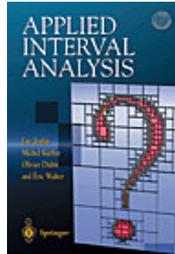


L'arithmétique des intervalles pour l'étude de propriétés topologiques d'un ensemble défini par des inégalités



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1 Motivations

1.1 Inversion ensembliste

Modèle : $y_m(\mathbf{p}, t) = p_1 e^{-p_2 t}$.

Paramètres : p_1, p_2 .

Temps de mesure : t_1, t_2, \dots, t_m

Barres erreurs : $[y_1^-, y_1^+], [y_2^-, y_2^+], \dots, [y_m^-, y_m^+]$

Ensemble de vraisemblance :

$$\begin{aligned} \mathcal{S} &= \left\{ \mathbf{p} \in \mathbb{R}^2, \forall i \in \{1, \dots, m\}, y_m(\mathbf{p}, t_i) \in [y_i^-, y_i^+] \right\} . \\ &= \mathbf{y}_m^{-1}([\mathbf{y}]) . \end{aligned}$$

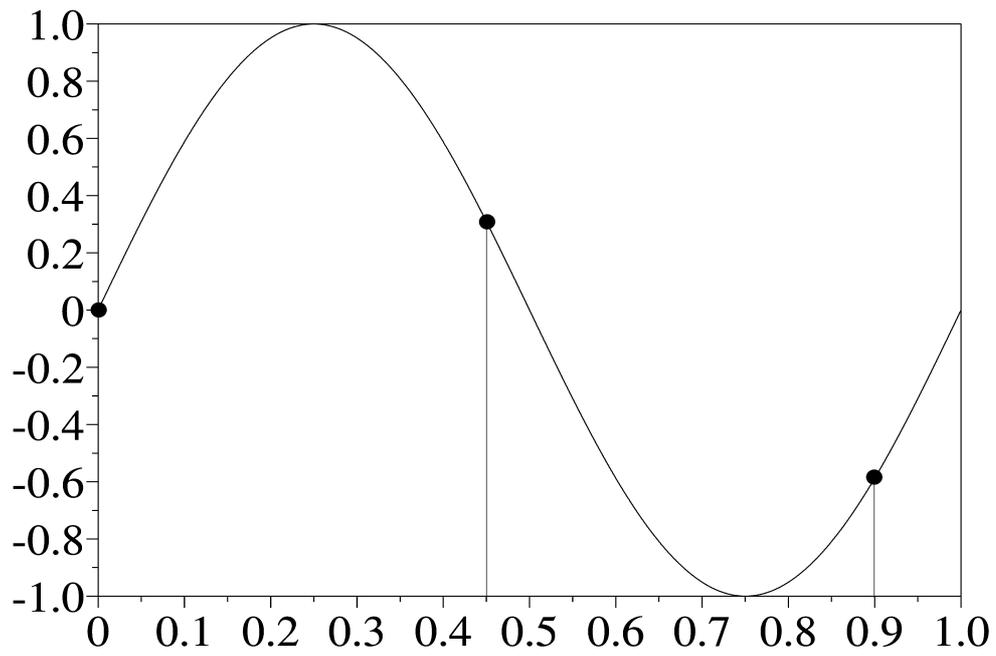
Illustration avec SetDemo

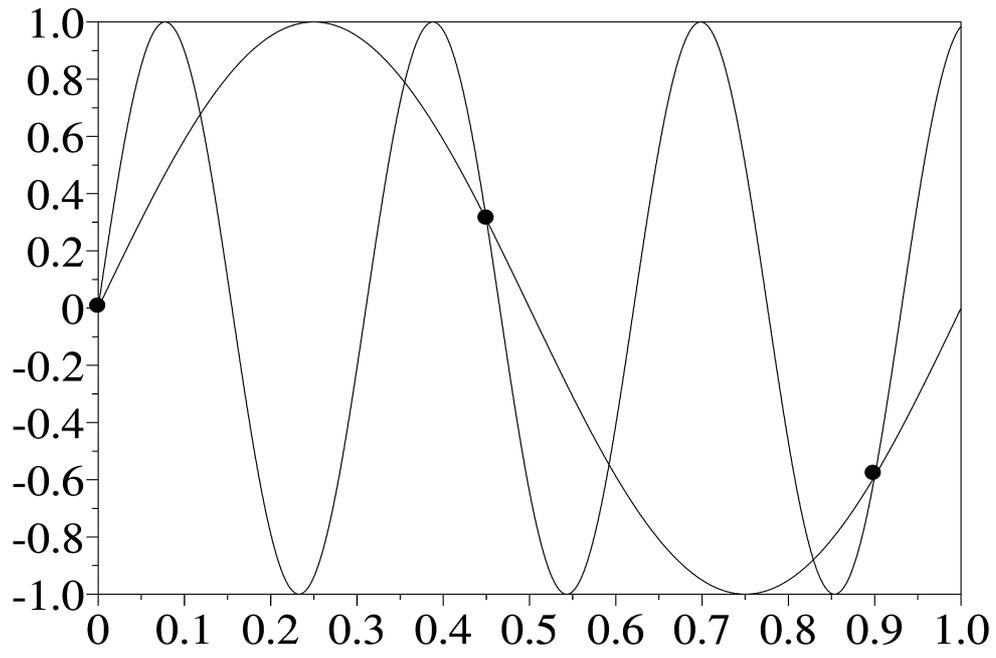
Modèle : $y_m(\mathbf{p}, t) = p_1 \sin(2\pi p_2 t)$.

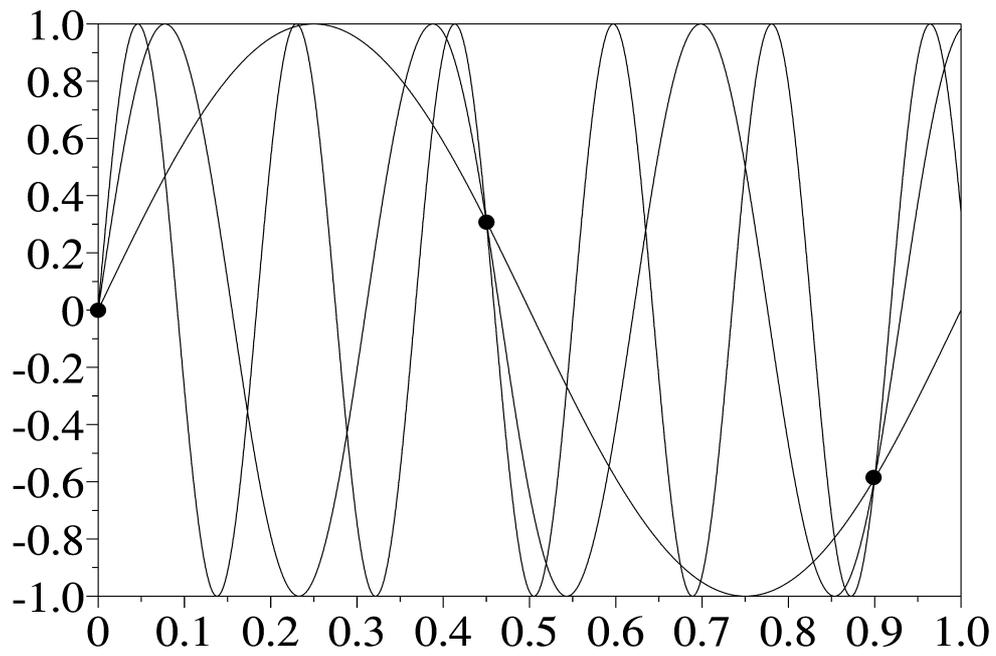
Paramètres : p_1, p_2 .

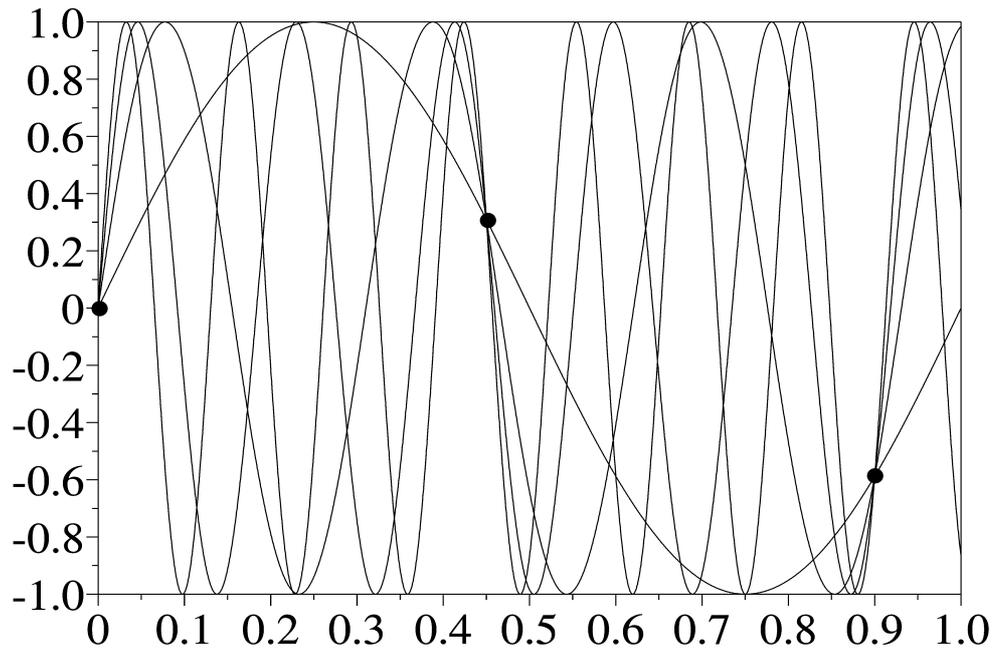
Temps de mesure : $t_1 = 0, t_2 = T_e, t_3 = 2T_e, \dots$

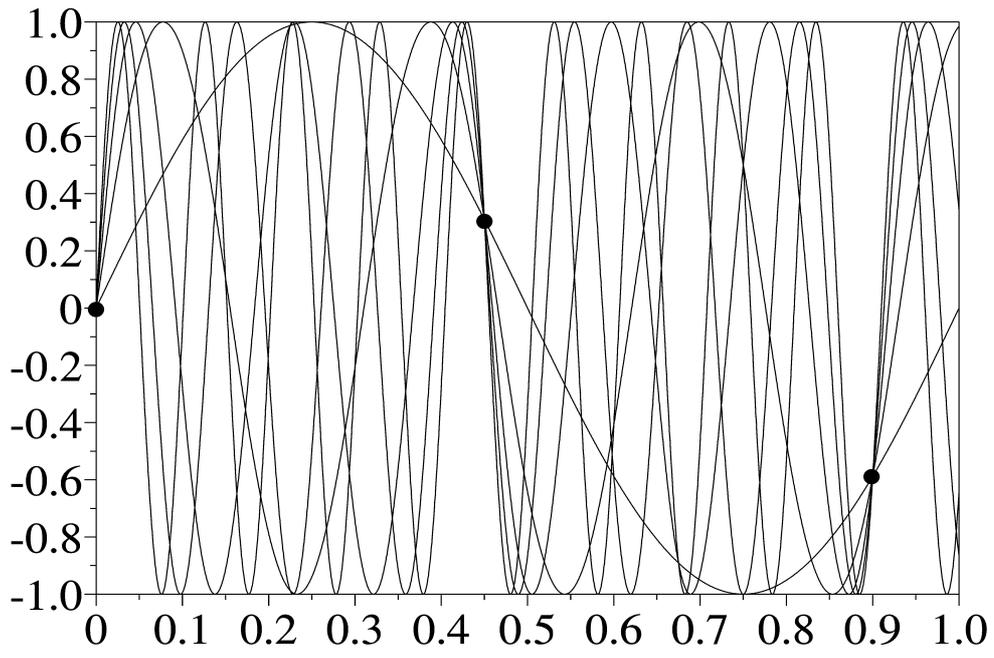
Barres erreurs : $[y_1^-, y_1^+], [y_2^-, y_2^+], [y_3^-, y_3^+], \dots$











2 Analyse par intervalles

2.1 Arithmétique des intervalles

Si $\diamond \in \{+, -, \cdot, /, \max, \min\}$, on définit

$$[x] \diamond [y] = [\{x \diamond y \mid x \in [x], y \in [y]\}].$$

Par exemple,

$$\begin{aligned} [-1, 3] + [2, 5] &= [?, ?], \\ [-1, 3] \cdot [2, 5] &= [?, ?], \\ [-1, 3] / [2, 5] &= [?, ?], \\ [-1, 3] \vee [2, 5] &= [?, ?]. \end{aligned}$$

Si $\diamond \in \{+, -, \cdot, /, \max, \min\}$, on définit

$$[x] \diamond [y] = [\{x \diamond y \mid x \in [x], y \in [y]\}].$$

Par exemple,

$$\begin{aligned} [-1, 3] + [2, 5] &= [1, 8], \\ [-1, 3] \cdot [2, 5] &= [-5, 15], \\ [-1, 3] / [2, 5] &= \left[-\frac{1}{2}, \frac{3}{2}\right], \\ [-1, 3] \vee [2, 5] &= [2, 5]. \end{aligned}$$

$$\begin{aligned} [x^-, x^+] + [y^-, y^+] &= [?, ?], \\ [x^-, x^+] \cdot [y^-, y^+] &= [?, ?], \\ [x^-, x^+] \vee [y^-, y^+] &= [?, ?]. \end{aligned}$$

$$\begin{aligned}
[x^-, x^+] + [y^-, y^+] &= [x^- + y^-, x^+ + y^+], \\
[x^-, x^+].[y^-, y^+] &= [x^-y^- \wedge x^+y^- \wedge x^-y^+ \wedge x^+y^+, \\
&\quad x^-y^- \vee x^+y^- \vee x^-y^+ \vee x^+y^+], \\
[x^-, x^+] \vee [y^-, y^+] &= [\vee(x^-, y^-), \vee(x^+, y^+)].
\end{aligned}$$

Si $f \in \{\cos, \sin, \text{sqr}, \text{sqrt}, \log, \exp, \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

Par exemple,

$$\begin{aligned}\sin([0, \pi]) &= [?, ?], \\ \text{sqr}([-1, 3]) &= [?, ?], \\ \text{abs}([-7, 1]) &= [?, ?], \\ \text{sqrt}([-10, 4]) &= [?, ?], \\ \log([-2, -1]) &= [?, ?].\end{aligned}$$

Si $f \in \{\cos, \sin, \text{sqr}, \text{sqrt}, \log, \exp, \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

Par exemple,

$$\sin([0, \pi]) = [0, 1],$$

$$\text{sqr}([-1, 3]) = [-1, 3]^2 = [0, 9],$$

$$\text{abs}([-7, 1]) = [0, 7],$$

$$\text{sqrt}([-10, 4]) = \sqrt{[-10, 4]} = [0, 2],$$

$$\log([-2, -1]) = \emptyset.$$

2.2 Projection de contraintes

Soient x, y, z , 3 variables telles que

$$x \in [-\infty, 5],$$

$$y \in [-\infty, 4],$$

$$z \in [6, \infty],$$

$$z = x + y.$$

Les valeurs < 2 pour x , < 1 pour y et > 9 pour z sont inconsistantes.

Projeter une contrainte (ici, $z = x + y$), c'est calculer le plus petit intervalle qui contient toutes les valeurs consistantes pour les variables. Pour notre exemple cela revient à projeter (suivant x, y et z) le sous-ensemble de \mathbb{R}^3 donné par

$$\mathbb{S} = \{(x, y, z) \in [-\infty, 5] \times [-\infty, 4] \times [6, \infty] \mid z = x + y\}.$$

2.3 Calcul de la projection d'une contrainte

Puisque $x \in [-\infty, 5]$, $y \in [-\infty, 4]$, $z \in [6, \infty]$ et $z = x + y$, nous avons

$$z = x + y \Rightarrow z \in [?, ?]$$

$$x = z - y \Rightarrow x \in [?, ?]$$

$$y = z - x \Rightarrow y \in [?, ?]$$

Puisque $x \in [-\infty, 5]$, $y \in [-\infty, 4]$, $z \in [6, \infty]$ et $z = x + y$, nous avons

$$z = x + y \Rightarrow z \in [6, \infty] \cap ([-\infty, 5] + [-\infty, 4]) \\ = [6, \infty] \cap [-\infty, 9] = [6, 9].$$

$$x = z - y \Rightarrow x \in [-\infty, 5] \cap ([6, \infty] - [-\infty, 4]) \\ = [-\infty, 5] \cap [2, \infty] = [2, 5].$$

$$y = z - x \Rightarrow y \in [-\infty, 4] \cap ([6, \infty] - [-\infty, 5]) \\ = [-\infty, 4] \cap [1, \infty] = [1, 4].$$

Notation :

$$z = x + y \Leftrightarrow +(z, x, y)$$

$$z = x * y \Leftrightarrow *(z, x, y)$$

$$y = \sin(x) \Leftrightarrow \sin(y, x)$$

$$y = x^2 \Leftrightarrow \text{sqr}(y, x)$$

Si $\diamond \in \{+, *, \max, \sin, \cos, \log, \dots\}$, est une contrainte primitive,

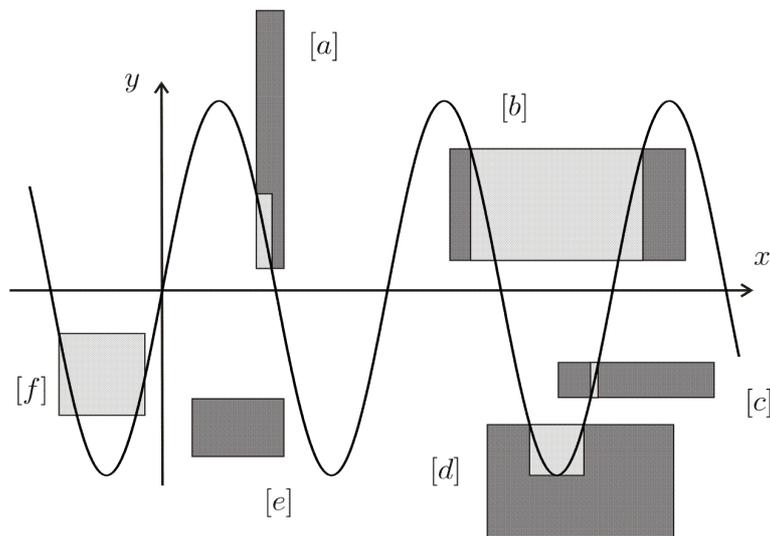
$$\diamond([x_1], [x_2], \dots) = [\{(x_1, x_2, \dots) \mid \diamond(x_1, x_2, \dots)\}].$$

Par exemple

opération		résultat
$+(11, 7, 4)$	est	vrai
$+(0, 7, 4)$	est	faux
$+[11], [7], [4]$	est	$([11], [7], [4])$
$+[0], [7], [4]$	est	\emptyset
$+[-\infty, 5], [-\infty, 4], [6, \infty]$	est	$[2, 5] \times [1, 4] \times [6, 9]$

Sinus

$$([y'], [x']) = \sin([y], [x])$$



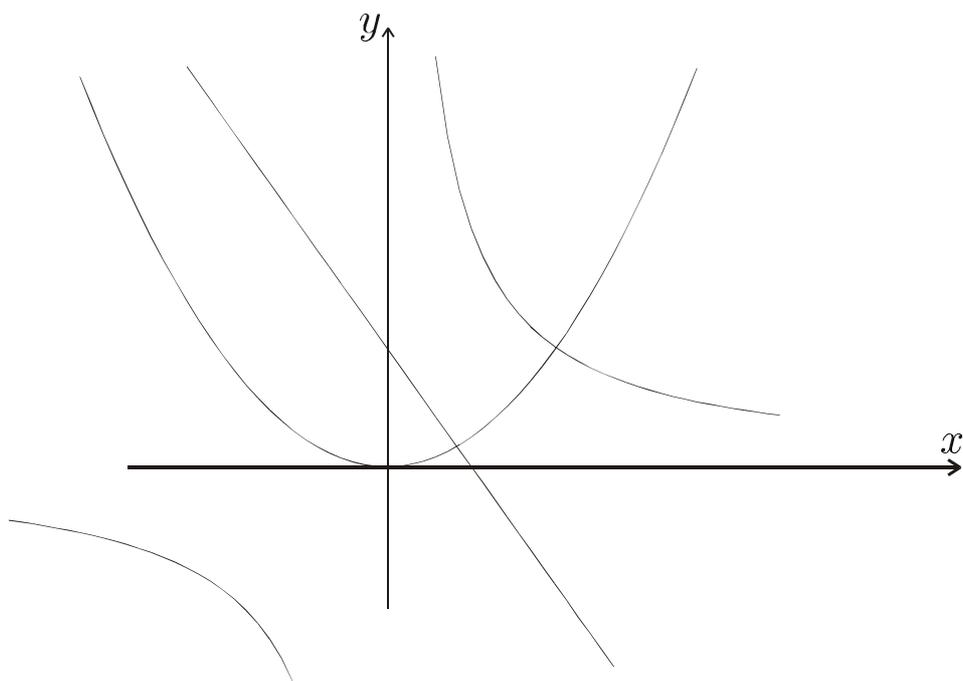
2.4 Propagation de contraintes

Considérons les trois contraintes

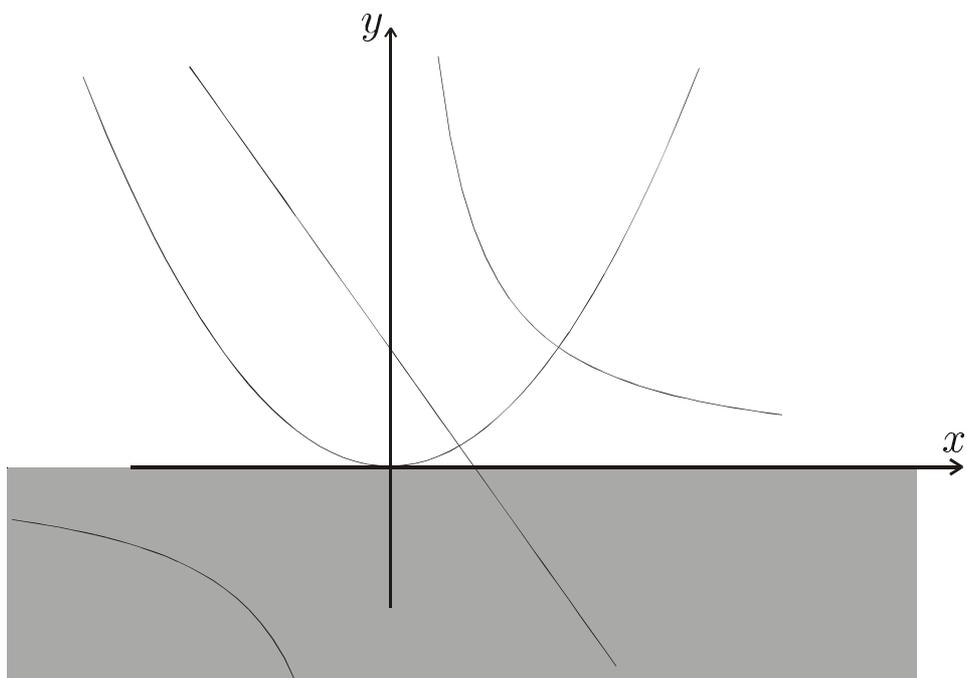
$$\begin{cases} (C_1) : & y = x^2 \\ (C_2) : & xy = 1 \\ (C_3) : & y = -2x + 1 \end{cases}$$

A chaque variable, on associe le domaine $[-\infty, \infty]$.

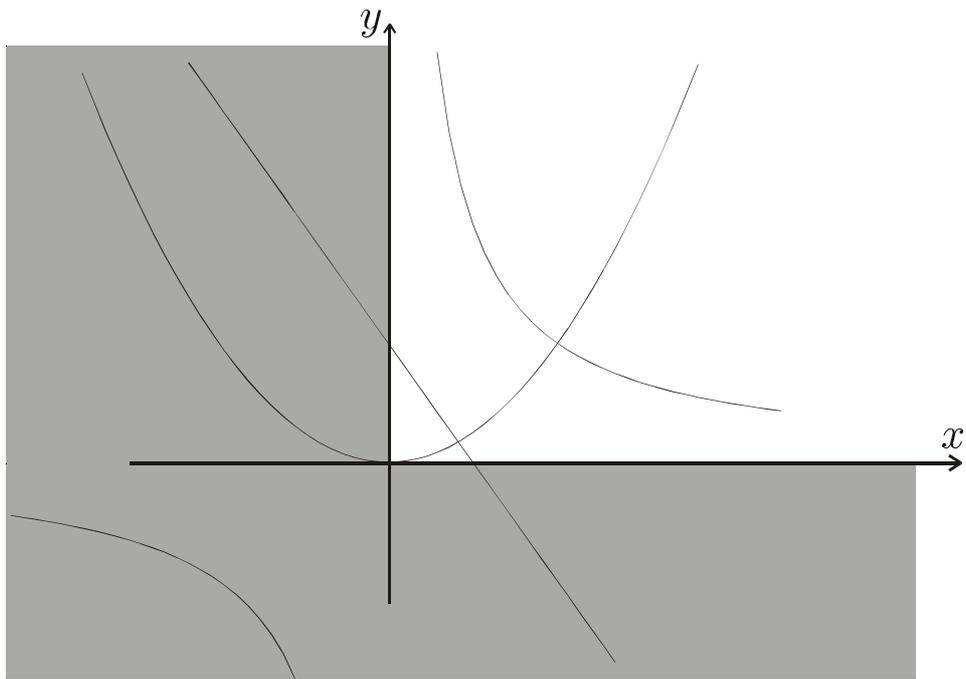
La *propagation de contrainte* revient à projeter les contraintes jusqu'à l'équilibre.



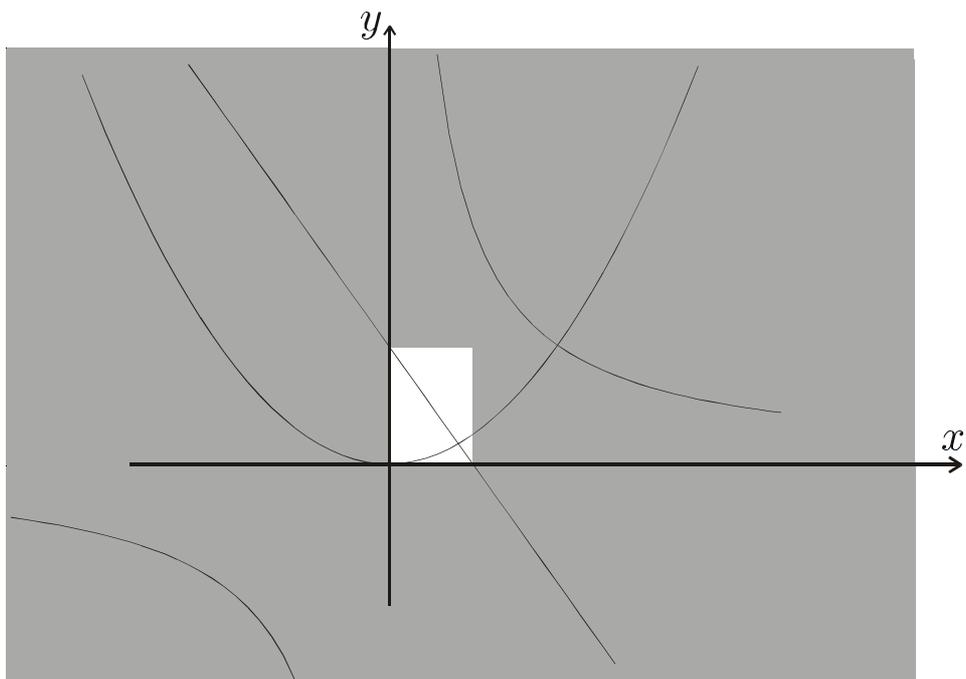
$$\left\{ \begin{array}{l} x \in [-\infty, \infty] \\ y \in [-\infty, \infty] \end{array} \right. \xRightarrow{(y=x^2)} \left\{ \begin{array}{l} x \in [-\infty, \infty] \\ y \in [0, \infty] \end{array} \right.$$



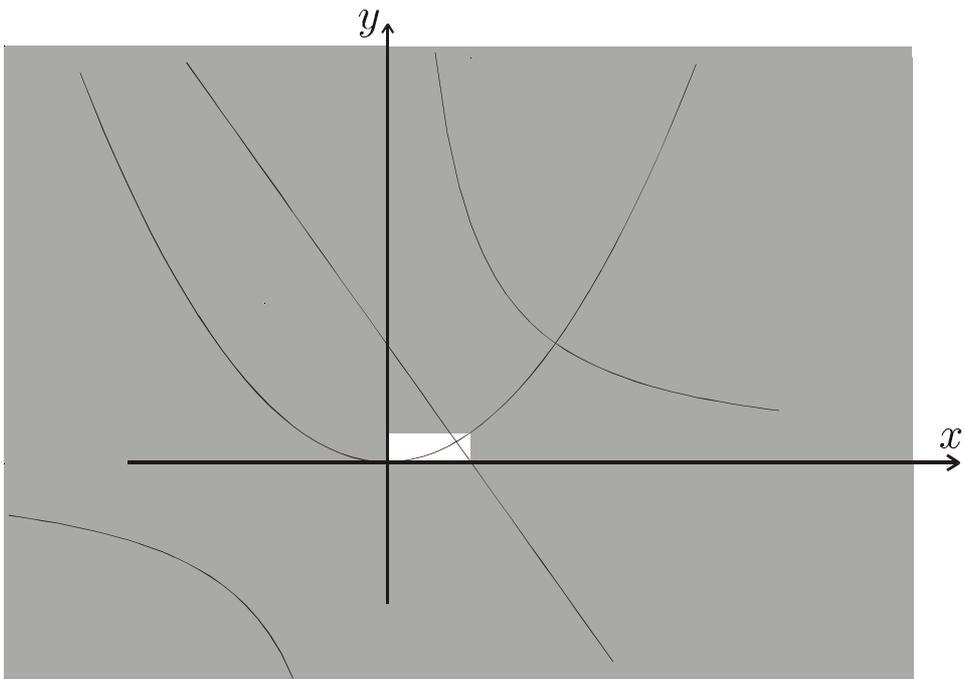
$$\left\{ \begin{array}{l} x \in [-\infty, \infty] \\ y \in [0, \infty] \end{array} \right. \quad (xy=1) \quad \Longrightarrow \quad \left\{ \begin{array}{l} x \in [0, \infty] \\ y \in [0, \infty] \end{array} \right.$$



$$\begin{cases} x \in [0, \infty] \\ y \in [0, \infty] \end{cases} \quad (y = -2x + 1) \quad \Longrightarrow \quad \begin{cases} x \in [0, 1/2] \\ y \in [0, 1] \end{cases}$$



$$\begin{cases} x \in [0, 1/2] \\ y \in [0, 1] \end{cases} \quad (y=x^2) \quad \Longrightarrow \quad \begin{cases} x \in [0, 1/2] \\ y \in [0, 1/4] \end{cases}$$



$$\begin{cases} x \in [0, 1/2] \\ y \in [0, 1/4] \end{cases} \stackrel{(xy=1)}{\implies} \begin{cases} x \in \emptyset \\ y \in \emptyset \end{cases}$$

$$\begin{aligned}
(C_1) &\Rightarrow y \in [-\infty, \infty]^2 = [0, \infty] \\
(C_2) &\Rightarrow x \in 1/[0, \infty] = [0, \infty] \\
(C_3) &\Rightarrow y \in [0, \infty] \cap ((-2) \cdot [0, \infty] + 1) \\
&\quad = [0, \infty] \cap ([-\infty, 1]) = [0, 1] \\
&\quad x \in [0, \infty] \cap (-[0, 1]/2 + 1/2) \\
&\quad = [0, 1/2] \\
(C_1) &\Rightarrow y \in [0, 1] \cap [0, 1/2]^2 = [0, 1/4] \\
(C_2) &\Rightarrow x \in [0, 1/2] \cap 1/[0, 1/4] = \emptyset \\
&\quad y \in [0, 1/4] \cap 1/\emptyset = \emptyset.
\end{aligned}$$

2.5 Décomposition

Pour des contraintes plus complexes, nous devons effectuer une décomposition. Par exemple :

$$\begin{aligned}x + \sin(y) - xz &\leq 0, \\x \in [-1, 1], y \in [-1, 1], z \in [-1, 1]\end{aligned}$$

peut être décomposé comme suit

$$\left\{ \begin{array}{lll} a = \sin(y) & x \in [-1, 1] & a \in [-\infty, \infty] \\ b = x + a & y \in [-1, 1] & b \in [-\infty, \infty] \\ c = xz & z \in [-1, 1] & c \in [-\infty, \infty] \\ b - c = d & & d \in [-\infty, 0] \end{array} \right. ,$$

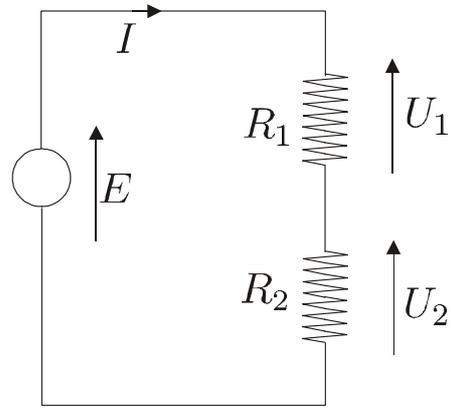
Pour contracter les domaines pour x, y, z , il faut effectuer

$$\left\{ \begin{array}{l} \sin([a], [y]) \\ +([b], [x], [a]) \\ *([c], [x], [z]) \\ +([b], [c], [d]) \end{array} \right.$$

jusqu'à l'équilibre.

3 Applications

3.1 Estimation



Contraintes

$$P = EI; E = (R_1 + R_2) I;$$

$$U_1 = R_1 I; U_2 = R_2 I; E = U_1 + U_2.$$

Domaines initiaux

$$\begin{aligned} R_1 &\in [0, \infty] \Omega, & R_2 &\in [0, \infty] \Omega, \\ E &\in [23, 26] \text{V}, & I &\in [4, 8] \text{A}, \\ U_1 &\in [10, 11] \text{V}, & U_2 &\in [14, 17] \text{V}, \\ P &\in [124, 130] \text{W}, \end{aligned}$$

Contraintes

$$\begin{aligned} P &= EI; E = (R_1 + R_2) I; \\ U_1 &= R_1 I; U_2 = R_2 I; E = U_1 + U_2. \end{aligned}$$

ou de façon équivalente

$$\begin{aligned} &*(P, E, I); +(R, R_1, R_2), *(E, R, I); \\ &*(U_1, R_1, I); *(U_2, R_2, I); +(E, U_1, U_2). \end{aligned}$$

Pour contracter les domaines, nous appliquons la procédure

$$\begin{aligned} &*([P], [E], [I]); +([R], [R_1], [R_2]), *([E], [R], [I]); \\ &*([U_1], [R_1], [I]); *([U_2], [R_2], [I]); +([E], [U_1], [U_2]). \end{aligned}$$

jusqu'à l'équilibre.

Nous obtenons les domaines contractés

$$\begin{aligned} R_1 &\in [1.84, 2.31] \Omega, & R_2 &\in [2.58, 3.35] \Omega, \\ E &\in [24, 26] V, & I &\in [4.769, 5.417] A, \\ U_1 &\in [10, 11] V, & U_2 &\in [14, 16] V, \\ P &\in [124, 130] W, \end{aligned}$$

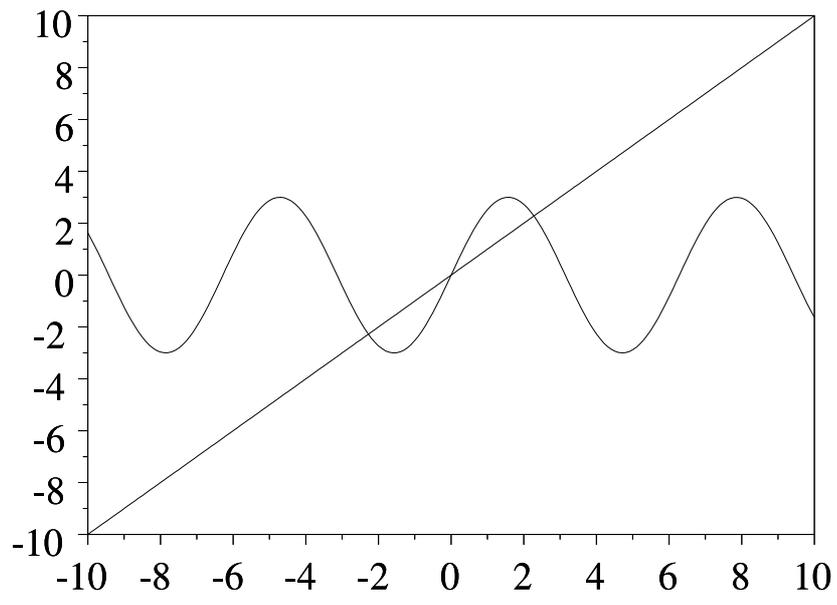
au lieu des domaines

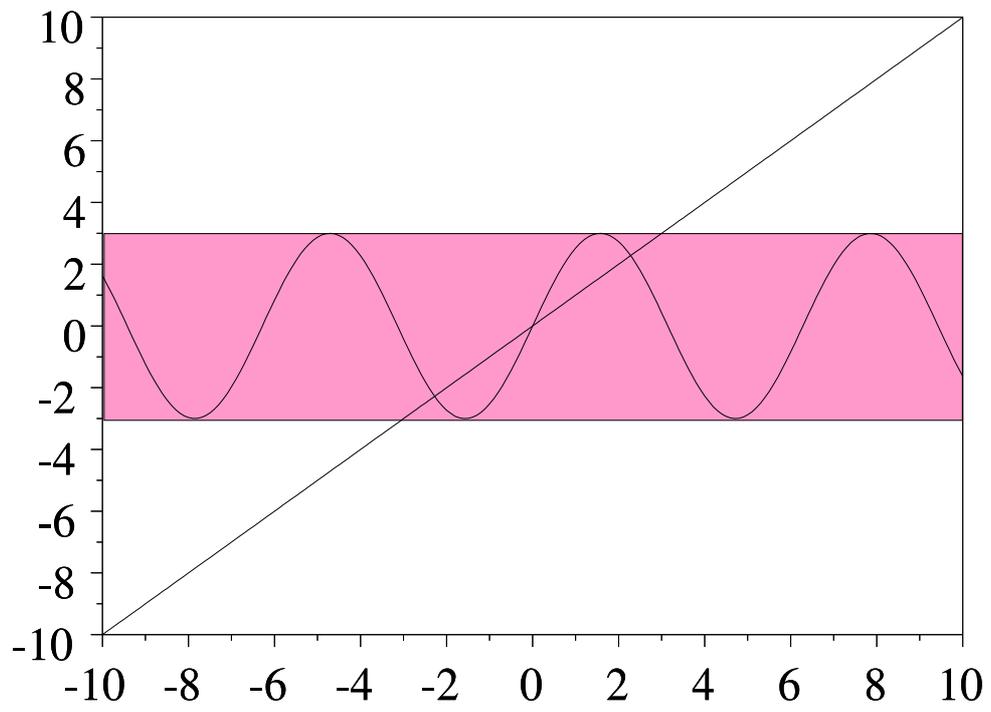
$$\begin{aligned} R_1 &\in [0, \infty] \Omega, & R_2 &\in [0, \infty] \Omega, \\ E &\in [23, 26] V, & I &\in [4, 8] A, \\ U_1 &\in [10, 11] V, & U_2 &\in [14, 17] V, \\ P &\in [124, 130] W, \end{aligned}$$

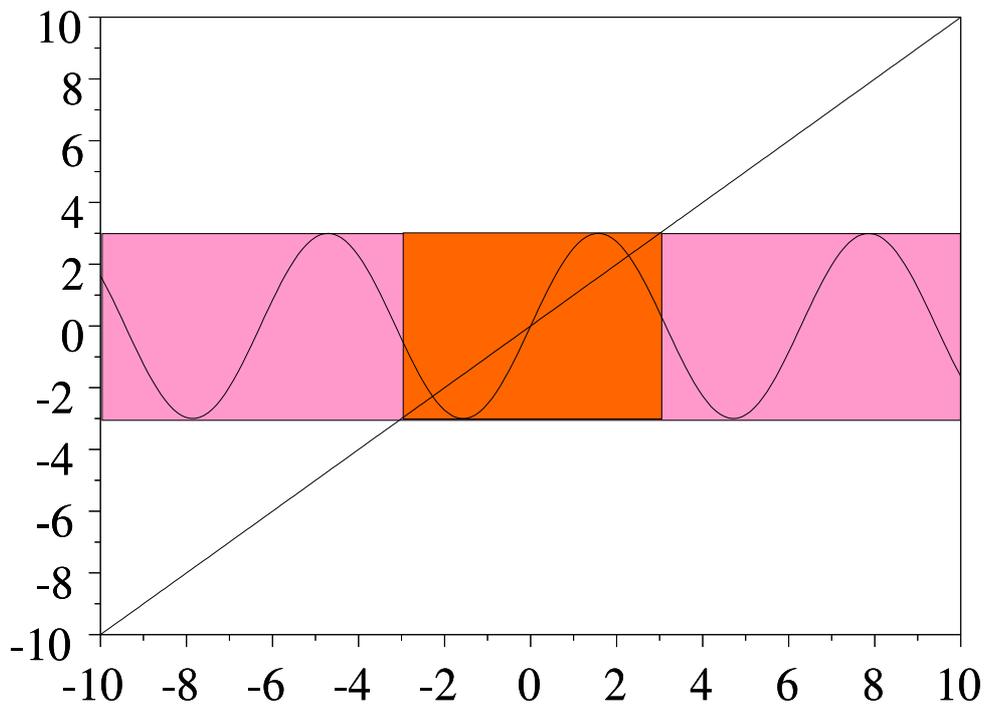
3.2 Résolution d'équations non linéaires (M. Dao)

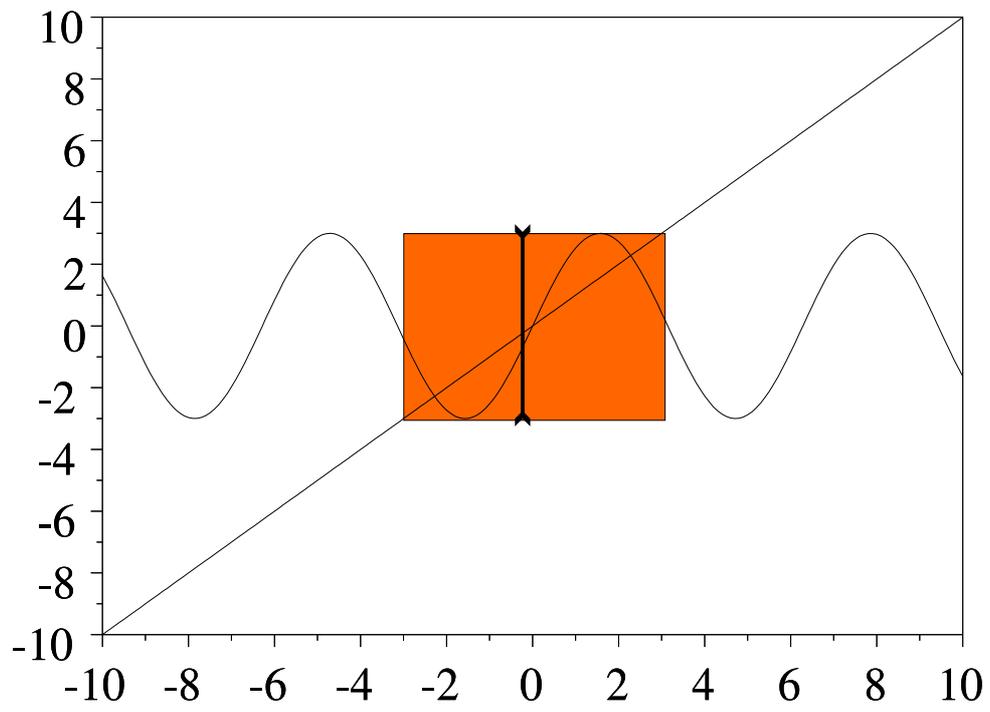
Soit le système d'équation

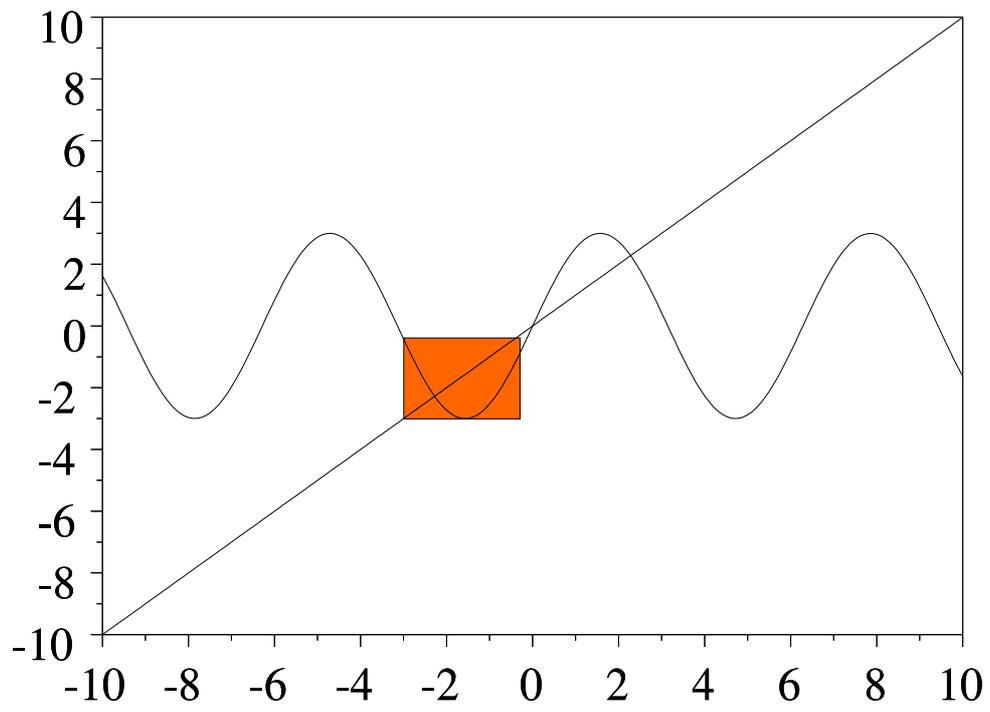
$$\begin{cases} y = 3 \sin(x) \\ y = x \end{cases} \quad x \in \mathbb{R}, y \in \mathbb{R}.$$

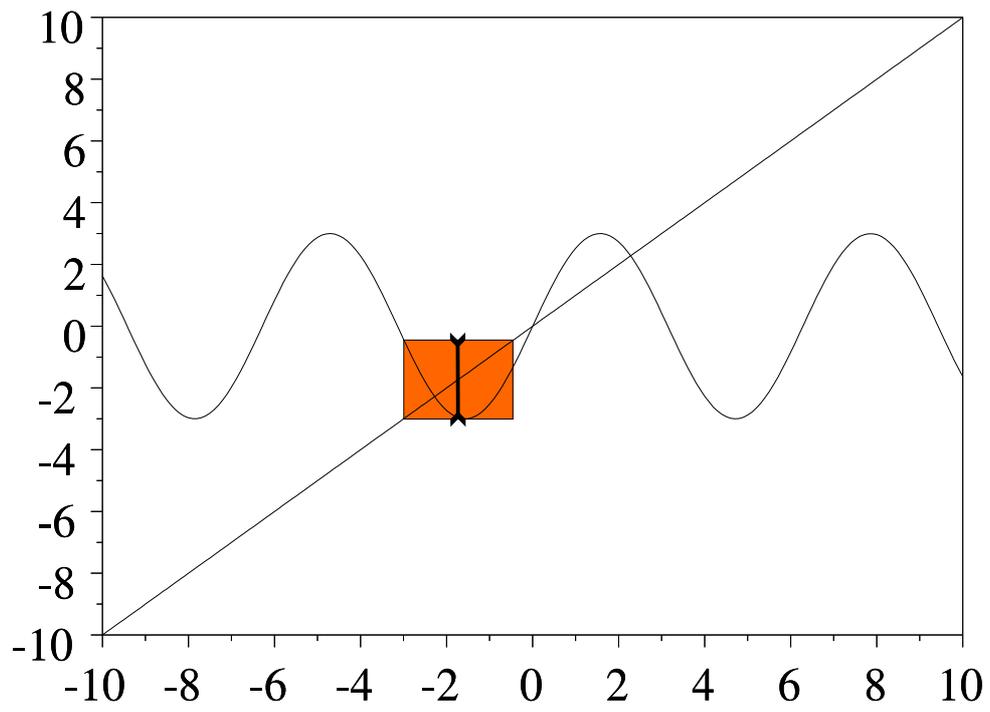


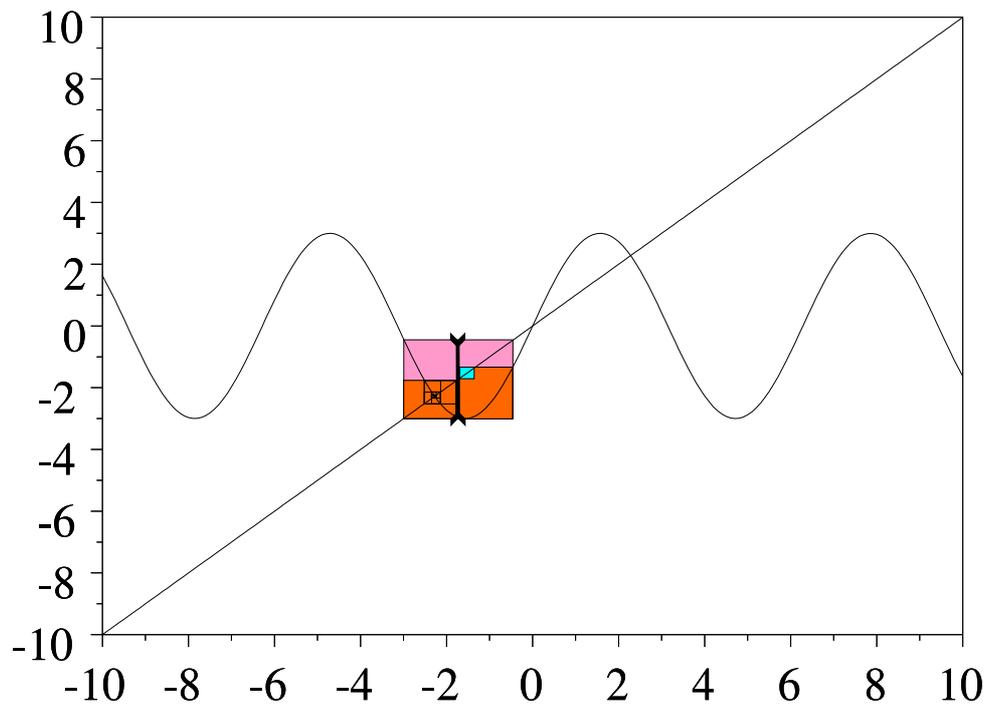












(Illustration avec Proj2D)

3.3 Prouver l'infailibilité de contraintes

Idée : On utilise la contraposée des contraintes.

Par exemple, prouver que

$$\forall x \in [x], \forall y \in [y], f(x, y) \leq 0 \text{ et } g(x, y) \leq 0,$$

revient à montrer que

$$\{(x, y) \in [x] \times [y] \mid f(x, y) > 0 \text{ or } g(x, y) > 0\} = \emptyset$$

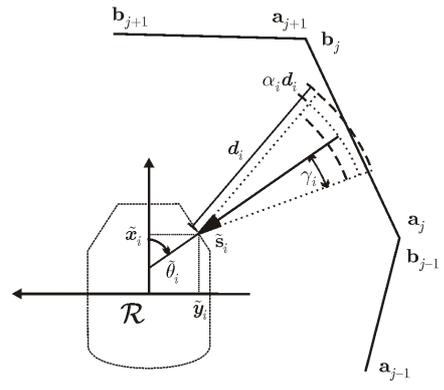
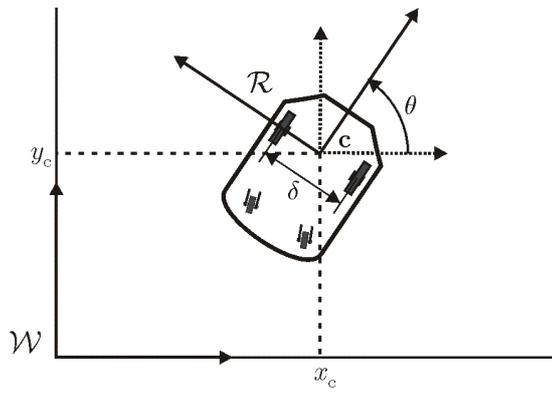
c'est-à-dire,

$$\{(x, y) \in [x] \times [y] \mid \max(f(x, y), g(x, y)) > 0\} = \emptyset.$$

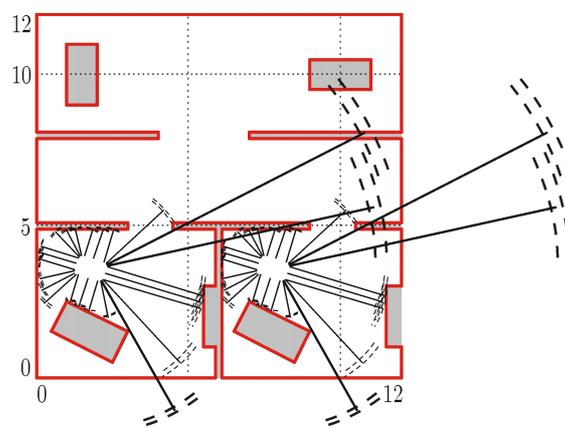
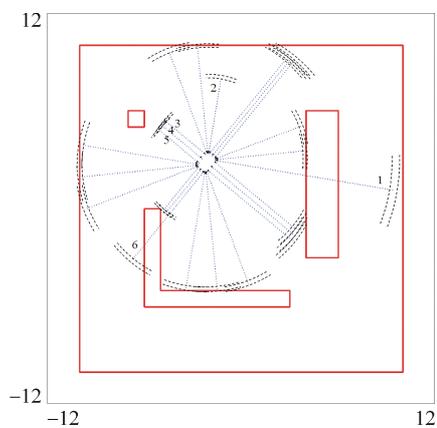
3.4 Localisation

Robot avec 24 télémètres à ultrason



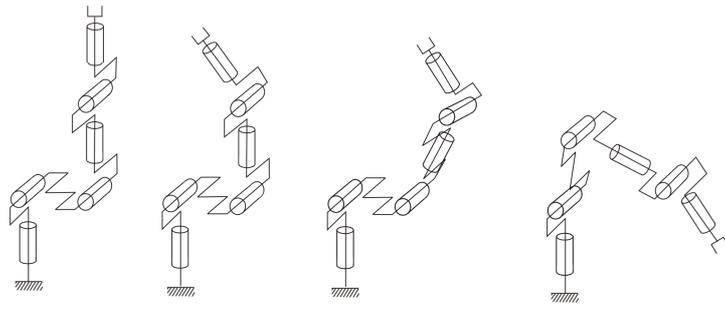


Après inversion ensembliste



3.5 Calibration de robots (X. Baguenard)



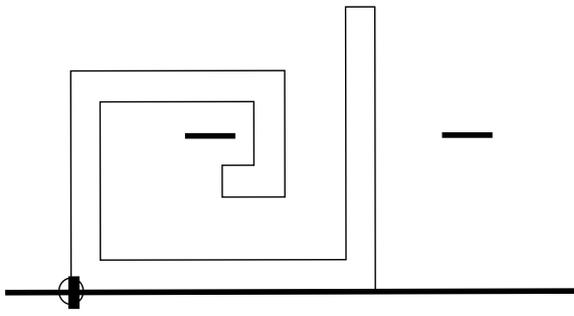


j	α_j	d_j	θ_j^o	r_j
0	-	-	$\frac{\pi}{2}$	0.5
1	0.1	0	0	0
2	$-\frac{\pi}{2}$	0	0	0
3	0	0.5	0	0
4	$\frac{\pi}{2}$	0	0	0.5
5	$-\frac{\pi}{2}$	0	0	0
6	$\frac{\pi}{2}$	0	-	-

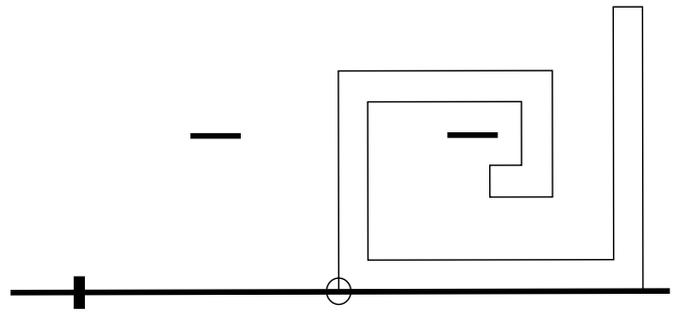
i	b_x^i	b_y^i	b_z^i
1	0.1	0.2	0.1
2	0.1	0.1	0.2
3	0.2	0.1	0.1

	Domaines initiaux	Domaines contractés
r_0	[0.4, 0.6]	[0.494046, 0.50101]
d_1	[0, 0.1]	[0, 0.000558009]
r_1	[0, 0.1]	[0, 0.00693694]
d_3	[0.49, 0.51]	[0.498385, 0.501133]
r_4	[0.49, 0.51]	[0.499216, 0.50114]
b_x^1	[0, 0.2]	[0.0996052, 0.100629]
b_y^1	[0.1, 0.3]	[0.199502, 0.200455]
b_z^1	[0, 0.2]	[0.0997107, 0.100714]
b_x^2	[0, 0.2]	[0.0996747, 0.100712]
b_y^2	[0, 0.2]	[0.0994585, 0.10031]
b_z^2	[0.1, 0.3]	[0.199535, 0.200642]
b_x^3	[0.1, 0.3]	[0.199689, 0.200578]
b_y^3	[0, 0.2]	[0.0997562, 0.100319]
b_z^3	[0, 0.2]	[0.0995661, 0.100557]

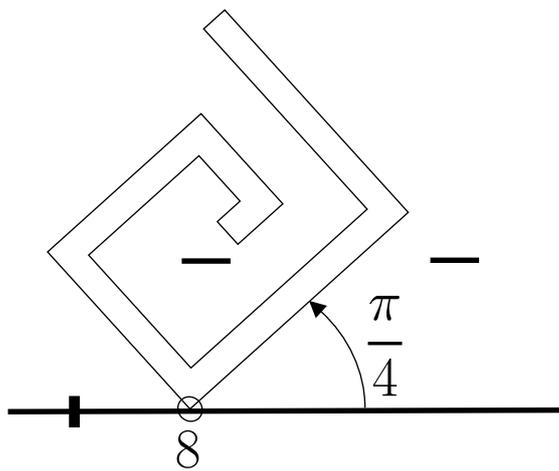
3.6 Planification de chemin



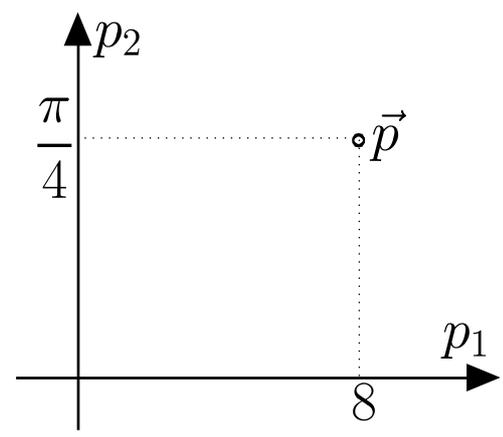
Initial configuration: $\vec{p} = (0 \ 0)^T$



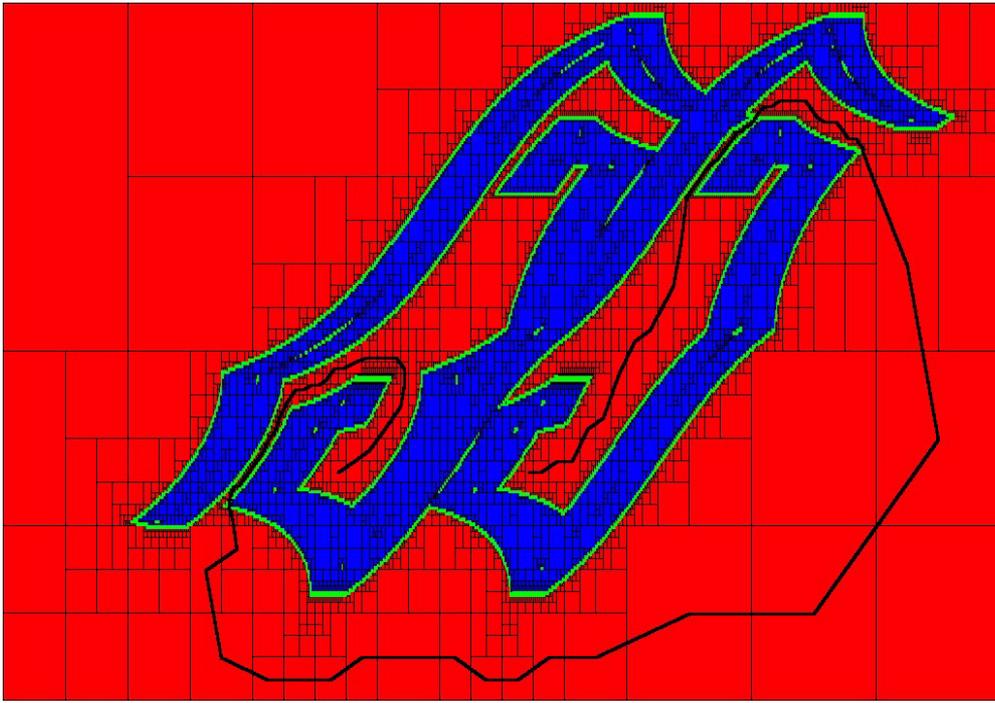
Goal configuration: $\vec{p} = (17 \ 0)^T$

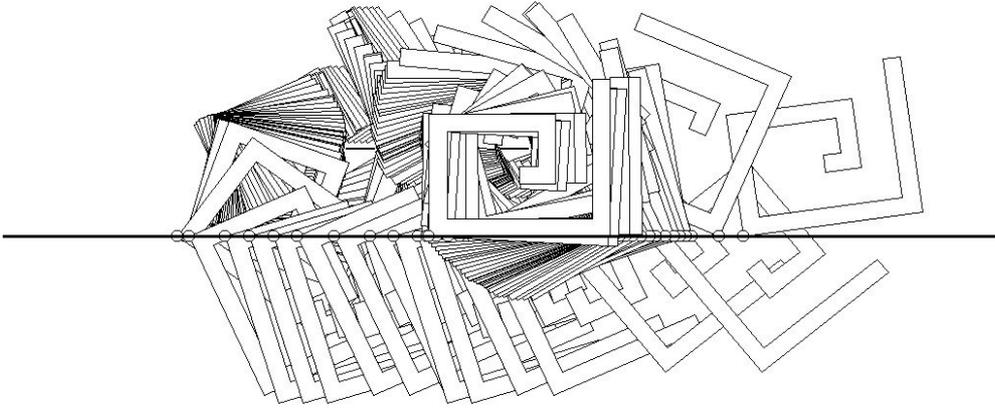


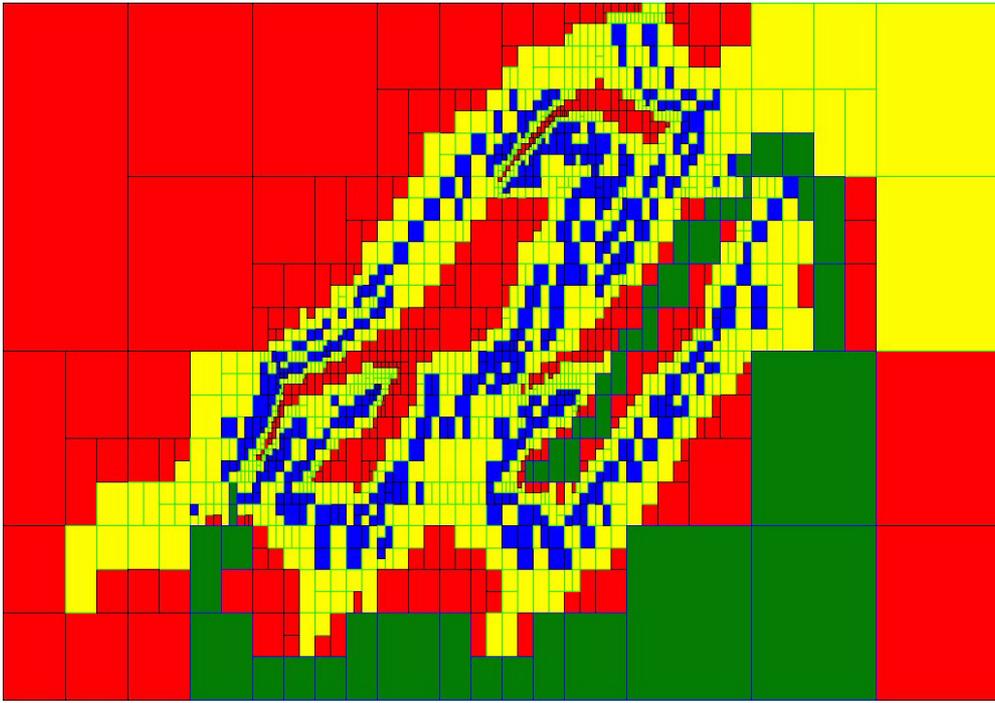
Room

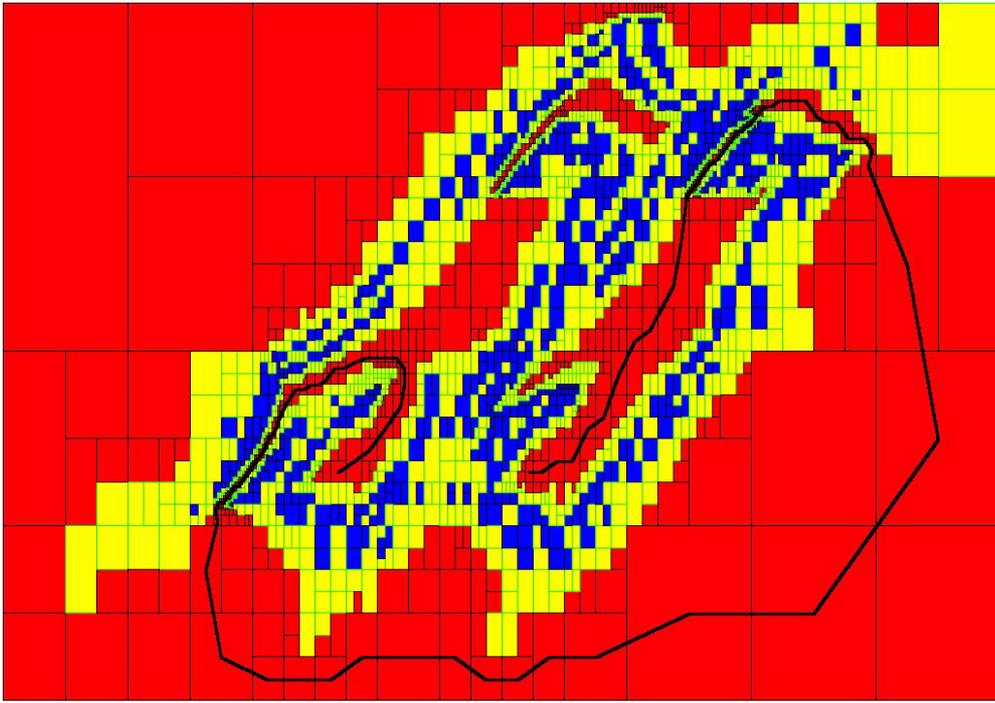


Configuration space









4 Topologie avec les intervalles (N. Delanoue)