf{n}$ can be expressed by the unitary quaternion :

$$\mathbf{Q} = \left[cos\left(\frac{\alpha}{2}\right) \quad \mathbf{n}^T \cdot sin\left(\frac{\alpha}{2}\right)\right]^T, \|\mathbf{n}\| = 1 \rightarrow \|\mathbf{Q}\| = 1$$

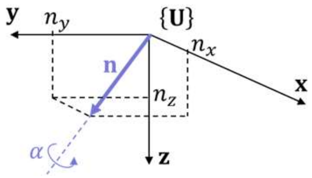
- The composition of 2 rotations, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be expressed using the (non commutative) quaternionic multiplication, resulting in the $\mathbf{Q_3}$ quaternion such that :

$$\mathbf{Q}_3 = \mathbf{Q}_2 \otimes \mathbf{Q}_1$$

- The conjugate of a quaternion $\mathbf{Q}$ is denoted $\mathbf{Q}^*$ and exhibits the following properties:

$$\mathbf{Q}^* = \left[cos\left(\frac{\alpha}{2}\right) \quad -\mathbf{n}^T \cdot sin\left(\frac{\alpha}{2}\right)\right]^T;$$

$$(\mathbf{Q}_1 \otimes \mathbf{Q}_2)^* = \mathbf{Q}_2^* \otimes \mathbf{Q}_1^* \ ;$$




$\mathbf{Q} \otimes \mathbf{Q}^* = \|\mathbf{Q}\| \cdot \mathbf{1_Q}$, where $\mathbf{1_Q} = [\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}]^{\boldsymbol{T}}$ is called *identity quaternion*

- A vector $\mathbf{v} \in \mathbb{R}^3$ can be expressed as a pure imaginary (non unitary) quaternion as:

$$\mathbf{V} = [0, \mathbf{v}^T]^T$$

- The rotation $\mathbf{Q}$ applied on a vector $\mathbf{v_1} \in \mathbb{R}^3$ results in a vector $\mathbf{v_2}$ expressed as:

$\mathbf{V_2} = \mathbf{Q} \otimes \mathbf{V_1} \otimes \mathbf{Q}^*$, where $\mathbf{V_1} = \left[0, \mathbf{v_1^T}\right]^T$ and $\mathbf{V_2} = \left[0, \mathbf{v_2^T}\right]^T$

# Quaternion, basic relations

- An object, onto which a frame $\{\mathbf{B}\}$: $(\mathbf{X_B}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ is rigidly attached, in rotation w.r.t an intertial frame $\{\mathbf{U}\}$: $(\mathbf{O}, \mathbf{x}, \mathbf{y}, \mathbf{z})$, has an angular velocity vector denoted $\boldsymbol{\omega}$. The orientation of $\{\mathbf{B}\}$ w.r.t $\{\mathbf{U}\}$ is denoted with quaternion $\mathbf{Q}$. Hence the following relations hold :

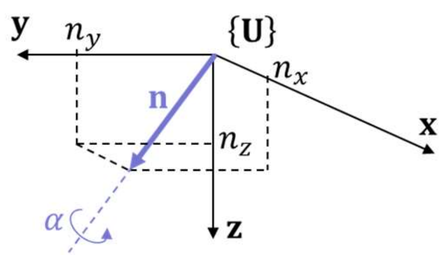
$$\boldsymbol{\Omega}_B = \mathbf{2} \cdot \mathbf{Q}^* \otimes \dot{\mathbf{Q}}, \text{ where } \boldsymbol{\Omega}_B = [0, \boldsymbol{\omega}_B^T]^T \text{ and } \boldsymbol{\omega}_B = [p, q, r]^{\boldsymbol{T}} \text{ is } \boldsymbol{\omega} \text{ expressed in } \{\mathbf{B}\}, \text{ and}$$

$$\boldsymbol{\Omega}_\mathbf{0} = \mathbf{2} \cdot \dot{\mathbf{Q}} \otimes \mathbf{Q}^*, \text{ where } \boldsymbol{\Omega}_0 = [0, \boldsymbol{\omega}_0^T]^T \text{ and } \boldsymbol{\omega}_0 \text{ is } \boldsymbol{\omega} \text{ expressed in } \{\mathbf{U}\}$$

- The left-multiplication of previous relation by $\mathbf{Q}$ yields the kinematic rotational model of the moving object as:

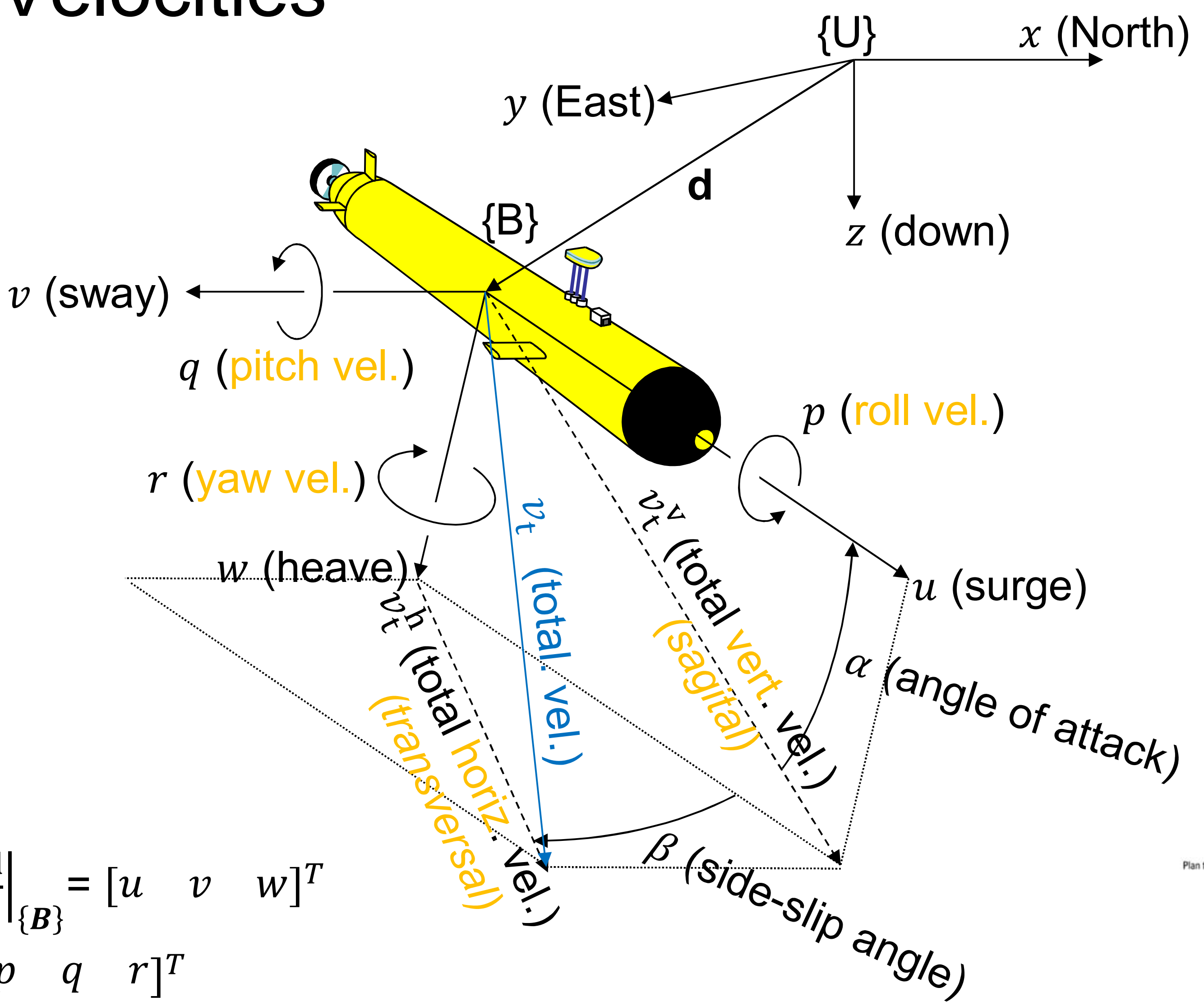
$$\dot{\mathbf{Q}} = \frac{\mathbf{1}}{\mathbf{2}} \cdot \mathbf{Q} \otimes \boldsymbol{\Omega}_B,$$

where $\boldsymbol{\Omega}_B = [0, \boldsymbol{\omega}_B^T]^T$, $\boldsymbol{\omega}_B = [p, q, r]^{\boldsymbol{T}}$
and $p$, $q$ and $r$ denote the object rotational velocities expressed in its own $\{\mathbf{B}\}$ frame, as described in the sequel.



# Representation Formalism

- Velocities




$$\mathbf{v}_{\boldsymbol{B}} = \left.\frac{\mathrm{d}\mathbf{d}}{\mathrm{d}t}\right|_{\{\boldsymbol{B}\}} = [u \quad v \quad w]^T$$

$$\boldsymbol{\omega}_{\boldsymbol{B}} = [p \quad q \quad r]^T$$

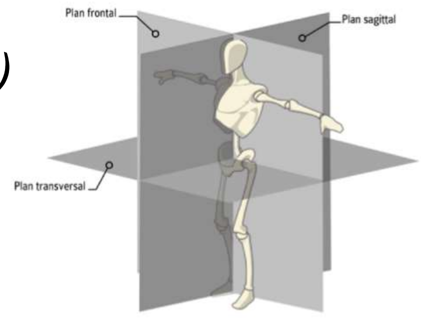



Figure 1 : Plans anatomiques (crédit photo sci-sport.com)

# Representation Formalism

| | Euler angles | Quaternions | Rotation Matrix |
|---|---|---|---|
| Position | $\mathbf{X} = [x \quad y \quad z]^T$ | | |
| Attitude | $\mathbf{\Phi}^{\mathrm{eul}} = [\phi \quad \theta \quad \psi]^T$ | $\mathbf{Q}$ | $\mathbf{R}$ |
| Velocities | $\mathbf{V}_{\boldsymbol{B}} = [u \quad v \quad w]^T, \boldsymbol{\omega}_B = [p \quad q \quad r]^T$ | | |
| State | $\boldsymbol{\chi} = \begin{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} \mathbf{X} \\ \mathbf{\Phi}^{\mathrm{eul}} \end{bmatrix} \\ \boldsymbol{\nu} = \begin{bmatrix} \mathbf{V}_{\mathbf{B}} \\ \boldsymbol{\omega}_B \end{bmatrix} \end{bmatrix}$ | $\boldsymbol{\chi} = \begin{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Q} \end{bmatrix} \\ \boldsymbol{\nu} = \begin{bmatrix} \mathbf{V}_{\mathbf{B}} \\ \boldsymbol{\omega}_B \end{bmatrix} \end{bmatrix}$ | $\boldsymbol{\chi} = \{\mathbf{X}, \mathbf{R}, \mathbf{V}_{\boldsymbol{B}}, \boldsymbol{\omega}_B\}$ |
| Kinematic Model $\dot{\boldsymbol{\eta}} = \mathbf{f}(\boldsymbol{\nu})$ | $\begin{bmatrix} \dot{\mathbf{X}} \\ \dot{\mathbf{\Phi}}^{\mathrm{eul}} \end{bmatrix} = \mathbf{R}_{\mathrm{cin}}^{\mathrm{eul}} \cdot \begin{bmatrix} \mathbf{V}_{\mathbf{B}} \\ \boldsymbol{\omega}_B \end{bmatrix}$ | $\begin{cases} [0, \dot{\mathbf{X}}^T]^T = \mathbf{Q} \otimes [0, \mathbf{V}_{\boldsymbol{B}}^T]^T \otimes \mathbf{Q}^* \\ \dot{\mathbf{Q}} = \frac{1}{2} \cdot \mathbf{Q} \otimes \underbrace{[0, \boldsymbol{\omega}_B^T]^T}_{\mathbf{\Omega}_B} \end{cases}$ | $\begin{cases} \dot{\mathbf{X}} = \mathbf{R} \cdot \mathbf{V}_{\mathbf{B}} \\ \dot{\mathbf{R}} = \mathbf{R} \cdot (\boldsymbol{\omega}_{\mathbf{B}} \wedge) \end{cases}$ |

$$\mathbf{R}_{\mathrm{cin}}^{\mathrm{eul}} = \begin{bmatrix} \cos\psi \cdot \cos\theta & \cos\psi \cdot \sin\theta \cdot \sin\phi - \sin\psi \cdot \cos\phi & \cos\psi \cdot \sin\theta \cdot \cos\phi + \sin\psi \cdot \sin\phi & 0 & 0 & 0 \\ \sin\psi \cdot \cos\theta & \sin\psi \cdot \sin\theta \cdot \sin\phi + \cos\psi \cdot \cos\phi & \sin\psi \cdot \sin\theta \cdot \cos\phi - \cos\psi \cdot \sin\phi & 0 & 0 & 0 \\ -\sin\theta & \cos\theta \cdot \sin\phi & \cos\theta \cdot \cos\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sin\phi \cdot \tan\theta & \cos\phi \cdot \tan\theta \\ 0 & 0 & 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & 0 & 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix}$$

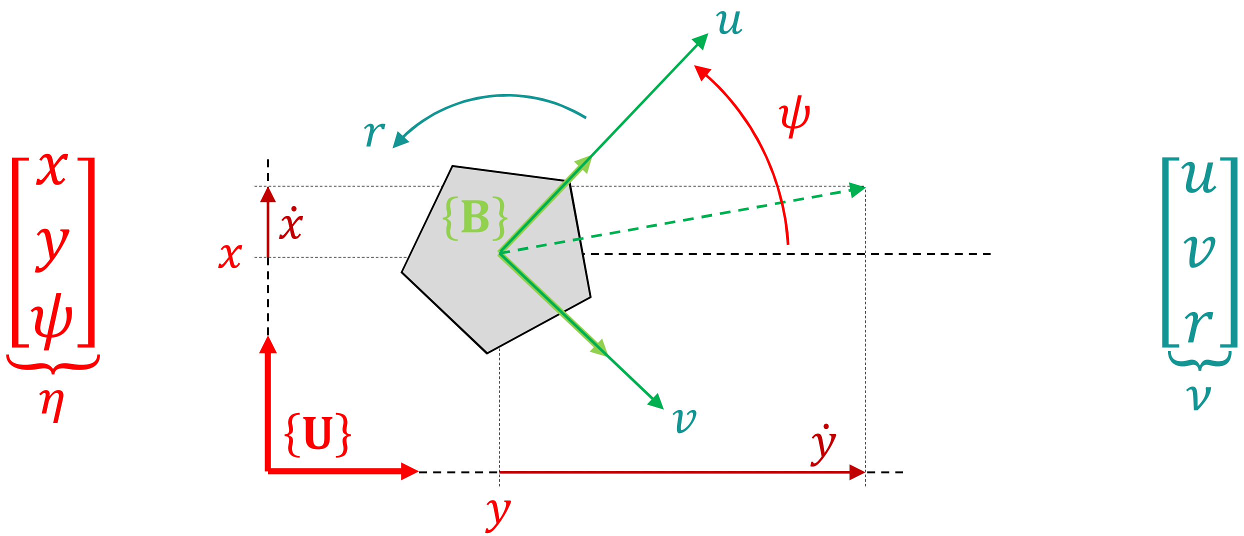
Gimbal lock, if $\theta = \pm\frac{\pi}{2}$

$$\boldsymbol{\omega}_B \wedge = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix}$$

# II – Models
## b) Kinematic Model

# Kinematic Model

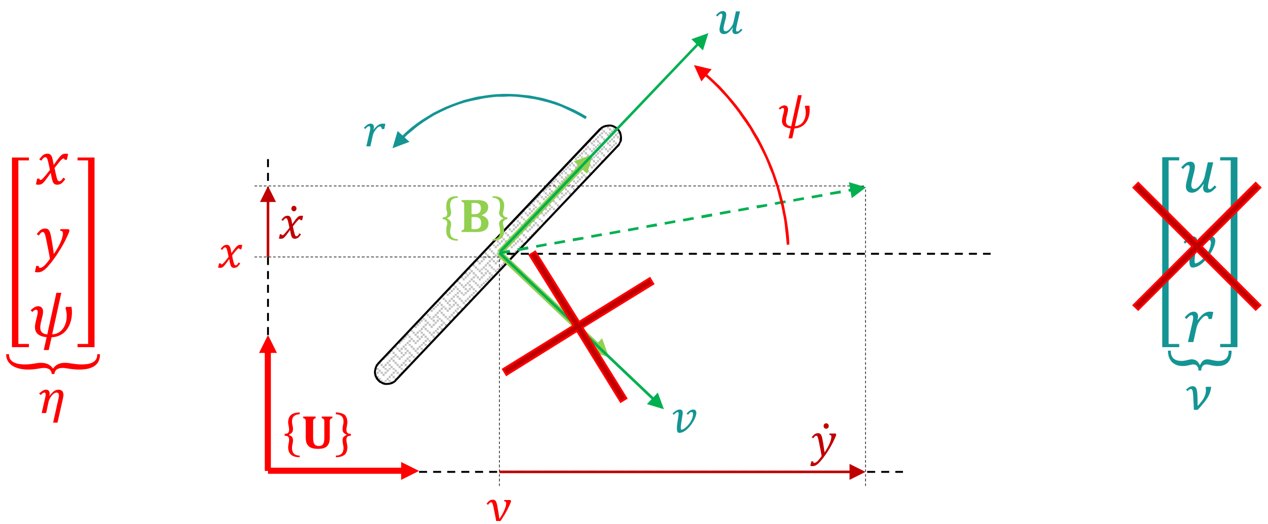
- 2D, Cartesian, no constraint




$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ r \end{bmatrix}}_{\boldsymbol{\nu}} \rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

# Kinematic Model

- 2D, Cartesian, the Wheel, nonholonomic constraint : $v = \dot{x} \cdot \cos\psi + \dot{y} \cdot \sin\psi = 0$

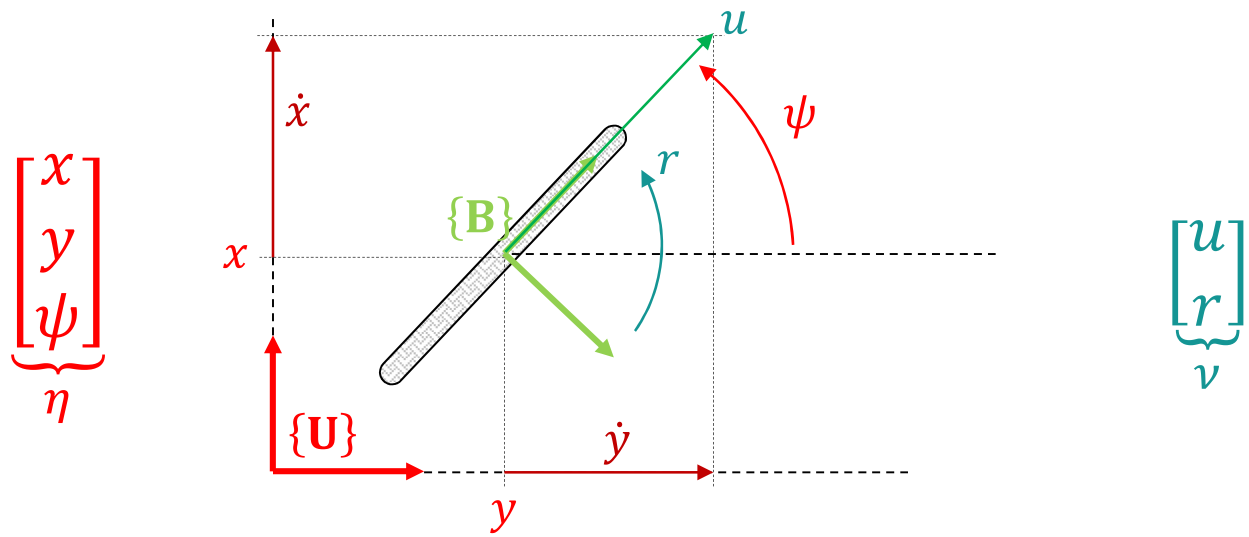



Non-holonomy

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ r \end{bmatrix}}_{\boldsymbol{\nu}} \rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

# Kinematic Model

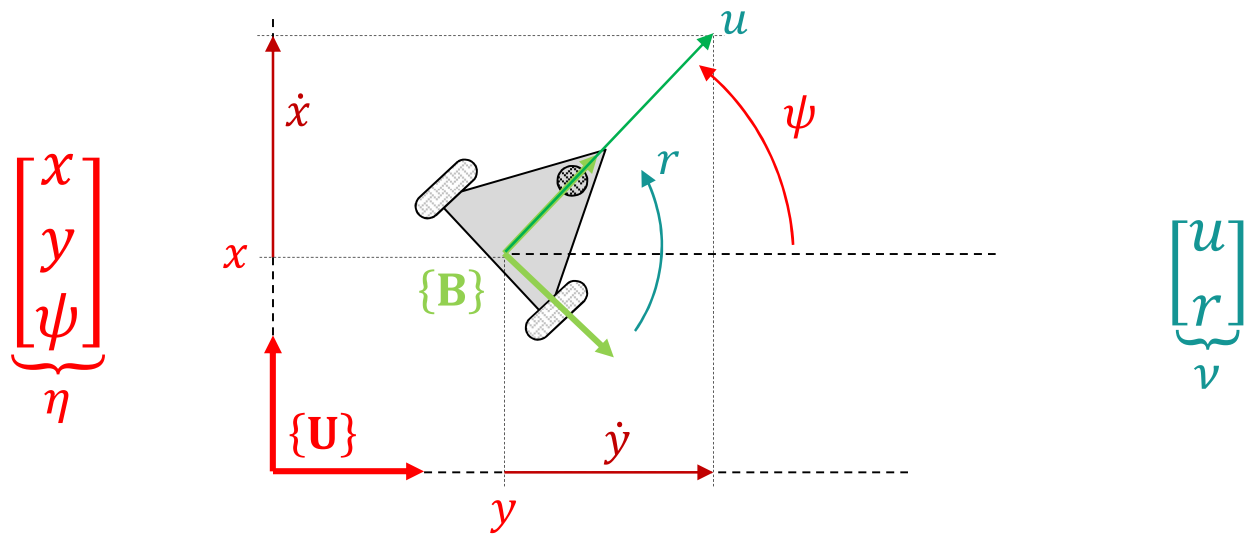
- 2D, Cartesian, the Wheel, nonholonomic constraint : $v = \dot{x} \cdot \cos\psi + \dot{y} \cdot \sin\psi = 0$




$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{J}} \cdot \underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}} \rightarrow \dot{\boldsymbol{\eta}} = \mathbf{J} \cdot \boldsymbol{\nu}$$

# Kinematic Model

- 2D, Cartesian, the Unicycle, nonholonomic constraint : $v = \dot{x} \cdot \cos\psi + \dot{y} \cdot \sin\psi = 0$

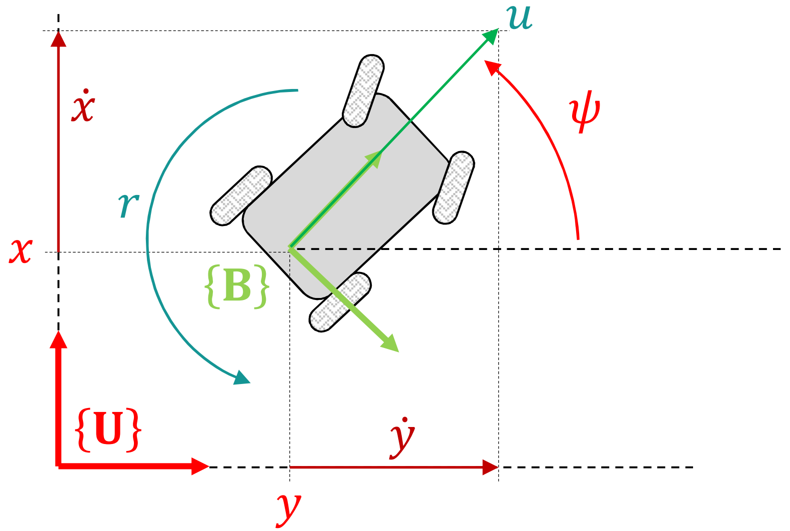



$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{J}} \cdot \underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}} \rightarrow \dot{\boldsymbol{\eta}} = \mathbf{J} \cdot \boldsymbol{\nu}$$

# Kinematic Model

- 2D, Cartesian, the Car, nonholonomic constraint : $v = \dot{x} \cdot \cos\psi + \dot{y} \cdot \sin\psi = 0$

$$\underbrace{\begin{bmatrix} x \\ y \\ \psi \end{bmatrix}}_{\eta}$$




$$\underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\nu}$$

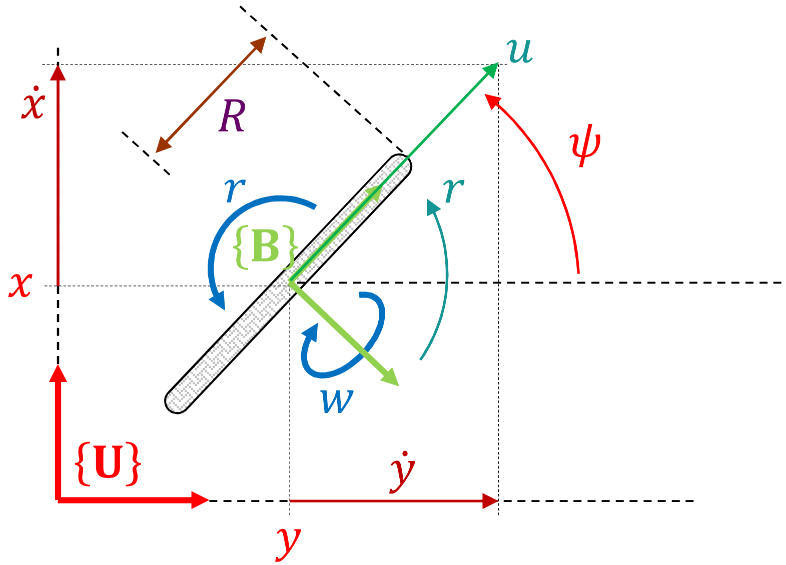
$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{J}} \cdot \underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}} \rightarrow \dot{\boldsymbol{\eta}} = \mathbf{J} \cdot \boldsymbol{\nu}$$

# II – Models
## c) Actuation Model

# Actuation Model

- 2D, the Wheel, pure rolling constraint :

$$u = \dot{x} \cdot \cos\psi + \dot{y} \cdot \sin\psi = R \cdot w$$

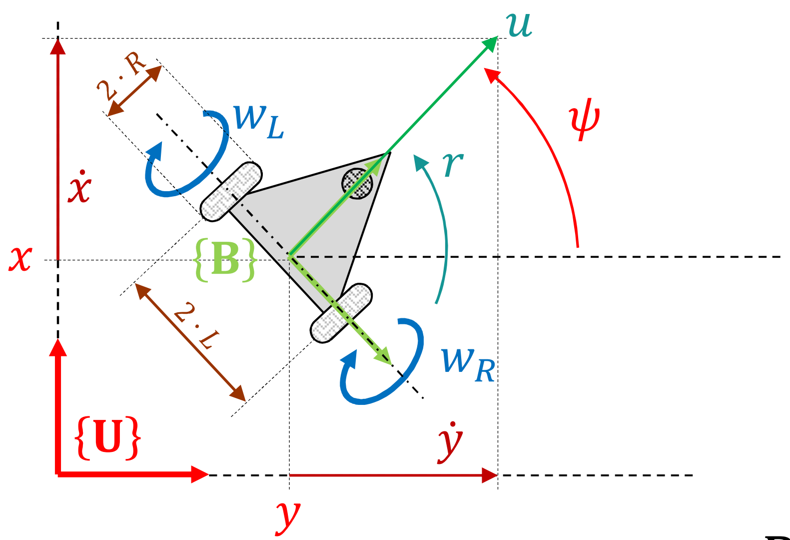



$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{J}} \cdot \underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}} = \underbrace{\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} w \\ r \end{bmatrix}}_{\mathbf{U}} \rightarrow \boldsymbol{\nu} = \mathbf{A} \cdot \mathbf{U}$$

# Actuation Model

- 2D, the Unicycle



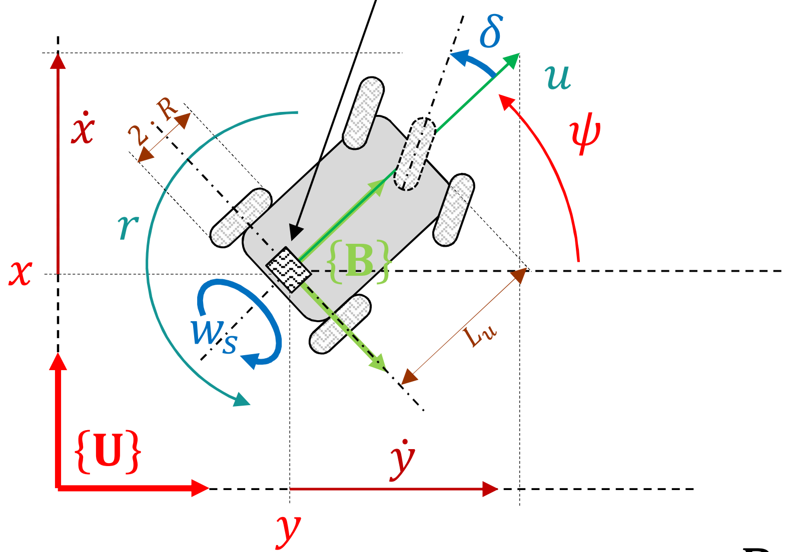

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{J}} \cdot \underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}} = \underbrace{\begin{bmatrix} \frac{R}{2} & \frac{R}{2} \\ \frac{R}{2 \cdot L} & \frac{-R}{2 \cdot L} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} w_R \\ w_L \end{bmatrix}}_{\mathbf{U}} \rightarrow \boldsymbol{\nu} = \mathbf{A} \cdot \mathbf{U}$$

# Actuation Model

- 2D, the Car

differential drive ($K_R$)

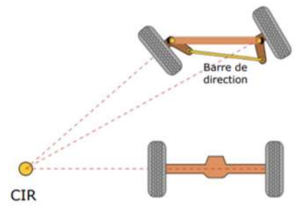


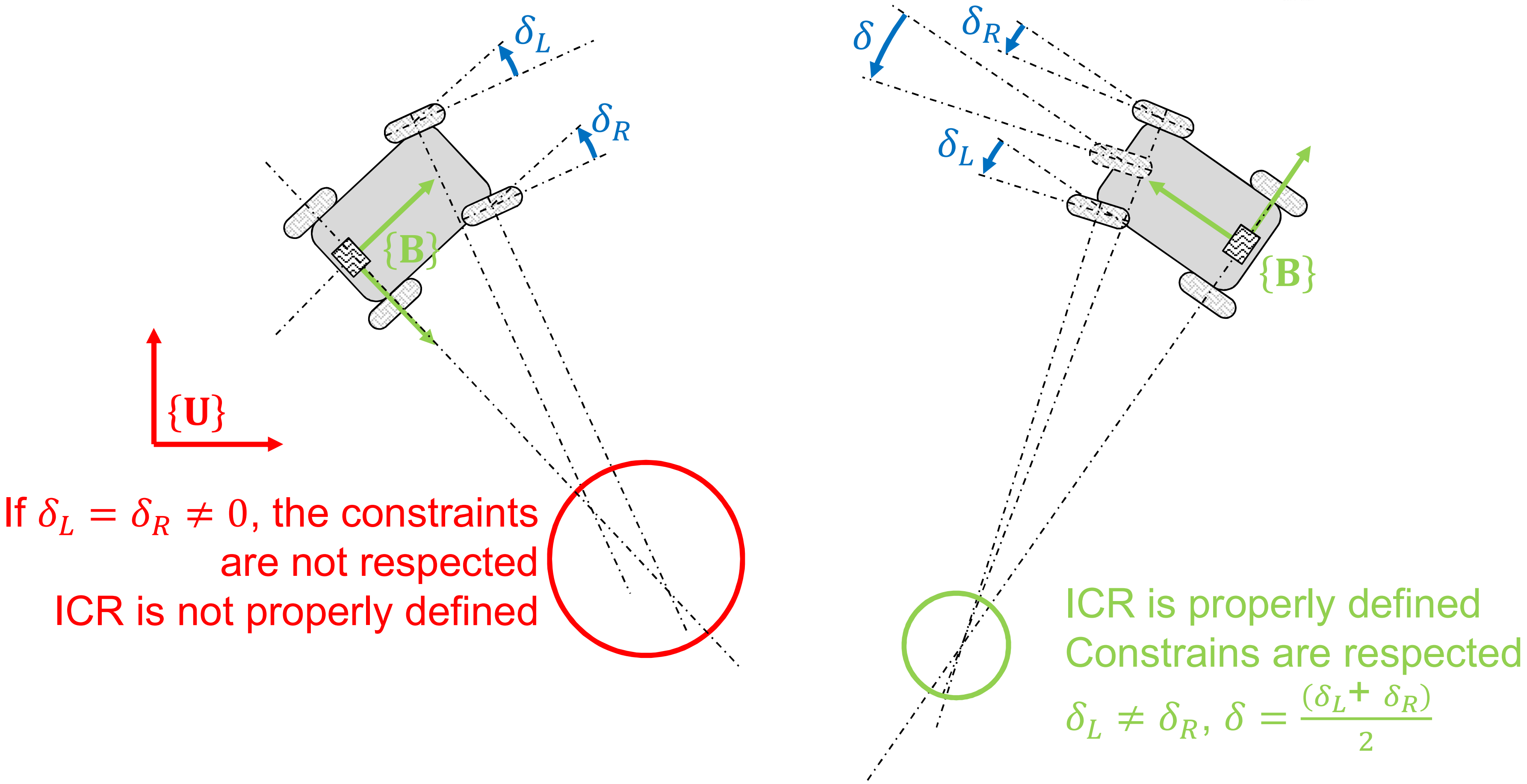

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{J}} \cdot \underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\underbrace{\begin{bmatrix} u \\ r \end{bmatrix}}_{\boldsymbol{\nu}} = \begin{bmatrix} \dfrac{R}{K_R} \cdot w_s \\ \dfrac{u \cdot \tan\delta}{L_u} \end{bmatrix} \rightarrow \boldsymbol{\nu} = \mathbf{A}(\mathbf{U})$$

# Actuation Model

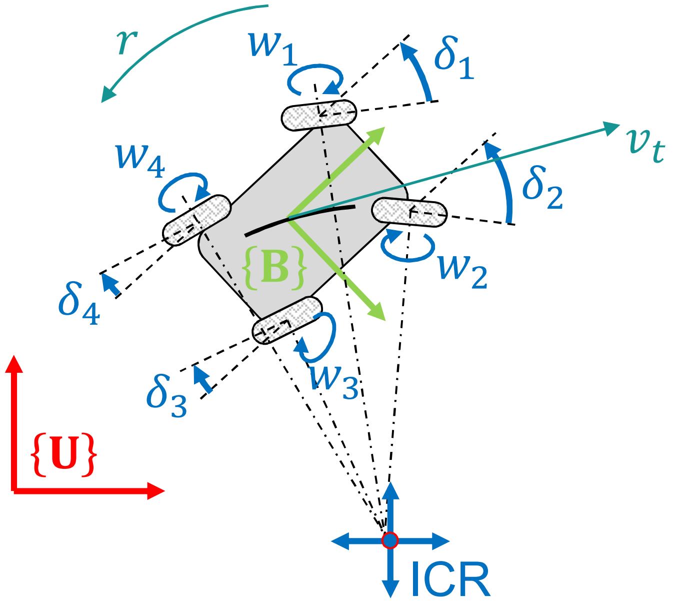
- 2D, the Car, non-sliding contraints

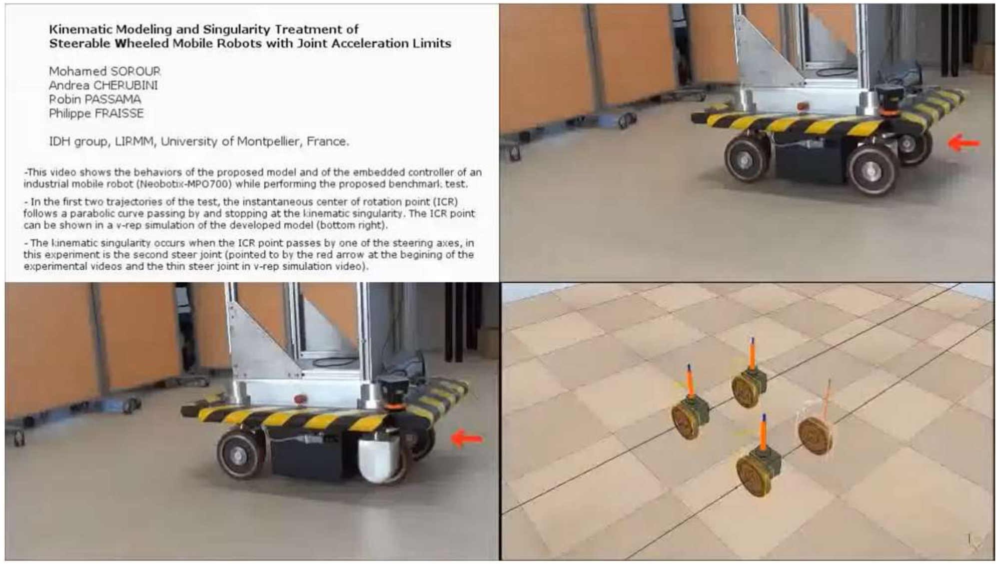


# Actuation Model

- 2D, the Car, Pseudo-Omni-Directional Wheeled Robots

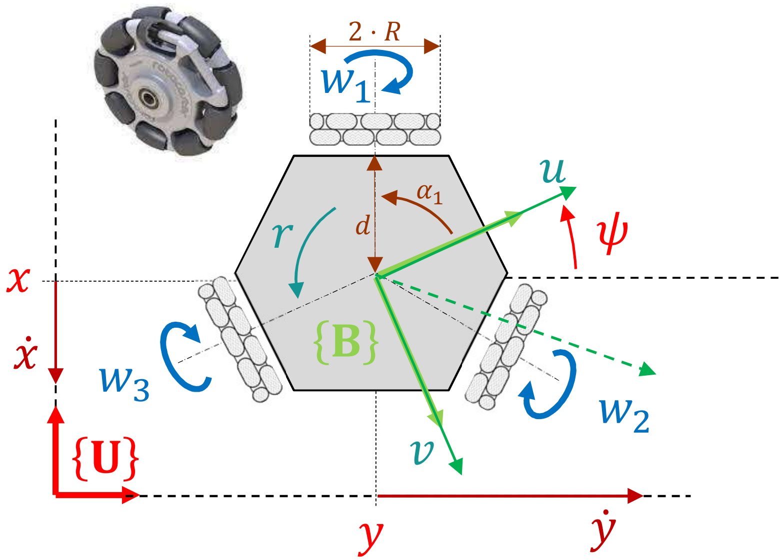



$$\begin{cases} \mathbf{W} = [w_1 \quad w_2 \quad w_3 \quad w_4]^T \\ \mathbf{\Delta} = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4]^T \end{cases}, \begin{cases} \mathbf{W} = \mathbf{f}(v_t, r), \text{subject to } \Phi_{\mathbf{W}}(\mathbf{W}) = 0 \\ \mathbf{\Delta} = g(v_t, r), \text{subject to } \Phi_{\mathbf{\Delta}}(\mathbf{\Delta}) = 0 \end{cases}, \left\| \boldsymbol{X}_{ICR}\big|_{\{\mathbf{B}\}} \right\|^{-1} = \frac{r}{v_t}$$

# Actuation Model

- 2D, Cartesian, Omni-directional *sweedish wheels* system




$$\underbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\psi}\end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix}u\\ v\\ r\end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

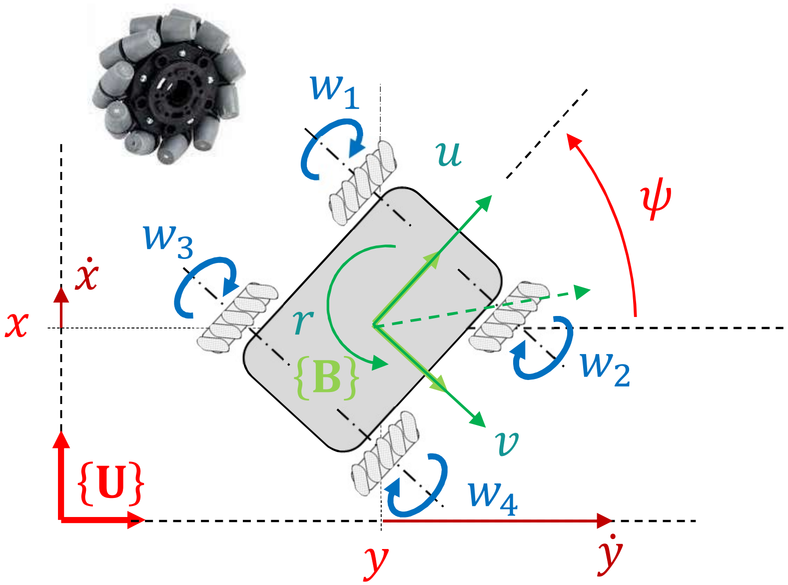
$$\underbrace{\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}}_{\mathbf{U}} = -\frac{1}{R} \cdot \underbrace{\begin{bmatrix}-\sin\alpha_1 & \cos\alpha_1 & d\\ -\sin\alpha_2 & \cos\alpha_2 & d\\ -\sin\alpha_3 & \cos\alpha_3 & d\end{bmatrix}}_{\mathbf{A}^{-1}} \cdot \underbrace{\begin{bmatrix}u\\ v\\ r\end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \boldsymbol{\nu} = \mathbf{A} \cdot \mathbf{U}$$

$\alpha_1 = \frac{\pi}{3}, \alpha_2 = -\frac{\pi}{3}, \alpha_3 = \pi$

# Actuation Model

- 2D, Cartesian, Omni_directional *Mecanum wheels* system





$$\underbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\psi}\end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix}u\\ v\\ r\end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

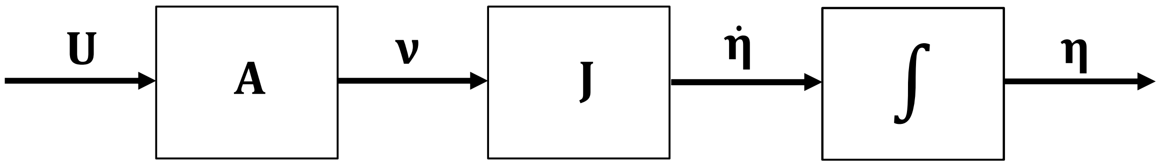
$$\underbrace{\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}}_{\mathbf{U}} = \underbrace{\begin{bmatrix}1 & 1 & -(L_1+L_2)\\ -1 & 1 & -(L_1+L_2)\\ 1 & -1 & -(L_1+L_2)\\ -1 & -1 & -(L_1+L_2)\end{bmatrix}}_{\mathbf{A}^{-1}} \cdot \underbrace{\begin{bmatrix}u\\ v\\ r\end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \boldsymbol{\nu} = \mathbf{A} \cdot \mathbf{U}$$

@ Forklift

# Simulation

- Kinematic simulation




Video sim Unicycle

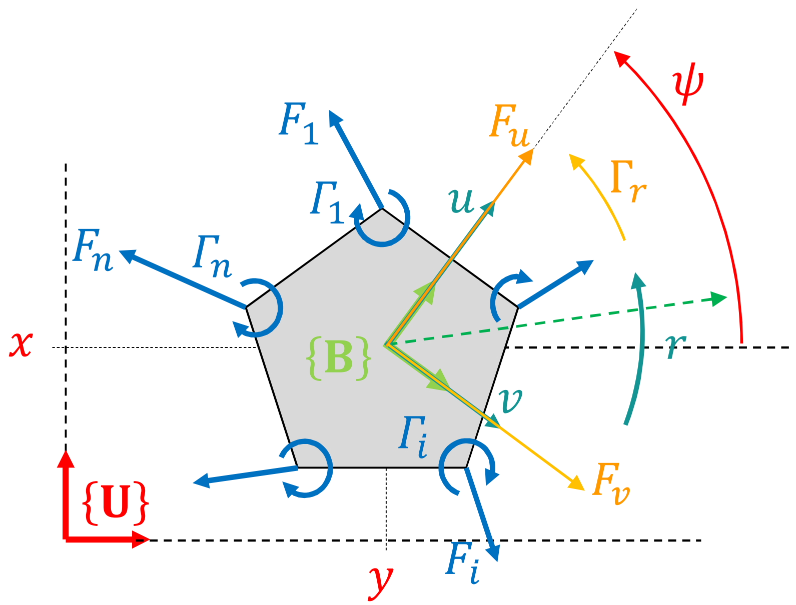
Video sim Omni 3 roues

Video sim Car

Video sim Omni Mecanum

# Actuation Model

- 2D, Cartesian, Propulsive actuation




$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R}(\boldsymbol{\eta}) \cdot \boldsymbol{\nu}$$

Dynamic Model:
$$\dot{\boldsymbol{\nu}} = \mathbf{f}_{\mathrm{D}}(\boldsymbol{\Theta}, \boldsymbol{\eta}, \boldsymbol{\nu}, \mathbf{F}_B)$$

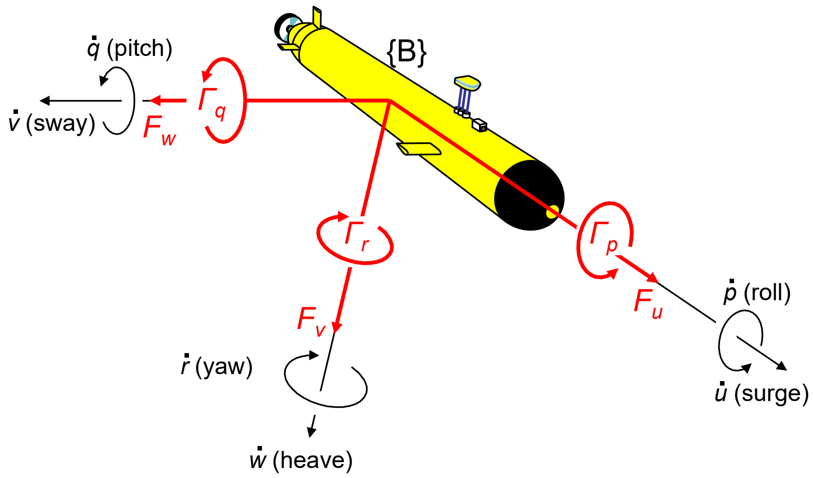
$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ \Gamma_r \end{bmatrix}}_{\mathbf{F}_B} = \mathbf{A} \cdot \underbrace{[F_1, \Gamma_1, \dots, F_i, \Gamma_i, \dots, F_n, \Gamma_n]^T}_{\mathbf{F}_{\mathrm{m}}}$$

$$\rightarrow \mathbf{F}_B = \mathbf{A} \cdot \mathbf{F}_{\mathrm{m}}$$

The hydrodynamic effects model are easily expressed in the body frame

# Dynamic Model

$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ F_w \\ \Gamma_p \\ \Gamma_q \\ \Gamma_r \end{bmatrix}}_{\mathbf{F_B}}$$




$$\underbrace{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}}_{\dot{\nu}}$$

Hydro-dynamic phenomena (fluid/structure interaction) :

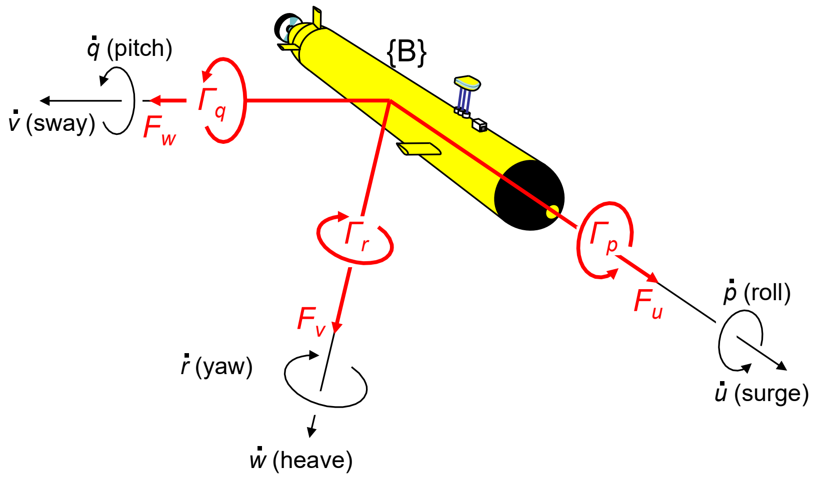
- Buoyancy
- Lift and Drag
- Added Mass

The hydrodynamic effects model are easily expressed in the body frame

$$\mathbf{F}_B = \mathbf{f}_D(\boldsymbol{\Theta}, \boldsymbol{\nu}, \boldsymbol{\eta}, \dot{\boldsymbol{\nu}})$$

# Dynamic Model

$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ F_w \\ \Gamma_p \\ \Gamma_q \\ \Gamma_r \end{bmatrix}}_{\mathbf{F_B}}$$



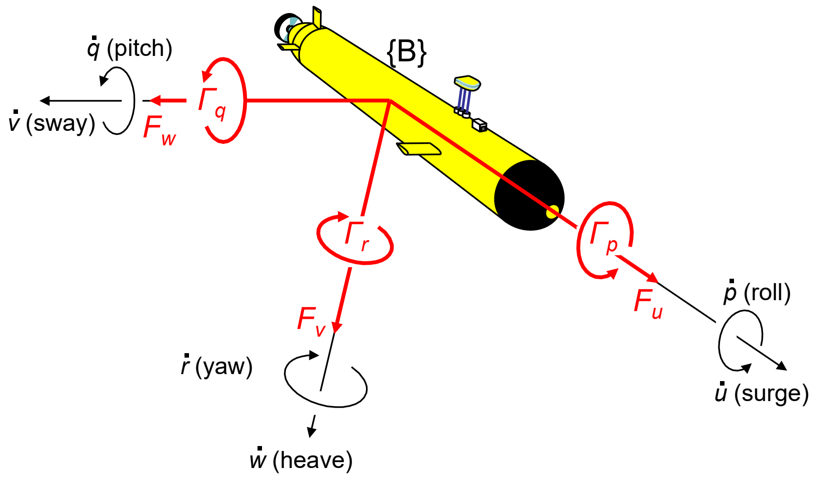

$$\underbrace{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}}_{\dot{v}}$$

$$\begin{aligned}
F_u &= X_{\dot{u}} \cdot \dot{u} + X_{u \cdot |u|} \cdot u \cdot |u| + X_{w \cdot q} \cdot w \cdot q + X_{v \cdot r} \cdot v \cdot r + X_G(\eta) \\
F_v &= Y_{\dot{v}} \cdot \dot{v} + Y_{v \cdot |v|} \cdot v \cdot |v| + Y_{u \cdot r} \cdot u \cdot r + Y_{w \cdot p} \cdot w \cdot p + Y_G(\eta) \\
F_w &= Z_{\dot{w}} \cdot \dot{w} + Z_{w \cdot |w|} \cdot w \cdot |w| + Z_{u \cdot q} \cdot u \cdot q + Z_{v \cdot p} \cdot v \cdot p + Z_G(\eta) \\
\Gamma_p &= K_{\dot{p}} \cdot \dot{p} + K_{p \cdot |p|} \cdot p \cdot |p| + K_{q \cdot r} \cdot q \cdot r + K_v \cdot v + K_w \cdot w + K_G(\eta) \\
\Gamma_q &= M_{\dot{q}} \cdot \dot{q} + M_{q \cdot |q|} \cdot q \cdot |q| + M_{p \cdot r} \cdot p \cdot r + M_u \cdot u + M_w \cdot w + M_G(\eta) \\
\Gamma_r &= N_{\dot{r}} \cdot \dot{r} + N_{r \cdot |r|} \cdot r \cdot |r| + N_{p \cdot q} \cdot p \cdot q + N_u \cdot u + N_v \cdot v + N_G(\eta)
\end{aligned}$$

$$\mathbf{F}_B = \mathbf{f}_D(\boldsymbol{\Theta}, \boldsymbol{\nu}, \boldsymbol{\eta}, \dot{\boldsymbol{\nu}}) \Longrightarrow \dot{\boldsymbol{\nu}} = \mathbf{f}_D^{-1}(\boldsymbol{\Theta}, \boldsymbol{\nu}, \boldsymbol{\eta}, \mathbf{F}_B) \quad \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu} \Longrightarrow \ddot{\boldsymbol{\eta}} = \mathbf{R} \cdot \dot{\boldsymbol{\nu}} + \dot{\mathbf{R}} \cdot \boldsymbol{\nu}$$

# Dynamic Model

$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ F_w \\ \Gamma_p \\ \Gamma_q \\ \Gamma_r \end{bmatrix}}_{\mathbf{F_B}}$$




$$\underbrace{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}}_{\dot{\nu}}$$

A NONLINEAR UNIFIED STATE-SPACE MODEL FOR SHIP MANEUVERING AND CONTROL IN A SEAWAY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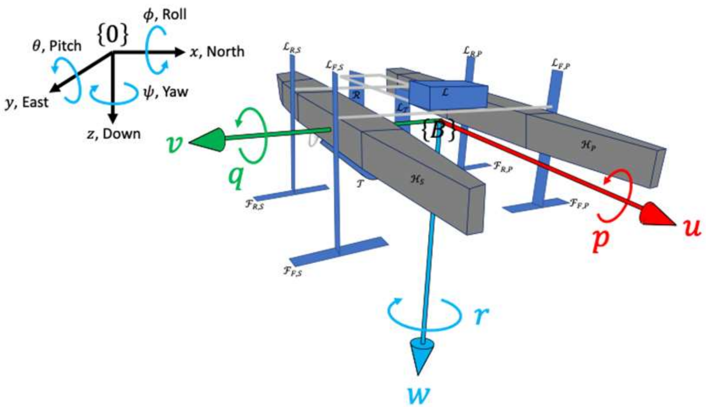
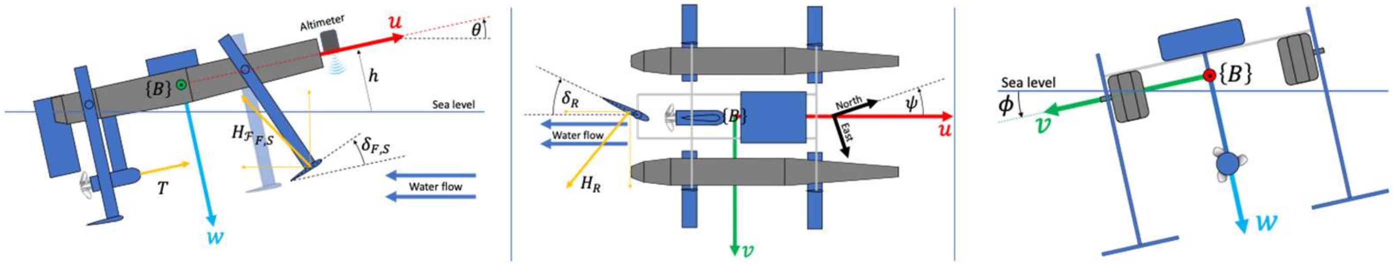
This article presents a unified state-space model for ship maneuvering, station-keeping, and control in a seaway. The frequency-dependent potential and viscous damping terms, which in classic theory results in a convolution integral not suited for real-time simulation, is compactly represented by using a state-space formulation. The separation of the vessel model into a low-frequency model (represented by zero-frequency added mass and damping) and a wave-frequency model (represented by motion transfer functions or RAOs), which is commonly used for simulation, is hence made superfluous.



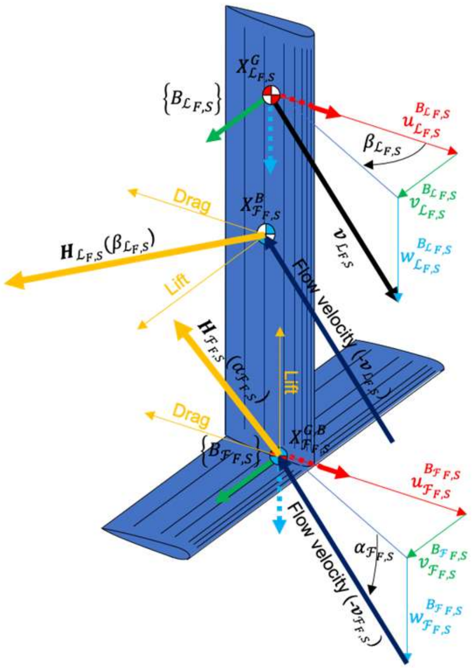
*Keywords:* ship modelling, equations of motion, hydrodynamics, maneuvering, seakeeping, autopilots, dynamic positioning.

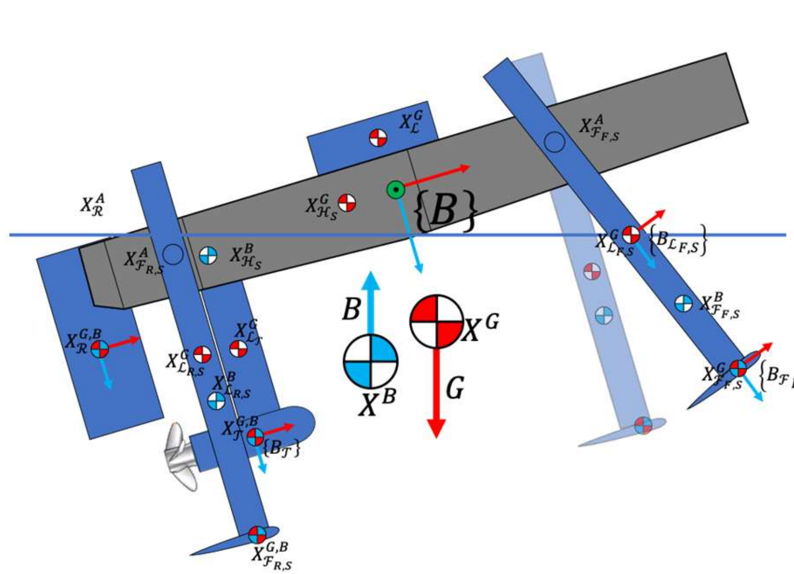
# Dynamic Model



# Dynamic Model




$$\sum \mathbf{F} = \mathbf{M}(\boldsymbol{\eta}) \cdot \ddot{\boldsymbol{\eta}}$$

$$\boldsymbol{M}^*(\boldsymbol{\eta})\ddot{\boldsymbol{\eta}} + \boldsymbol{C}^*(\boldsymbol{v}, \boldsymbol{\eta})\dot{\boldsymbol{\eta}} + \boldsymbol{D}^*(\boldsymbol{v}, \boldsymbol{\eta})\dot{\boldsymbol{\eta}} + \boldsymbol{g}^*(\boldsymbol{\eta}) + \boldsymbol{g}_o^*(\boldsymbol{\eta}) = \boldsymbol{\tau}^* + \boldsymbol{\tau}_{\text{wind}}^* + \boldsymbol{\tau}_{\text{wave}}^*$$

$$\boldsymbol{M}^*(\boldsymbol{\eta}) = \boldsymbol{J}_\Theta^{-\top}(\boldsymbol{\eta})\boldsymbol{M}\boldsymbol{J}_\Theta^{-1}(\boldsymbol{\eta})$$

$$\boldsymbol{C}^*(\boldsymbol{v}, \boldsymbol{\eta}) = \boldsymbol{J}_\Theta^{-\top}(\boldsymbol{\eta})[\boldsymbol{C}(\boldsymbol{v}) - \boldsymbol{M}\boldsymbol{J}_\Theta^{-1}(\boldsymbol{\eta})\dot{\boldsymbol{J}}_\Theta(\boldsymbol{\eta})]\boldsymbol{J}_\Theta^{-1}(\boldsymbol{\eta})$$

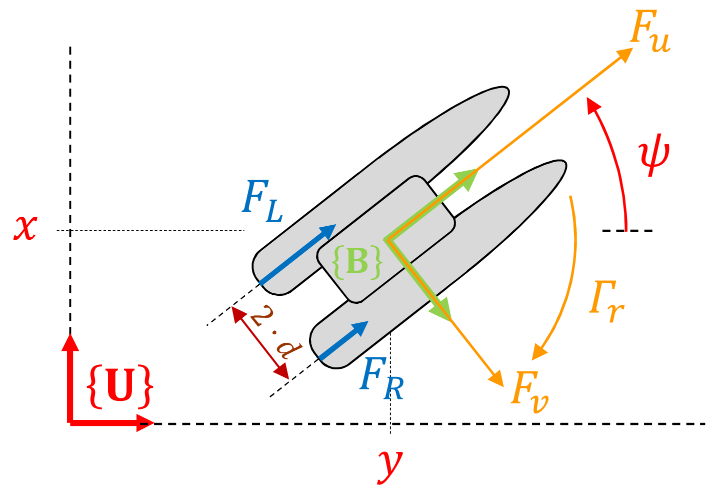
$$\boldsymbol{D}^*(\boldsymbol{v}, \boldsymbol{\eta}) = \boldsymbol{J}_\Theta^{-\top}(\boldsymbol{\eta})\boldsymbol{D}(\boldsymbol{v})\boldsymbol{J}_\Theta^{-1}(\boldsymbol{\eta})$$

$$\boldsymbol{g}^*(\boldsymbol{\eta}) + \boldsymbol{g}_o^*(\boldsymbol{\eta}) = \boldsymbol{J}_\Theta^{-\top}(\boldsymbol{\eta})[\boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o]$$

$$\boldsymbol{\tau}^* + \boldsymbol{\tau}_{\text{wind}}^* + \boldsymbol{\tau}_{\text{wave}}^* = \boldsymbol{J}_\Theta^{-\top}(\boldsymbol{\eta})(\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}})$$

# Actuation Model

- 2D, Cartesian, Propulsive actuation, ASV




$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ \Gamma_r \end{bmatrix}}_{\mathbf{F}_B} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ d & -d \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} F_L \\ F_R \end{bmatrix}^T}_{\mathbf{F}_m}$$

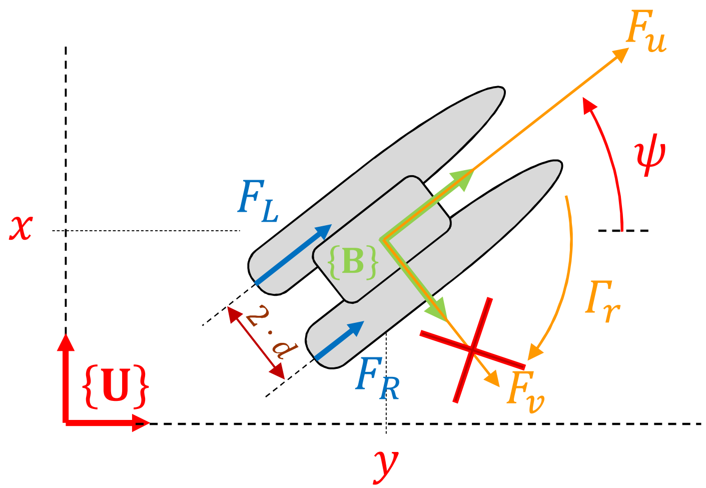
$$\rightarrow \mathbf{F}_B = \mathbf{A} \cdot \mathbf{F}_m$$

$$\underbrace{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix}}_{\dot{\boldsymbol{\nu}}} = \begin{cases} \dfrac{1}{X_{\dot{u}}} \cdot (F_u - X_u \cdot u \cdot |u|) \\ \dfrac{1}{Y_{\dot{v}}} \cdot (F_v - Y_v \cdot v \cdot |v| - Y_{ur} \cdot u \cdot r) \\ \dfrac{1}{N_{\dot{u}}} \cdot (\Gamma_r - N_r \cdot r \cdot |r|) \end{cases}$$

$$\rightarrow \dot{\boldsymbol{\nu}} = \mathbf{f}_D(\boldsymbol{\Theta}, \boldsymbol{\nu}, \boldsymbol{\eta}, \mathbf{F}_B)$$

# Actuation Model

- 2D, Cartesian, Propulsive actuation, ASV




$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ \Gamma_r \end{bmatrix}}_{\mathbf{F}_B} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ d & -d \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} F_L \\ F_R \end{bmatrix}^T}_{\mathbf{F}_m}$$

$$\rightarrow \mathbf{F}_B = \mathbf{A} \cdot \mathbf{F}_m$$

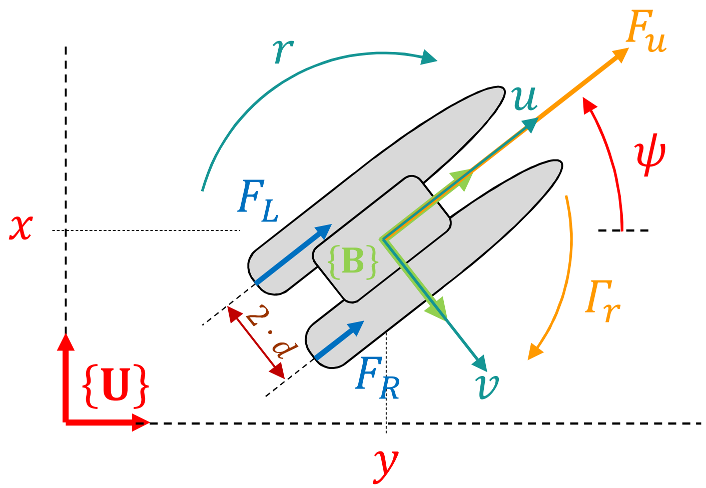
$$\underbrace{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix}}_{\dot{\boldsymbol{\nu}}} = \begin{cases} \dfrac{1}{X_{\dot{u}}} \cdot (F_u - X_u \cdot u \cdot |u|) \\ \dfrac{1}{Y_{\dot{v}}} \cdot (F_v - Y_v \cdot v \cdot |v| - Y_{ur} \cdot u \cdot r) \\ \dfrac{1}{N_{\dot{u}}} \cdot (\Gamma_r - N_r \cdot r \cdot |r|) \end{cases}$$

$$\rightarrow \dot{\boldsymbol{\nu}} = \mathbf{f}_D(\boldsymbol{\Theta}, \boldsymbol{\nu}, \boldsymbol{\eta}, \mathbf{F}_B)$$

Under-actuation

# Actuation Model

- 2D, Cartesian, Propulsive actuation, ASV




$$\underbrace{\begin{bmatrix} F_u \\ \Gamma_r \end{bmatrix}}_{\mathbf{F}_B} = \underbrace{\begin{bmatrix} 1 & 1 \\ d & -d \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} F_L \\ F_R \end{bmatrix}^T}_{\mathbf{F}_m}$$

$$\rightarrow \mathbf{F}_B = \mathbf{A} \cdot \mathbf{F}_m$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

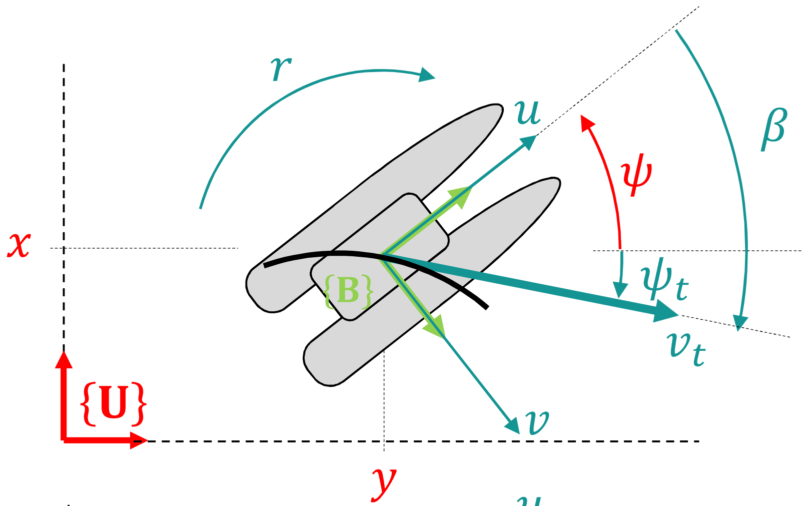
$$\rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

$$\underbrace{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix}}_{\dot{\boldsymbol{\nu}}} = \begin{cases} \dfrac{1}{X_{\dot{u}}} \cdot (F_u - X_u \cdot u \cdot |u|) \\ \dfrac{1}{Y_{\dot{v}}} \cdot (-Y_v \cdot v \cdot |v| - Y_{ur} \cdot u \cdot r) \\ \dfrac{1}{N_{\dot{u}}} \cdot (\Gamma_r - N_r \cdot r \cdot |r|) \end{cases}$$

$$\rightarrow \dot{\boldsymbol{\nu}} = \mathbf{f}_D(\boldsymbol{\Theta}, \boldsymbol{\nu}, \boldsymbol{\eta}, \mathbf{F}_B)$$

# Back to kinematics

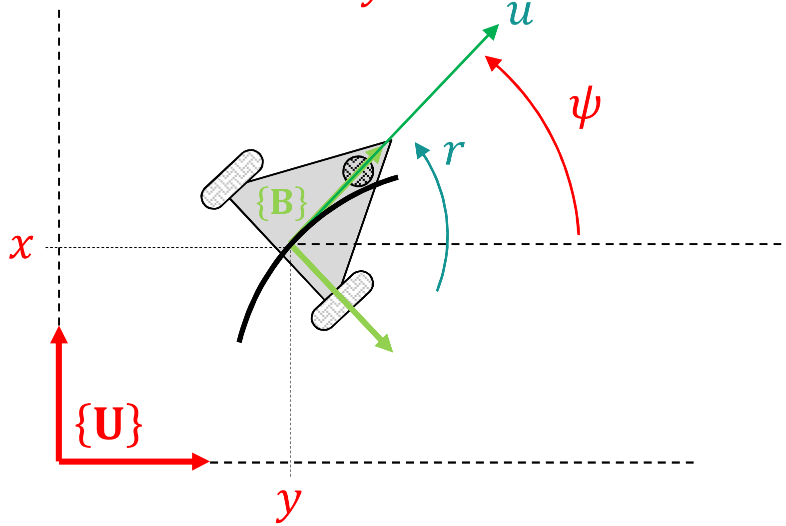
- 2D, Cartesian, ASV vs Unicycle




$$\underbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\psi}\end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix}u\\ v\\ r\end{bmatrix}}_{\boldsymbol{\nu}}$$

$$= \begin{bmatrix}\cos\psi_t & 0\\ \sin\psi_t & 0\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}v_t\\ \dot{\psi}_t\end{bmatrix}$$

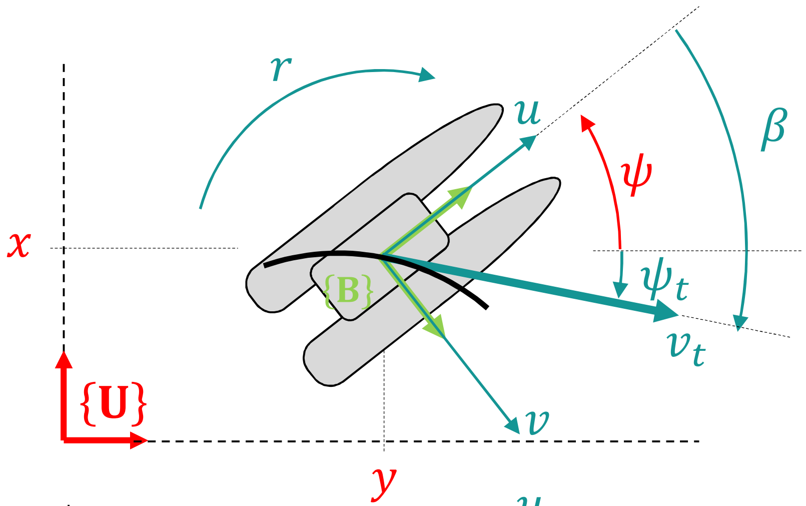
*The total velocity of a moving object is necessarily tangent to its own trajectory*

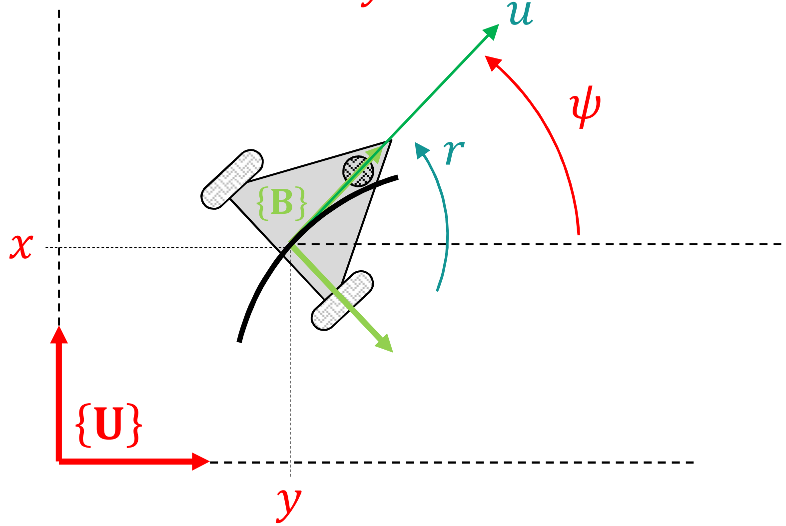



$$\underbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\psi}\end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix}\cos\psi & 0\\ \sin\psi & 0\\ 0 & 1\end{bmatrix}}_{\mathbf{J}} \cdot \underbrace{\begin{bmatrix}u\\ r\end{bmatrix}}_{\boldsymbol{\nu}}$$

# Back to kinematics

- 2D, Cartesian, ASV vs Unicycle




$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} == \begin{bmatrix} \cos\psi_t & 0 \\ \sin\psi_t & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_t \\ \dot{\psi}_t \end{bmatrix}$$

$$\dot{\psi}_t = r + \dot{\beta}, \text{where } \beta = \operatorname{atan}\frac{v}{u}$$

$$\begin{cases} F_u = m_u \cdot \dot{u} + d_u \\ 0 = m_v \cdot \dot{v} + m_{ur} \cdot u \cdot r + d_v \\ \Gamma_r = m_r \cdot \dot{r} + d_r \end{cases}$$

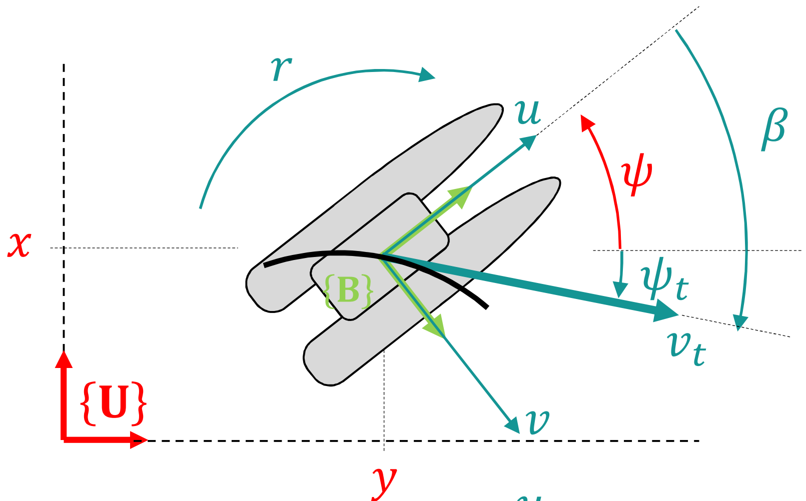
$$\rightarrow r = \frac{\dot{\psi}_t + f(\beta, \mathbf{v}, \dot{u})}{1 - \cos^2\beta \cdot \frac{m_{ur}}{m_v}}$$

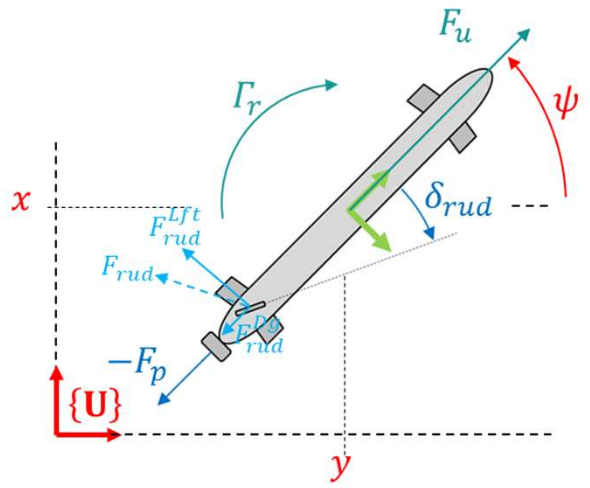
$$\text{well posed if } \frac{m_{ur}}{m_v} = \frac{m - Y_r}{m - Y_{\dot{v}}} < 1$$

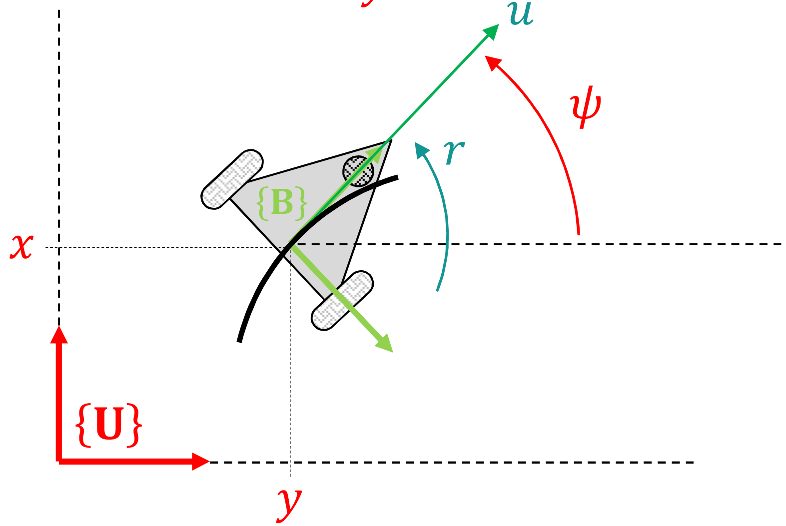
Covered by *stern dominancy* assumption (open-loop local stability)

# Back to kinematics

- 2D, Cartesian, ASV, AUV vs Unicycle



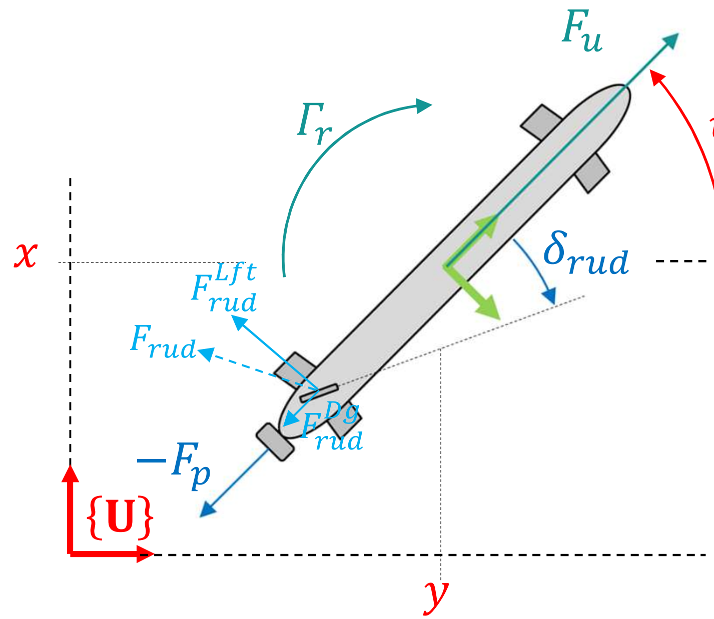




$$\underbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\psi}\end{bmatrix}}_{\dot{\boldsymbol{\eta}}} == \begin{bmatrix}\cos\psi_t & 0\\ \sin\psi_t & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix} v_t\\ r+\dot{\beta}\end{bmatrix}$$

Covered by *stern dominancy* assumption
(open-loop local stability)

# Actuation Model

- 2D, Cartesian, Propulsive actuation, AUV




$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix}}_{\dot{\boldsymbol{\eta}}} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ r \end{bmatrix}}_{\boldsymbol{\nu}}$$

$$\rightarrow \dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu}$$

$$\underbrace{\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix}}_{\dot{\boldsymbol{\nu}}} = \begin{cases} \dfrac{1}{X_{\dot{u}}} \cdot (F_u - X_u \cdot u \cdot |u|) \\ \dfrac{1}{Y_{\dot{v}}} \cdot (F_v - Y_v \cdot v \cdot |v| - Y_{ur} \cdot u \cdot r) \\ \dfrac{1}{N_{\dot{u}}} \cdot (\Gamma_r - N_r \cdot r \cdot |r|) \end{cases}$$

$$\rightarrow \dot{\boldsymbol{\nu}} = \mathbf{f}_D(\boldsymbol{\Theta}, \boldsymbol{\nu}, \boldsymbol{\eta}, \mathbf{F}_B)$$

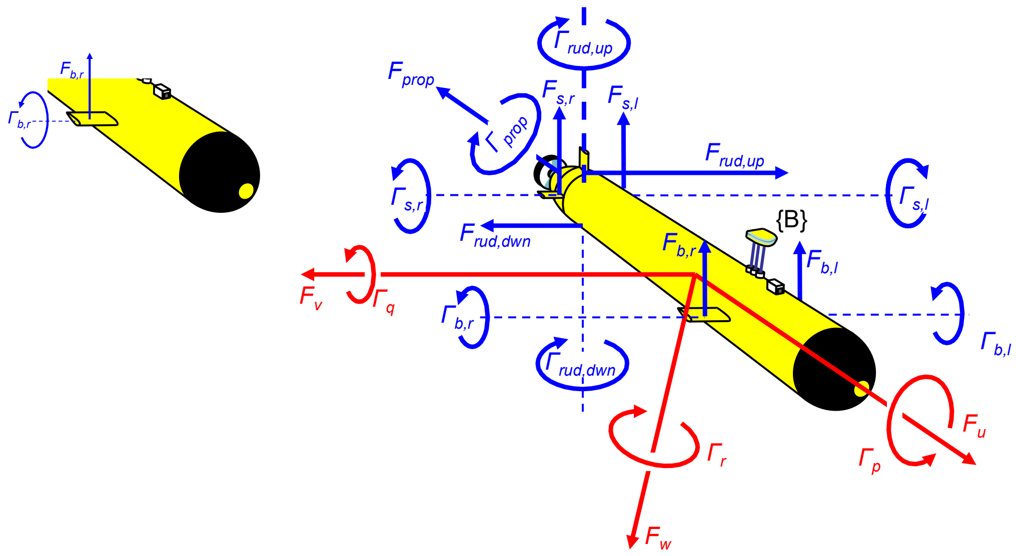
$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ \Gamma_r \end{bmatrix}}_{\mathbf{F}_B} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & d \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} F_p \\ \left.\begin{matrix} F_{rud}^{Dg} \\ F_{rud}^{Lft} \end{matrix}\right\} = \mathbf{f}_{rud}(v_t, \delta_{rud}) \end{bmatrix}^T}_{\mathbf{F}_m}$$

$$\rightarrow \mathbf{F}_B = \mathbf{A} \cdot \mathbf{F}_m$$

# Actuation Model

- 3D, Cartesian, Propulsive actuation, AUV

$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ F_w \\ \Gamma_p \\ \Gamma_q \\ \Gamma_r \end{bmatrix}}_{\mathbf{F_B}}$$




$$\underbrace{\begin{bmatrix} F_{b,l} \\ \Gamma_{b,l} \\ F_{b,r} \\ \Gamma_{b,r} \\ F_{s,l} \\ \Gamma_{s,l} \\ F_{s,r} \\ \Gamma_{s,r} \\ F_{rud,up} \\ \Gamma_{rud,up} \\ F_{rud,dwn} \\ \Gamma_{rud,dwn} \\ F_{prop} \\ \Gamma_{prop} \end{bmatrix}}_{\mathbf{F_m}}$$

$$\mathbf{F_B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R & 0 & -R & 0 & R & 0 & -R & 0 & R & 0 & R & 0 & 0 & 1 \\ d_s & 1 & -d_s & -1 & d_b & 1 & -d_b & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_b & 1 & d_b & -1 & 0 & 0 \end{bmatrix} \cdot \mathbf{F_m}$$

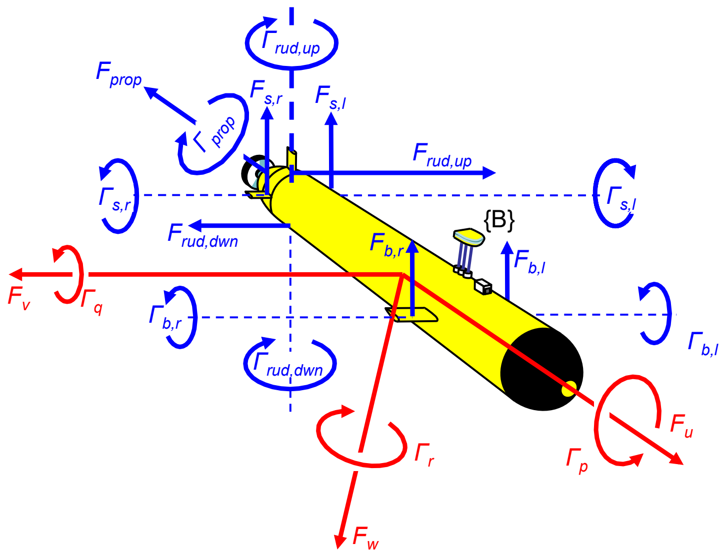
(please check !)

$$\mathbf{F_B} = \mathbf{A} \cdot \mathbf{F_m}$$

# Actuation Model

- 3D, Cartesian, Propulsive actuation, AUV

$$\underbrace{\begin{bmatrix} F_u \\ F_v \\ F_w \\ \Gamma_p \\ \Gamma_q \\ \Gamma_r \end{bmatrix}}_{\mathbf{F_B}}$$

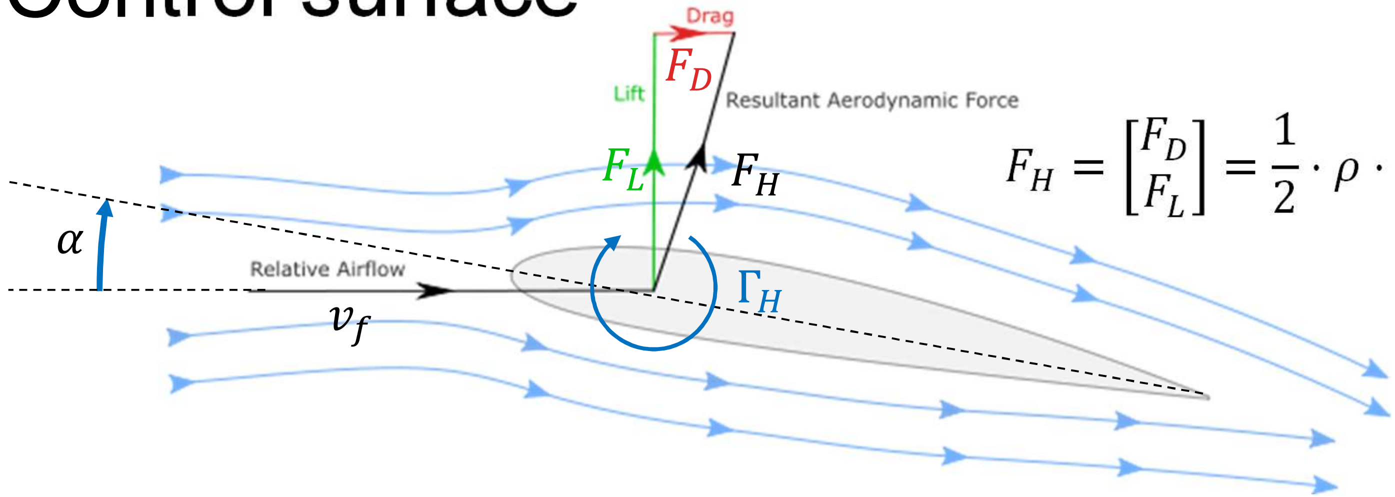



$$\mathbf{F_B = A \cdot F_m}$$

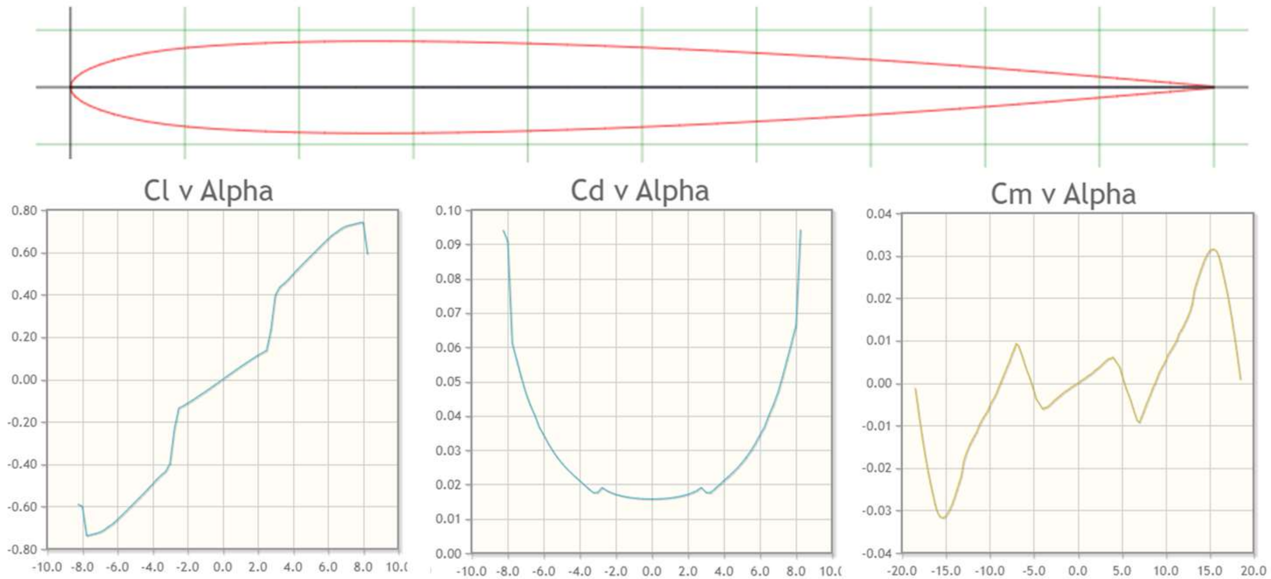
$$\underbrace{\begin{bmatrix} F_{b,l} \\ \Gamma_{b,l} \\ F_{b,r} \\ \Gamma_{b,r} \\ F_{s,l} \\ \Gamma_{s,l} \\ F_{s,r} \\ \Gamma_{s,r} \\ F_{rud,up} \\ \Gamma_{rud,up} \\ F_{rud,dwn} \\ \Gamma_{rud,dwn} \\ F_{prop} \\ \Gamma_{prop} \end{bmatrix}}_{\mathbf{F_m}} \longleftrightarrow \boxed{\text{Actuators characteristics}} \longleftrightarrow \begin{bmatrix} \delta_{rud,up} \\ \delta_{rud,dwn} \\ \delta_{s,r} \\ \delta_{s,l} \\ \delta_{b,r} \\ \delta_{b,l} \\ \omega_{prop} \end{bmatrix}$$

# Actuator characteristics

- Control surface



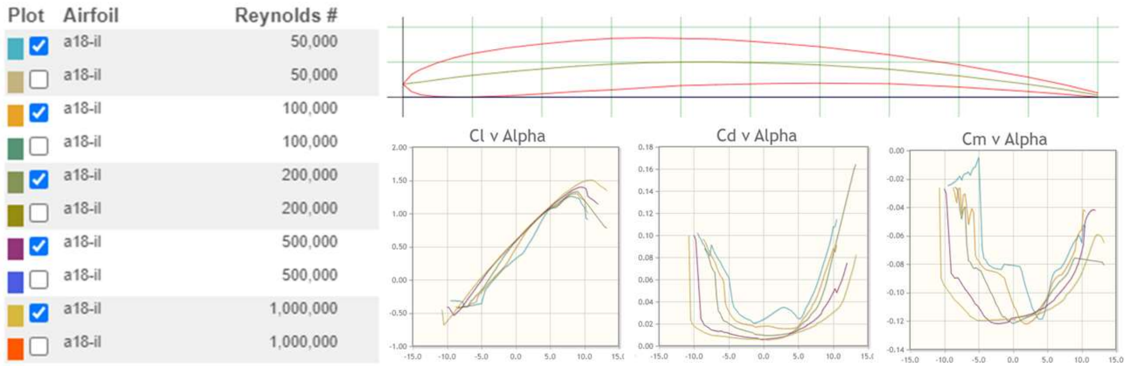

$$F_H = \begin{bmatrix} F_D \\ F_L \end{bmatrix} = \frac{1}{2} \cdot \rho \cdot S \cdot v_f^2 \cdot \begin{bmatrix} C_D(\alpha) \\ C_L(\alpha) \end{bmatrix}$$

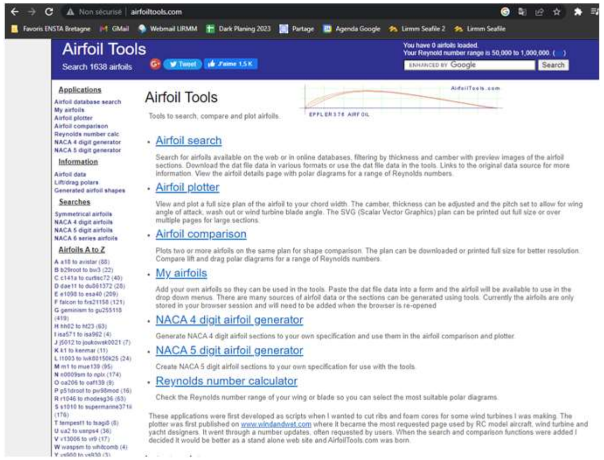



$$\begin{bmatrix} F_D \\ F_L \\ \Gamma_H \end{bmatrix} = \mathbf{f}_{\mathrm{m}}\big(\delta(c_{\mathrm{m}})\big)$$

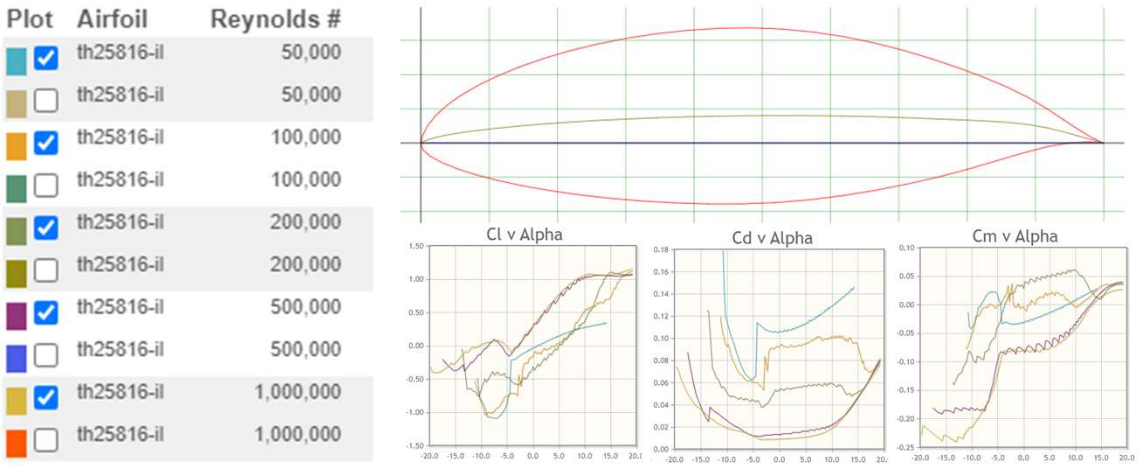
# Actuator characteristics

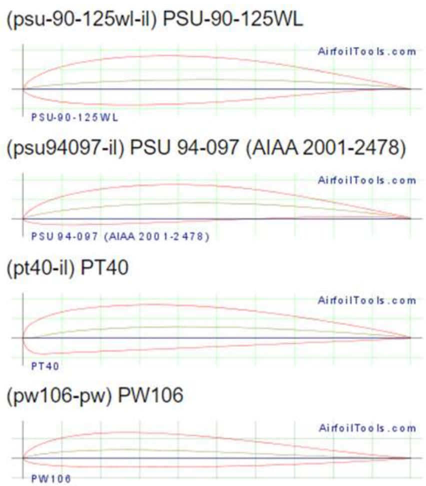
- Control surface




http://airfoiltools.com/airfoil/naca4digit

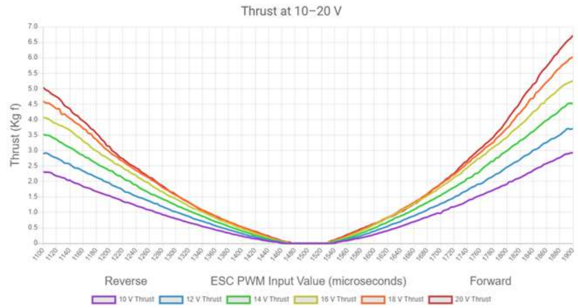



$$\mathbf{F}_{\mathrm{m}} = [F_D, F_L, \Gamma_M]^{\mathrm{T}} = \mathbf{f}_{\mathrm{m}}(C_L(\alpha), C_D(\alpha), C_M(\alpha), \nu, \delta(c_{\mathrm{m}}), S, \rho)$$

# Actuator characteristics

- Thruster (T200, BlueRobotics)



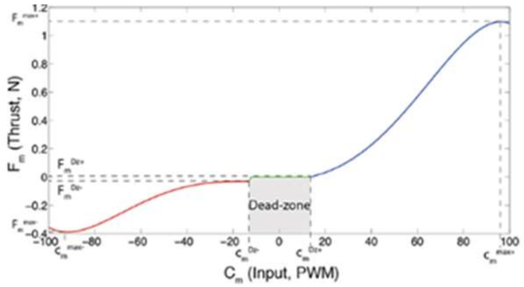
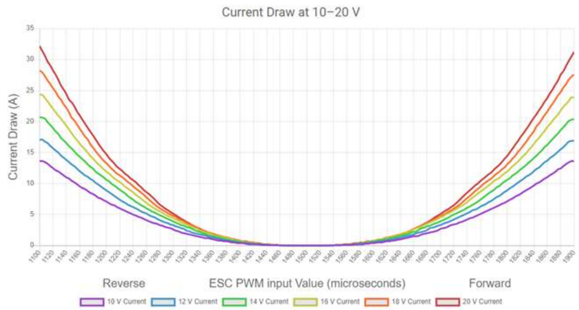
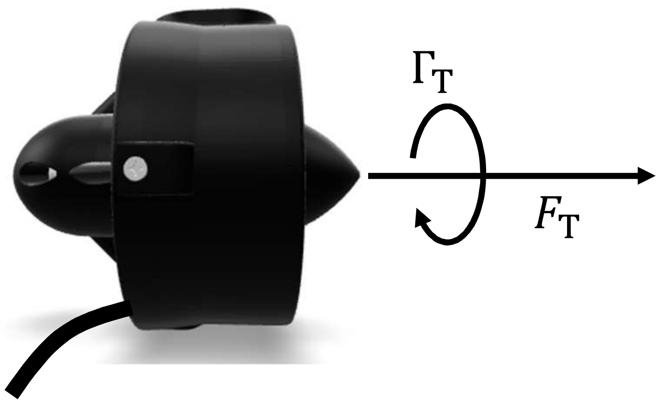



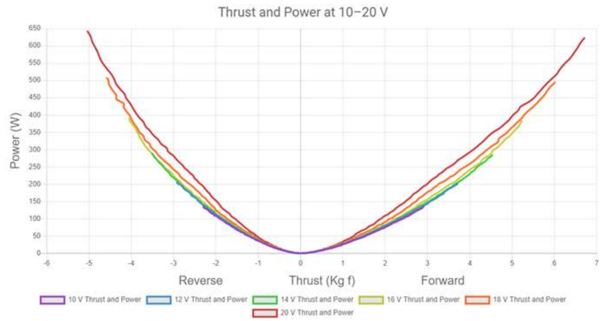
Fig. 4. Motor characteristic identification.



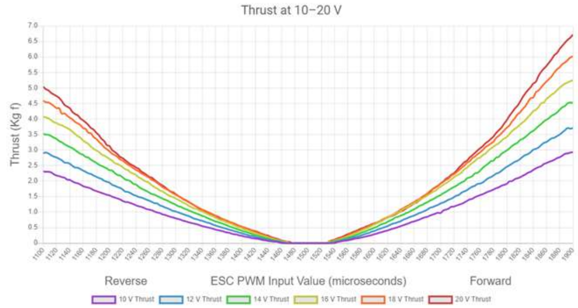

$c_m$(PWM)

$$\begin{bmatrix} F_T \\ \Gamma_T \end{bmatrix} = \mathbf{f}_m(c_m)$$



# Actuator characteristics

- Thruster (T200, BlueRobotics)



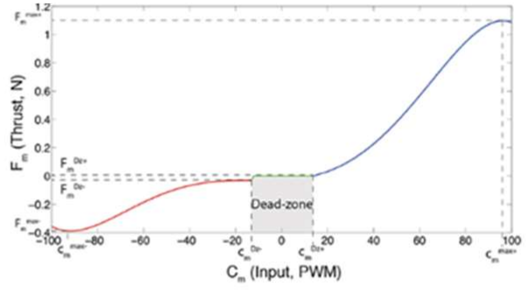
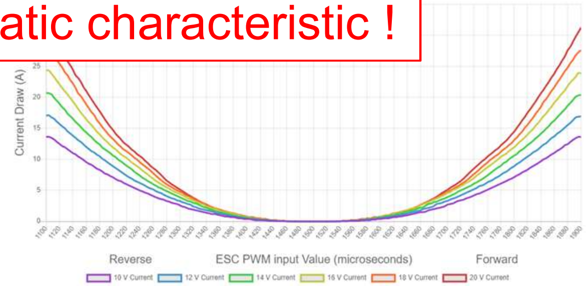
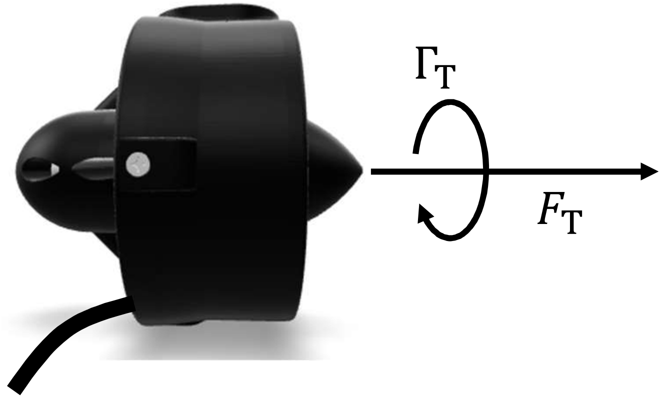




Fig. 4. Motor characteristic identification.

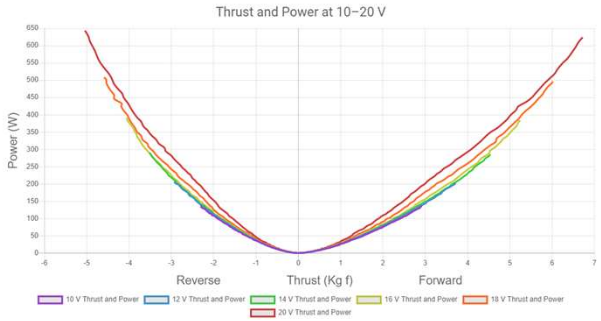
! Static characteristic !



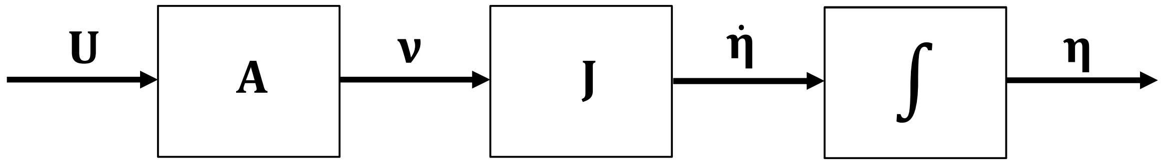

$c_m(\text{PWM})$

$$\begin{bmatrix} F_T \\ \Gamma_T \end{bmatrix} = \mathbf{f}_m(c_m)$$

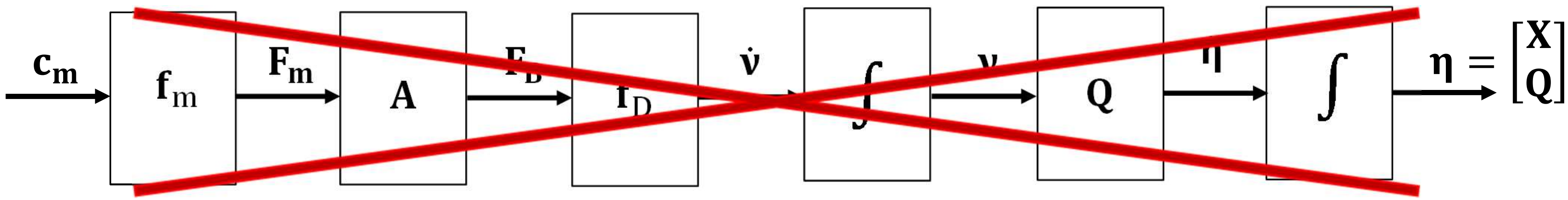


# Simulator

- Kinematics

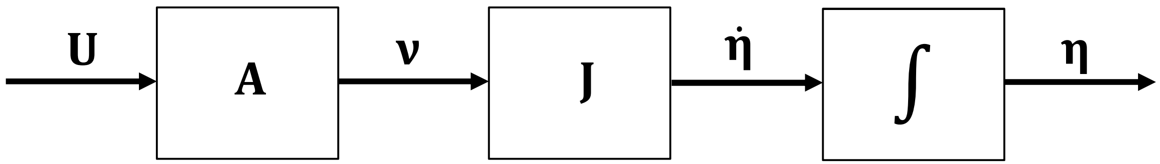



- Dynamics

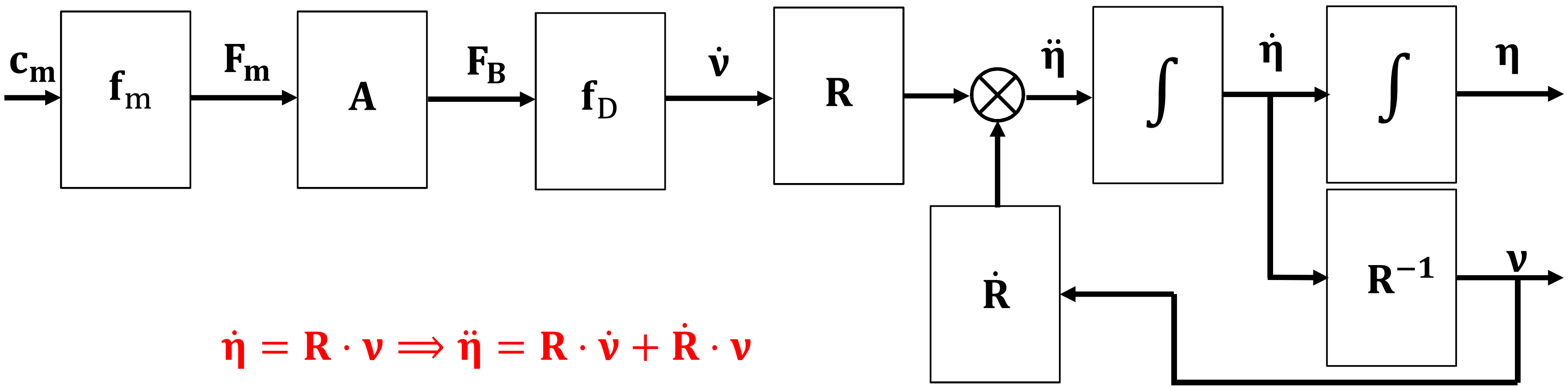


# Simulation

- Kinematics




- Dynamics




$$\dot{\boldsymbol{\eta}} = \mathbf{R} \cdot \boldsymbol{\nu} \Longrightarrow \ddot{\boldsymbol{\eta}} = \mathbf{R} \cdot \dot{\boldsymbol{\nu}} + \dot{\mathbf{R}} \cdot \boldsymbol{\nu}$$

- Exo
  - Generic simulation of wheeled systems
  - Elements for Simulation of an AUV