On Continuation Methods for Non-Linear Multi-Objective Optimization

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2 State of the Art
   - Scalarizing Methods
   - Parametric Optimization
   - Continuation Methods

3 Bi-Objective Constrained Certified Continuation Method
   - Parallelotope-based Certified Continuation
   - Handling Inequality Constraints
   - Experiments

4 Conclusion
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Non-Linear Multi-Objective Optimization

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Non-Linear Multi-Objective Optimization

\[ f_1, \ldots, f_k : x \in X \subseteq \mathbb{R}^n \rightarrow f(X) \]

- \( X^* \) set of non-dominated solutions: Pareto solutions (plain lines)
- \( f(X^*) \) set of non-dominated outcomes: Pareto set (plain lines)
Non-Linear Multi-Objective Optimization

General Non-Linear Multi-Objective Optimization (NLMOO) problem:

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g(x) \leq 0 \\
& \quad h(x) = 0 \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]  

(1)

Let \( X = \{ x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0 \} \).

- Objective functions: \( f : \mathbb{R}^n \rightarrow \mathbb{R}^k \),
- Inequality constraints: \( g : \mathbb{R}^n \rightarrow \mathbb{R}^p \),
- Equality constraints: \( h : \mathbb{R}^n \rightarrow \mathbb{R}^q \).

Functions may be non-linear.
What is continuation?

Unformal definition

Local approximation/coverage of a manifold of solutions.
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Unformal definition

Local approximation/coverage of a manifold of solutions.

- Local mean the use of local informations/observations,
- Solutions: of a system of equations, an optimization problem, ...; inducing (implicit) parameters,
- In NLMOO, when regular:
  - Two objectives $\rightarrow$ Manifold of dimension 1 (curves of solutions),
  - Three objectives $\rightarrow$ Manifold of dimension 2 (surfaces of solutions),
  - ...
Continuation in Non-Linear Multi-Objective Optimization

$X^*$ manifold of non-dominated solutions (plain lines)

$f(X^*)$ manifold of non-dominated outcomes (plain lines)
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Scalarizing

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\begin{align*}
\min & \quad \hat{f}(x, v) \\
\text{s.t} & \quad \hat{g}(x, v) \leq 0 \\
& \quad \hat{h}(x, v) = 0 \\
& \quad g(x) \leq 0 \\
& \quad h(x) = 0
\end{align*}
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Sequence of Mono-objective problems, \( v \in \{v_1, v_2, \ldots \} \)
Scalarizing Methods: Examples

- Weighted Sum:
  Minimize $\lambda f_1(x) + (1 - \lambda) f_2(x)$

- $\epsilon$-Constraint:
  Minimize $f_2(x)$, s.t. $f_1(x) \leq \epsilon$

- Normal Boundary Intersection:
  Maximize $t$, s.t. $f(x) = \mu_\lambda + dt$
  $\mu_\lambda = \lambda \hat{y}^2 + (1 - \lambda) \hat{y}^1$. Vector $d$ normal to the utopia plane

- Normal Constraint:
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  \mu_\lambda = \lambda \hat{y}_2 + (1 - \lambda) \hat{y}_1 \text{ and } d = \hat{y}_1 - \hat{y}_2
  \]
Scalarizing Methods: Examples

- **Weighted Sum:**
  Minimize $\lambda f_1(x) + (1 - \lambda)f_2(x)$

- **$\epsilon$-Constraint:**
  Minimize $f_2(x)$, s.t. $f_1(x) \leq \epsilon$

- **Normal Boundary Intersection:**
  Maximize $t$, s.t. $f(x) = \mu_\lambda + dt$, $\mu_\lambda = \lambda \hat{y}^2 + (1 - \lambda)\hat{y}^1$. Vector $d$ normal to the utopia plane

- **Normal Constraint** [Messac et al., 2003]:
  Minimize $f_2(x)$, s.t. $d^T f(x) - d^T \mu_\lambda \geq 0$, $\mu_\lambda = \lambda \hat{y}^2 + (1 - \lambda)\hat{y}^1$ and $d = \hat{y}^1 - \hat{y}^2$
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Scalarizing Methods: Examples

\[
\begin{bmatrix}
\min & f(x) \\
\text{s.t} & g(x) \leq 0 \\
& h(x) = 0 \\
\end{bmatrix}
\]

Scalarizing

\[
\begin{bmatrix}
\min & \hat{f}(x, v) \\
\text{s.t} & \hat{g}(x, v) \leq 0 \\
& \hat{h}(x, v) = 0 \\
\end{bmatrix}
\]

- **Weighted Sum:**
  Minimize \( \lambda f_1(x) + (1 - \lambda) f_2(x) \)

- **\( \epsilon \)-Constraint:**
  Minimize \( f_2(x) \), s.t. \( f_1(x) \leq \epsilon \)

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Parametric Optimization problem:

\[
\begin{bmatrix}
\text{min} & f(x, v) \\
\text{s.c} & h(x, v) = 0 \\
& g(x, v) \leq 0 \\
& x \in \mathbb{R}^n
\end{bmatrix}
\]

\(f : \mathbb{R}^{n+m} \to \mathbb{R}\) and \(v \in V \subseteq \mathbb{R}^m\) vector of parameters.

Continuation is natural in such applications:

- Parameters are explicit.
- Use of local informations interesting.
- Optimal solutions are usually computed as solutions to first order optimality conditions.
First order conditions:

\[
\nabla_x f(x, v) \lambda + \nabla_x g(x, v) r + \nabla_x h(x, v) s = 0 \\
(\forall i = 1, \ldots, p) \ g_i(x, v) r_i = 0 \\
(\forall i = 1, \ldots, q) \ h_i(x, v) = 0 \\
\lambda^T \lambda + r^T r + s^T s - 1 = 0
\]

With \( x \in X \subseteq \mathbb{R}^n, v \in \mathbb{R}^m, \lambda \in \mathbb{R}_+, r \in \mathbb{R}_+^p \) and \( s \in \mathbb{R}^q \).

System of \( n + m + 1 + p + q \) variables with \( n + p + q + 1 \) equations: \( m \)-dimensional manifold of solutions.
State of the art

Literature on Parametric Optimization:
- Singularity detections [Lundberg and Poore, 1993].
- Multi-Parametric [Domínguez et al., 2010].

Towards Multi-Objective Optimization:
- Tackling Multi(Bi)-Objective optimization [Rakowska et al., 1993].
State of the art

First order optimality conditions:
Parametric problem based on Weighted Sum

\[ \nabla_x f_1(x) \lambda_1 + \nabla_x f_2(x) \lambda_2 + \nabla_x g(x) r + \nabla_x h(x) s = 0 \]
\[ (\forall i = 1, \ldots, p) \ g_i(x) r_i = 0 \]
\[ (\forall i = 1, \ldots, q) \ h_i(x) = 0 \]
\[ \lambda^T \lambda + r^T r + s^T s - 1 = 0 \]

NLMOO first order conditions

\[ \nabla f_1(x) \lambda_1 + \nabla f_2(x) \lambda_2 + \nabla g(x) r + \nabla h(x) s = 0 \]
\[ (\forall i = 1, \ldots, p) \ g_i(x) r_i = 0 \]
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Continuation methods and applications

Continuation methods used to solve underconstrained systems of equations.

General problem

\[ F(x) = 0, \quad F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \]

Solutions form a \( m \)-dimensional manifold.

Appears in:

- Study of parameters in differential equations;
- Homotopy for solving polynomial systems;
- Non-Linear Optimization (interior-point methods, parametric optimization);
- . . .
Continuation methods example: Predictor/Corrector
Continuation methods example: Predictor/Corrector
Continuation methods example: Predictor/Corrector
Continuation methods example: Predictor/Corrector
Continuation methods example: Predictor/Corrector
Continuation methods example: Predictor/Corrector
Continuation methods example: Predictor/Corrector
Continuation methods example: Predictor/Corrector
NLMOO hybridized with Continuation Methods

Continuation methods:

- For first order conditions of NLMOO [Hillermeier, 2001].

Applications in Metaheuristics:

- Curve-based Genetic Algorithm [Harada et al., 2007]
- PSO and continuation [Schütze et al., 2008]
- Steepest Descent (HCS) as continuation [Schütze et al., 2009]

Applications in Global methods:

- Recovering algorithm [Schütze et al., 2005]
- Bi-objective method inspired by NBI [Pereyra, 2009, Pereyra et al., 2013]
Summary

⊕ Continuation methods + NLMOO promising,
⊕ Help for both metaheuristics and global algorithms,
Summary

- Continuation methods + NLMOO promising,
- Help for both metaheuristics and global algorithms,
- Few actually consider inequality constraints,
- Few gives certification of the continuity.
  - False representation of the manifold.
  - Loss of solutions.

Certification can be (numerically) achieved:

- Smale \( \alpha \)-theory or Kantorovich theorem $$\Rightarrow$$ maximal step
- Interval Analysis and parametric Interval Newton operators

Goal

Towards a certified and rigorous continuation method for (inequality) constrained NLMOO.

Here, restricted to the Bi-Objective case.
Summary
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Certification can be (numerically) achieved:

• Smale $\alpha$-theory or Kantorovich theorem $\rightarrow$ maximal step
  [Beltrán and Leykin, 2012, Faudot and Michelucci, 2007],
• Interval Analysis and parametric Interval Newton operators
  [Kearfott and Xing, 1994, Martin et al., 2012].

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4. Conclusion
ParCont: Certified Continuation with Parallelotopes

- Parallelotopes and parametric Interval Newton
  [Goldsztejn and Granvilliers, 2010]
  - Used in Constraint Programming,
  - Based on interval analysis,
  - Spouse the shape of the manifold.

- ParCont: Parallelotope-based Continuation
  [Martin et al., 2012]

- Singularities
ParCont: Certified Continuation with Parallelotopes

- Parallelotopes and parametric Interval Newton
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- ParCont: Parallelotope-based Continuation
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  - Equivalent PC method,
  - Builds locally new parallelotopes along the manifold,
  - Connects two consecutive parallelotopes.
- Singularities
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- Singularities
  - Can not certify singularities.
Bi-Objective Constrained Certified Continuation Method

Parallelotope-based Certified Continuation

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**Singularities**
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Optimality conditions

Let the system of first order optimality conditions:

\[
\nabla f_1(x) \lambda_1 + \nabla f_2(x) \lambda_2 + \nabla g(x) r + \nabla h(x) s = 0
\]

\[
(\forall i = 1, \ldots, p) \ g_i(x) r_i = 0
\]

\[
(\forall i = 1, \ldots, q) \ h_i(x) = 0
\]

\[
\lambda^T \lambda + r^T r + s^T s - 1 = 0
\]
Optimality conditions

Let the system of first order optimality conditions:

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\nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla g(x)r + \nabla h(x)s = 0
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(\forall i = 1, \ldots, p)\ g_i(x)r_i = 0
\]

\[
(\forall i = 1, \ldots, q)\ h_i(x) = 0
\]

\[
\lambda^T\lambda + r^Tr + s^Ts - 1 = 0
\]

Singularity when there exists \( i \) with \( r_i = 0 \) and \( g_i(x) = 0 \): change in the set of active constraints.

Problem

ParCont can not handle inequality constraints.

Towards a certified active set management strategy
Dealing with inequalities [Rakowska et al., 1993]

Definition

Let $\bar{A} \subseteq \{1, 2, \ldots, p\}$ be the set of active constraints at a feasible solution $x$. Let $\bar{g}$ and $\bar{r}$ be the induced inequality vector and weights.

To deal with singularities from change in the active constraint set:

- Solve the system:

$$
\nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla \bar{g}(x)\bar{r} + \nabla h(x)s = 0
$$

$$(\forall i \in \bar{A}) \quad g_i(x) = 0
$$

$$(\forall i = 1, \ldots, q) \quad h(x) = 0
$$

$$
\lambda^T\lambda + r^T r + s^T s - 1 = 0
$$

- Change the set $\bar{A}$ when activating/disactivating a constraint.
Detecting change in the active set

Example:
- Detection of a possible activation ($g_i(x) = 0$),
- Certify the activation: Interval Newton,
- Change $\bar{A}$, isolate the activation, orient the continuation ($r_i > 0$),
- Restart the Continuation.
Detecting change in the active set

Example:

- Detection of a possible activation \((g_i(x) = 0)\),
- **Certify the activation:** Interval Newton,
- Change \(\bar{A}\), isolate the activation, orient the continuation \((r_i > 0)\),
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Detecting change in the active set

Example:
- Detection of a possible activation ($g_i(x) = 0$),
- Certify the activation: Interval Newton,
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- Restart the Continuation.

\[ \bar{A} = \{i\} \]
Detecting change in the active set

Example:
- Detection of a possible activation \((g_i(x) = 0)\),
- Certify the activation: Interval Newton,
- Change \(\bar{A}\), isolate the activation, orient the continuation \((r_i > 0)\),
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Example:

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ParCont is implemented in C++, with:

- RealPaver API [Granvilliers and Benhamou, 2006],
- Gaol interval arithmetic library [Goualard, 2006],
- Lapack linear algebra library [Anderson et al., 1999],
- Crlibm verified rounding library.

Towards using Certified Continuation as a post-process of a metaheuristic (NSGAII [Deb et al., 2002]):

- As suggested in [Harada et al., 2007].
- Certify local optimality (and feasibility),
- Comparison of the efforts of the two methods.
CTP1 [Deb et al., 2001]: Standard bi-objective problem with 2 variables.

\[
\begin{align*}
\min \quad & f_1(x) = x_1 \\
\min \quad & f_2(x) = (1 + x_2) \exp(-x_1/(1 + x_2)) \\
\text{s.t} \quad & g_1(x) = 1 - f_2(x)/(0.858 \exp(-0.541f_1(x))) \leq 0 \\
& g_2(x) = 1 - f_2(x)/(0.728 \exp(-0.295f_1(x))) \leq 0 \\
& x_1, x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{align*}
\]

Start ParCont at \( f_2^* \).
Experiments

NSGAII:
- Computation time: 0.25 s,
- Population size: 200,
- Generation: 100.

ParCont:
- Computation time: 0.0925 s,
- 57 parallelotopes,
- 7 Active set changes.
Tanaka [Tanaka et al., 1995]: Bi-objective problem with 2 variables.

\[
\begin{align*}
\min & \quad f_1(x) = x_1 \\
\min & \quad f_2(x) = x_2 \\
\text{s.t} & \quad g_1(x) = -x_1^2 - x_2^2 + 1 + 0.1 \cos(16 \arctan(x_1/x_2)) \leq 0 \\
& \quad g_2(x) = 2x_1^2 + 2x_2^2 - 1 \leq 0 \\
& \quad x_1, x_2 \leq \pi \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Disconnected Pareto-front. ParCont started at:

\[
\begin{align*}
x^A &= \begin{pmatrix} 0.042 \\ 1.038 \end{pmatrix}, \\
x^B &= \begin{pmatrix} 0.586 \\ 0.774 \end{pmatrix}, \\
x^C &= \begin{pmatrix} 1.039 \\ 0.043 \end{pmatrix}
\end{align*}
\]
Experiments

NSGAII:
- Computation time: 0.6 s,
- Population size: 200,
- Generation: 300.

ParCont:
- Computation time: 0.46 s,
- 534 parallelotopes.

\[
\begin{align*}
\min f_1(x) &= x_1 + \frac{2}{n} \sum_{i=2}^{n} (x_i - 0.8x_1 \cos(6\pi x_1 + \frac{i\pi}{n}))^2 \\
\min f_2(x) &= 1 - x_1^2 + \frac{2}{n} \sum_{i=2}^{n} (x_i - 0.8x_1 \sin(6\pi x_1 + \frac{i\pi}{n}))^2 \\
\text{s.t} & \quad x_1 \leq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_i \leq 1, \; i = 2, \ldots, n \\
& \quad x_i \geq -1, \; i = 2, \ldots, n
\end{align*}
\]

ParCont start at $f_2^*$ with $n = 10$. 

Experiments

Projection on \((x_1, x_2, x_3)\)

NSGAII
ParCont
Experiments

- **NSGAII:**
  - Computation time: 1.3 s,
  - Population size: 200,
  - Generation: 400.

- **ParCont:**
  - Computation time: 11.2 s,
  - 2315 paralleloptopes.
Summary

Informations:
- Each step of ParCont has $O(n^3)$ time complexity,
- Compared to non-certified methods, it has to use a smaller step length.

Pros and Cons:
- ParCont ables to produce certified enclosures of Pareto-optimal solutions,
- Local optimality is proven for each enclosure,
- Use only local informations.
- Required twice continuously differentiable objectives and constraints,
- Some singularities not handled,
- Limited to 1-dimensional manifolds (bi-objective problems).
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Summary

We have seen that:

- Many state of art approaches attempt to parameterize Pareto-Optimal solutions,
- Continuation methods and NLMOO promising in different applications,
- A certified continuation method ParCont for bi-objective problems, dealing with change in active set of constraints, is proposed.

Next?

- Integration of ParCont in a global method: only one point per connected components is required,
Conclusion

Summary

We have seen that:

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- Adaptation of ParCont to 3-Objectives.
On Continuation Methods for Non-Linear Multi-Objective Optimization

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SWIM 2013
Small Workshop on Interval Methods
Brest, 5 - 7 June 2013


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