The Lorenz attractor exists

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The Lorenz equations

Introduced in 1963 by Edward Lorenz as a simplified model for convection:

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\begin{align*}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\
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Three fixed points: the origin and

\(C^\pm = (\pm \sqrt{\beta(\rho - 1)}, \pm \sqrt{\beta(\rho - 1)}, \rho - 1)\).
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Stability: The origin is a saddle point with eigenvalues

\[
0 < -\lambda_3 < \lambda_1 < -\lambda_2.
\]

The two symmetric fixed points $C^\pm$ are unstable spirals.
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Constant divergence:

$$\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = -(\sigma + \beta + 1).$$

The volume of a solid at time $t$ can be expressed as

$$V(t) = V(0)e^{-(\sigma+\beta+1)t} \approx V(0)e^{-13.7t},$$

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Maximal invariant set:

\[
\mathcal{A} = \bigcap_{t \geq 0} \varphi(\mathcal{U}, t).
\]

$\mathcal{A}$ must have zero volume, and $W^u(0) \subseteq \mathcal{A}$. 
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The geometric model:
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Return map: $R: \Sigma \setminus \Gamma \rightarrow \Sigma$.
The return plane $\Sigma$ is foliated by stable leaves. Projecting along these leaves gives a 1-d function:

$$f: [-1, 1] \rightarrow [-1, 1]$$
Properties: The function $f: [-1, 1] \rightarrow [-1, 1]$ satisfies:

1. $f(-x) = -f(x)$;
2. $\lim_{x \to 0} f'(x) = +\infty$;
3. $f''(x) < 0$ on $(0, 1]$;
4. $f'(x) > \sqrt{2}$;
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The real attractor seen from above $\Sigma$. 
More history:

1989  C. Robinson; M. Rychlik

Constructed *explicit* families of ODEs with geometric Lorenz attractors.

[*] Extra terms of degree 3 were needed,
[*] Arbitrarily small unfoldings,
[*] Lorenz equation *not* in the families.
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Computer-aided proof ⇒ homoclinic orbit.

1995 K. Mischaikow & M. Mrozek
Computer-aided proof ⇒ horseshoe.
[*] Non-classical parameter values,
[*] Objects have measure zero,
[*] Objects are not attracting.
What is a strange attractor?

We need to prove:

(1) There exists a compact $N \subset \Sigma$, such that

$$R(N \setminus \Gamma) \subset N.$$
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$$DR(x) \cdot \mathfrak{C}(x) \subset \mathfrak{C}(R(x)).$$
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(3) There exists $C > 0$ and $\lambda > 1$ such that for all $v \in \mathcal{C}(x)$, $x \in N$, we have

$$|DR^n(x)v| \geq C\lambda^n|v|, \quad n \geq 0.$$
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Open conditions - Perfect for interval methods!
How do we use these results?

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**Theorem:** For the classical parameter values, the Lorenz equations support a robust strange attractor $\mathcal{A}$ – the Lorenz attractor!
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**Theorem:** For the classical parameter values, the Lorenz equations support a robust strange attractor $\mathcal{A}$ – the Lorenz attractor!

By robust, we mean that a strange attractor exists in an open neighbourhood of the classical parameter values.
Strategy:

- Difficult to obtain global info about the flow. This is needed to define the Poincaré map and its derivative.
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- The linearizing process is very sensitive to changes in parameters.
- Don’t linearize, but make the flow closer to linear (normal form).
The flowing process
Let $N = \bigcup_{i=1}^{k} N_i$, and flow each initial rectangle $N_i$ between several codimension-1 surfaces.
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The return of $N_i$ is given by composing several distance-$d$ maps:

$$R(N_i) \subset \Pi^{(k(i))} \circ \cdots \circ \Pi^{(0)}(N_i).$$
The flowing process...
Use the fact that $\Pi^{(k)}$ – the “distance-$d$ map” – often is monotone. This allows us to shrink the flow regions.
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Flowing one step (seen from above):

$P^{(k)}(N_i)$ $\xrightarrow{\Pi^{(k)}}$ $P^{(k+1)}(N_i)$
The partitioning process

Idea: Dynamically split large images into smaller rectangles, and flow them separately.
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After $k$ steps the image of $N_i \subset \Sigma$ is enclosed by the union of many smaller rectangles:

$$P^{(k)}(N_i) \subseteq \bigcup_{j=1}^{n(i,k)} Q_{i,j}^{(k)}.$$
Finding the invariant set
At the return to $\Sigma$ we have information of the type

$$R(N_i) \subseteq \bigcup_{j=1}^{n(i)} Q_{i,j} \subseteq \bigcup_{j=1}^{m(i)} N_j.$$
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$$R(N) \subseteq N$$
Finding the invariant set
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$$R(N_i) \subseteq \bigcup_{j=1}^{n(i)} Q_{i,j} \subseteq \bigcup_{j=1}^{m(i)} N_j.$$  

Verify the cone condition:

$$Q_{i,j} \cap N_k \neq \emptyset \Rightarrow \mathcal{E}(Q_{i,j}) \subset \mathcal{E}(N_k).$$
Local theory and normal forms

Notation:

\[ x = (x_1, x_2, x_3, \ldots, x_n), \]

\[ |x| = \max\{|x_i| : i = 1, 2, 3, \ldots, n\}, \]

\[ \|f\|_r = \max\{|f(x)| : |x| \leq r\}. \]

\[ \{u \geq |x| : |(x)f|\}_{\text{max}} = u\|f\| \]

\[ \{3, 2, 1 = ? : |x|\}_{\text{max}} = |x| \]

\[ \epsilon \in I_{\mu} \Rightarrow x \equiv u x \equiv u x \]

\[ (\epsilon x, \epsilon x, \epsilon x) = x \]

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Flatness of order \( p \):

\[ x^n \in \mathcal{O}^p(x_1) \cap \mathcal{O}^p(x_2, x_3) \]

if \( n \in \mathbb{U}_p \overset{\text{def}}{=} \{ n \in \mathbb{N}^3 : n_1 \geq p \text{ and } n_2 + n_3 \geq p \} \).
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Change of variables:

\[
\begin{align*}
\dot{x} &= Ax + F(x) & x = y + \phi(y) \\
\dot{y} &= Ay + G(y)
\end{align*}
\]

original Lorenz \quad \quad normal form

where \( G(y) \in \mathcal{O}^{10}(y_1) \cap \mathcal{O}^{10}(y_2, y_3). \) \( G \) is almost linear.
Local theory and normal forms...

We find $\phi(y) = \sum a_n y^n$ by a simple power series substitution:

$$L_A \phi(y) = \{F(y + \phi(y))\}_{V_{10}},$$

where $V_{10} = \mathbb{N}^3 \setminus \mathbb{U}_{10}$, and

$$L_{A,i}(a_{i,n} y^n) = (n \lambda - \lambda_i) a_{i,n} y^n.$$
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Can we formally solve for the coefficients?
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Can we formally solve for the coefficients?

**Existence of a formal $\phi$:**

**Lemma:** Let $n \in \mathbb{V}_{10}$. Then, for $|n| \in [2, 57]$, we have $|n\lambda - \lambda_i| \geq 0.0112$. For $|n| \geq 58$, we have $|n\lambda - \lambda_i| \geq \frac{8}{3}|n|$. The proof requires the computation of the 19.386 first divisors (using interval arithmetic).
Local theory and normal forms...

We find \( \phi(y) = \sum a_n y^n \) by a simple power series substitution:

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L_A \phi(y) = \{F(y + \phi(y))\}_{V_0},
\]

where \( V_0 = \mathbb{N}^3 \setminus U_0 \), and

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L_{A,i}(a_i, ny^n) = \underbrace{\text{divisor}}_{n \lambda - \lambda_i} a_{i,n} y^n.
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The proof requires the computation of the 19.386 first divisors (using interval arithmetic).

**OK, what about convergence of \( \phi \)?**
Convergence of $\phi$:

Majorants: Find a $\hat{F} : \mathbb{R} \to \mathbb{R}$ such that $|F_i(r, r, r)| \leq \hat{F}(r)$, and let

$$\Omega(k) = \min_{|n|=k} \min_{i} \{|n\lambda - \lambda_i| : n \in \mathbb{V}_{10}\}.$$

Then $\phi$ converges whenever $\Psi(r) = \sum c_k r^k$ does, where

$$c_k = \frac{1}{\Omega(k)} \left[ \hat{F}(r + \sum_{j=2}^{k-1} c_j r^j) \right]_k.$$
**Convergence of** φ 

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\]

**Proposition:** The change of variables satisfies

\[
\|\phi\|_r \leq \frac{r^2}{2}, \quad r \leq 1,
\]

and the normal form satisfies

\[
\|G\|_r \leq 7 \cdot 10^{-9} \frac{r^{20}}{1 - 3r}, \quad r < \frac{1}{3}.
\]

For the proof we need the 186.576 first coefficients of \( \phi \).
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- I am very grateful to Jacob Palis and Lennart Carleson for suggesting this problem to me.
References


