Outer approximation of attractors using an interval quantization

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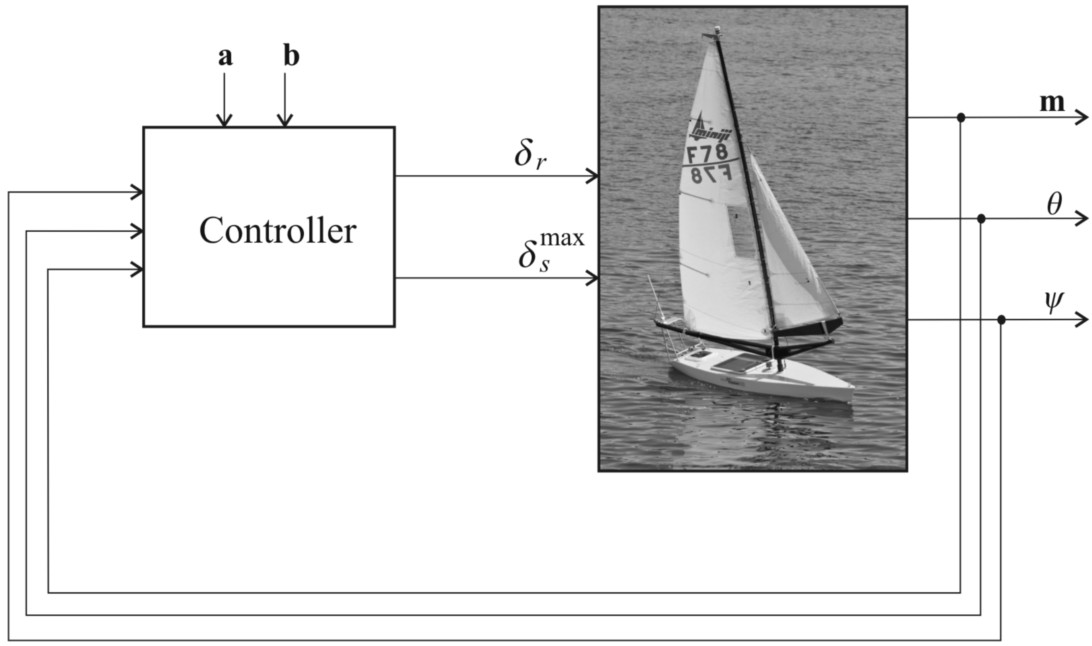
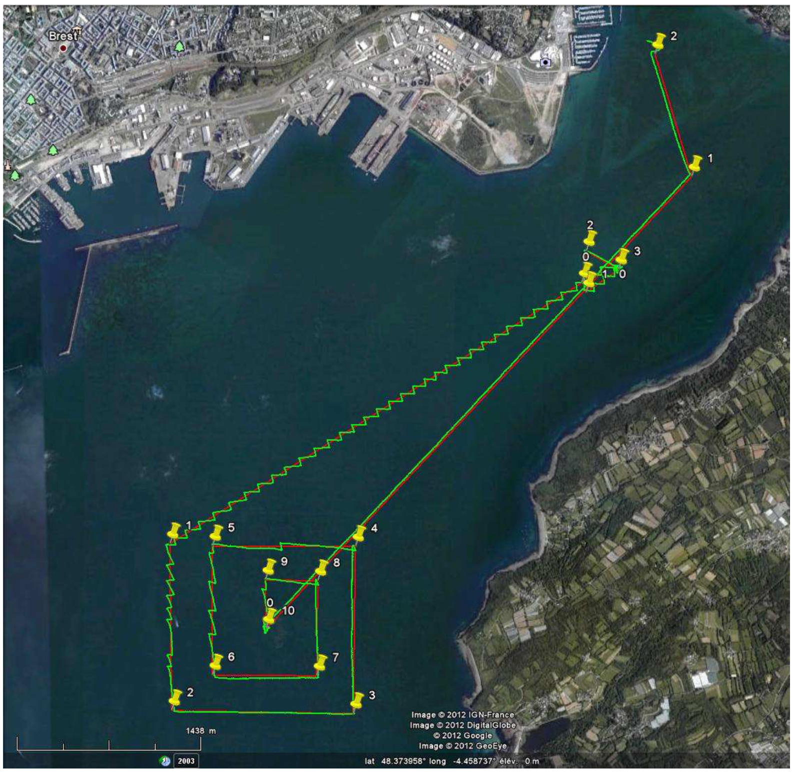
1 Motivation

Vaimos (IFREMER and ENSTA)

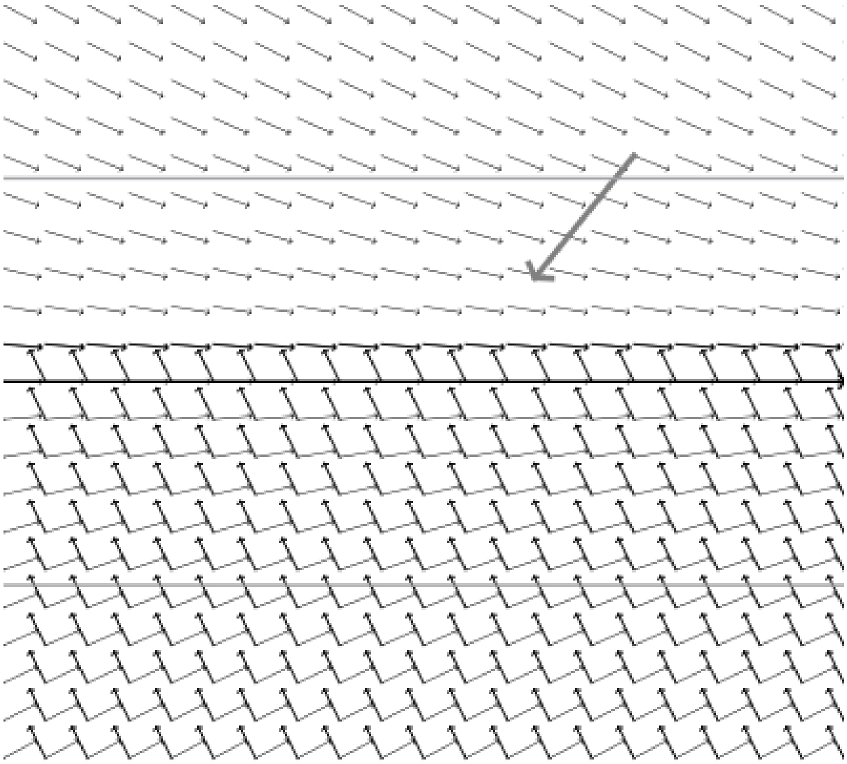

a
b
Controller
δ_r
$\delta_s^{\max}$
F78
m
θ
ψ

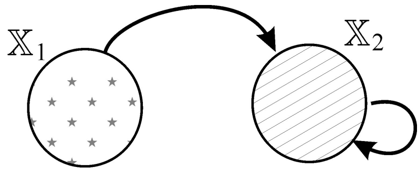

Brest
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
1438 m
2003
lat 48.373958° long -4.458737° élév. 0 m

2 V-stability

The autonomous sailboat robot satisfies

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \psi) \quad \text{with } \mathbf{x} = (x, y) \in \mathbb{R}^2.$$





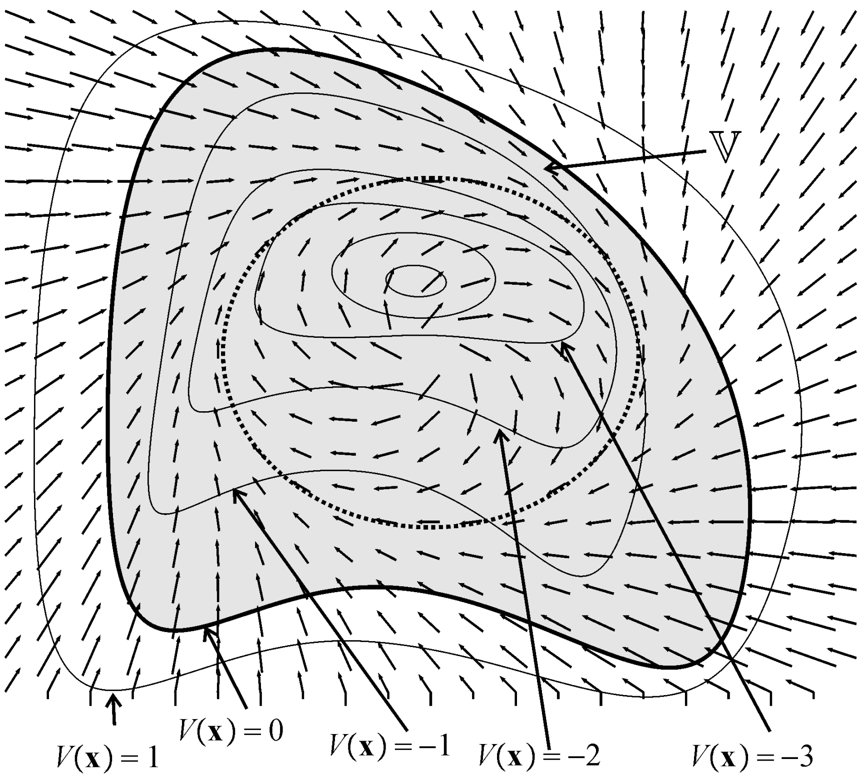
$\mathbb{X}_1$: outside the corridor.

$\mathbb{X}_2$: inside the corridor.

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system is V-stable if

$$\Big(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\Big).$$

Checking the V-stability can be done using using interval analysis.



$\mathbb{V}$
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Limitation. A function V such should be available.

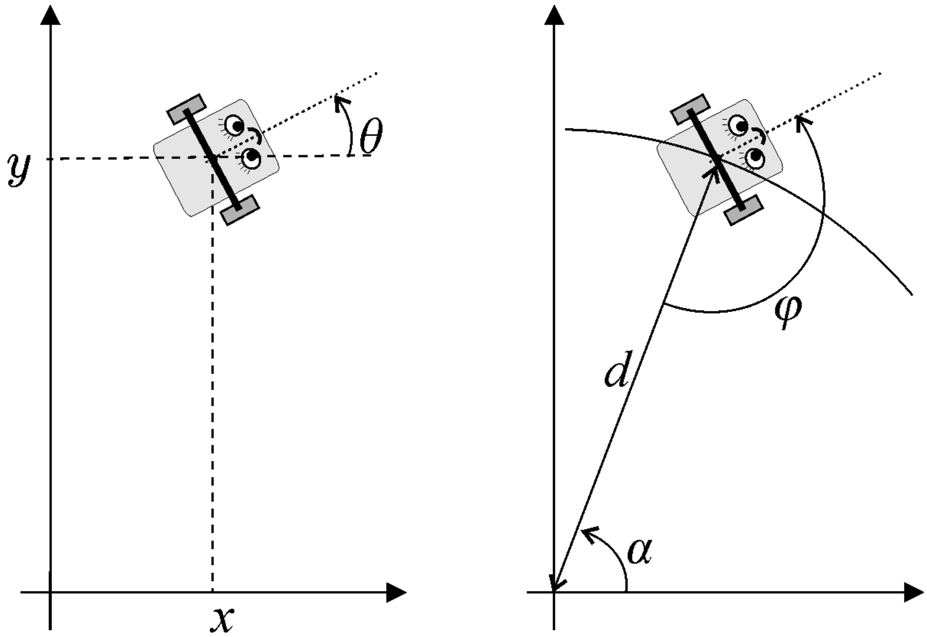
3 Station keeping problem

The problem of *station keeping* for a robot is to stay inside a disk around origin.

Consider a non holonomous robot described by

$$\begin{cases} \dot{x} & = & \cos\theta \\ \dot{y} & = & \sin\theta \\ \dot{\theta} & = & u. \end{cases}$$

Since $\dot{x}^2 + \dot{y}^2 = 1$, this robot cannot stop.



Transformation from Cartesian to polar

The polar form for the state equations is

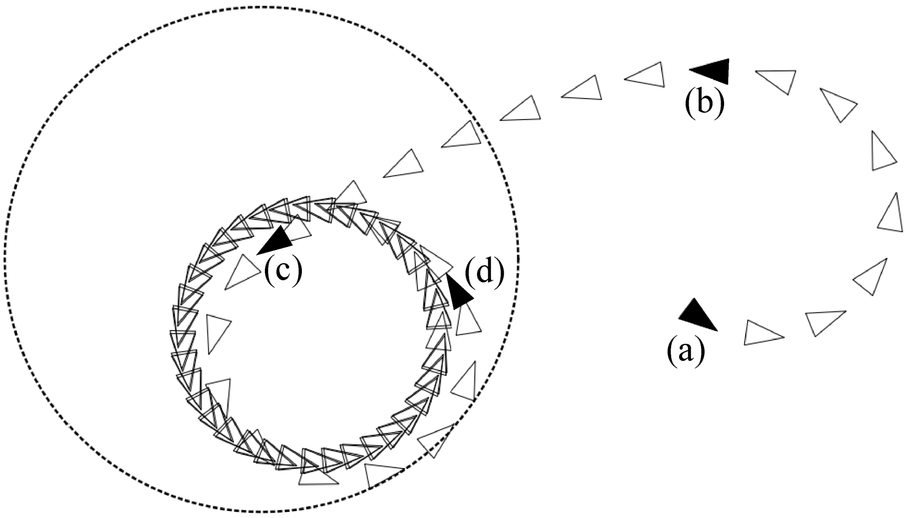
$$\begin{cases} \text{(i)} & \dot{\varphi} = \dfrac{\sin\varphi}{d} + u \\ \text{(ii)} & \dot{d} = -\cos\varphi. \\ \text{(iii)} & \dot{\alpha} = -\dfrac{\sin\varphi}{d}. \end{cases}$$

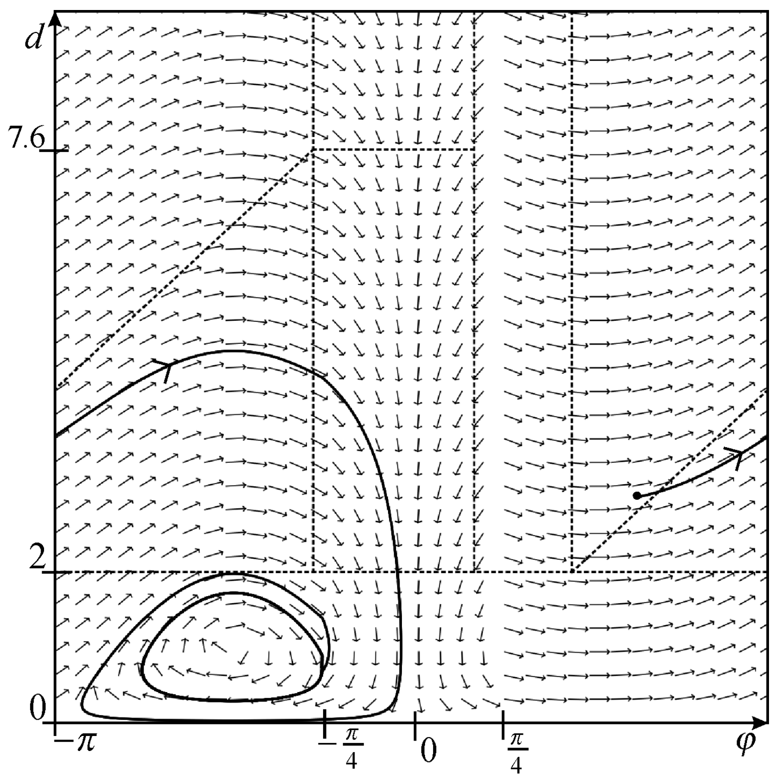
We propose here the following control

$$u = \begin{cases} +1 & \text{if } \cos\varphi \le \frac{1}{\sqrt{2}} \quad \text{(the robot turns left)} \\ -\sin\varphi & \text{otherwise} \quad \text{(the robot goes toward zero)} \end{cases}$$

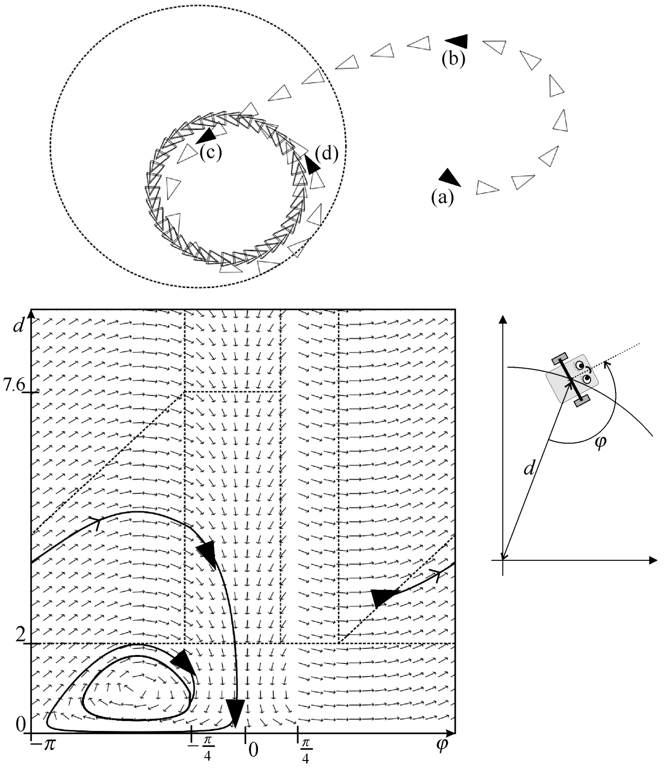
The closed loop state equations are

$$\begin{cases} \text{(i)} & \dot{\varphi} &= \begin{cases} \dfrac{\sin\varphi}{d} + 1 & \text{if } \cos\varphi \le \dfrac{1}{\sqrt{2}} \\ \left(\dfrac{1}{d} - 1\right)\sin\varphi & \text{otherwise} \end{cases} \\ \text{(ii)} & \dot{d} &= -\cos\varphi. \end{cases}$$


(a)
(b)
(c)
(d)



d
7.6
2
0
−π
−π/4
0
π/4
φ



(a)
(b)
(c)
(d)
d
7.6
2
0
$-\pi$
$-\frac{\pi}{4}$
0
$\frac{\pi}{4}$
φ
d
φ

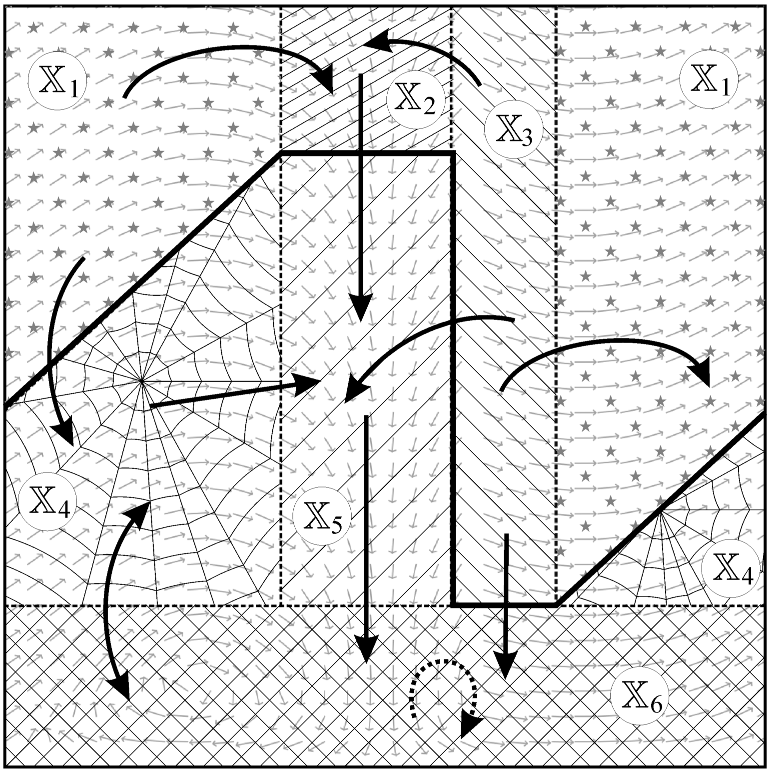
4 Quantization

Given a paving $\mathbb{X}_1, \ldots, \mathbb{X}_n$ of the state space $\mathbb{R}^n$.

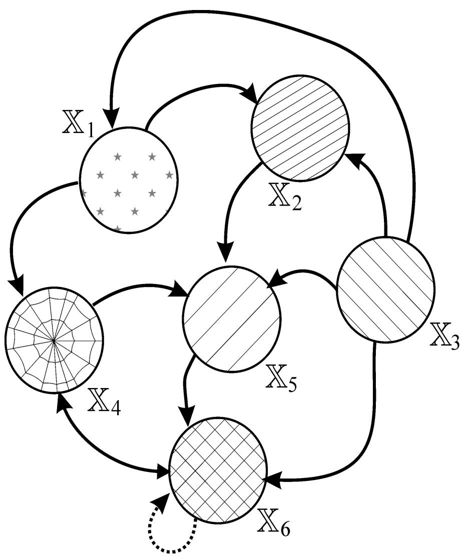
We define the relation $\hookrightarrow$ as follows

$\mathbb{X}_i \hookrightarrow \mathbb{X}_j$, $i \neq j$ iff there exists one trajectory crossing the frontier from $\mathbb{X}_i$ to $\mathbb{X}_j$.

$\mathbb{X}_i \hookrightarrow \mathbb{X}_i$ iff there exists one trajectory included in $\mathbb{X}_i$.



$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_1$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_4$
$\mathbb{X}_6$



Graph corresponding to the quantization

The relation $\hookrightarrow$ can be represented in a matrix form by

$$[\mathbf{G}] = \left[\underline{\mathbf{G}}, \overline{\mathbf{G}}\right] = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & [0,1] \end{pmatrix}$$

The transitive closure of this graph is given by

$$
\begin{aligned}
\left[\mathbf{G}^{+}\right] &= \left[\underline{\mathbf{G}}^{+}, \overline{\mathbf{G}}^{+}\right] \\
&= \left[\underline{\mathbf{G}} + \underline{\mathbf{G}}^{2} + \underline{\mathbf{G}}^{3} + \ldots,\ \overline{\mathbf{G}} + \overline{\mathbf{G}}^{2} + \overline{\mathbf{G}}^{3} + \ldots\right] \\
&= \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.
\end{aligned}
$$

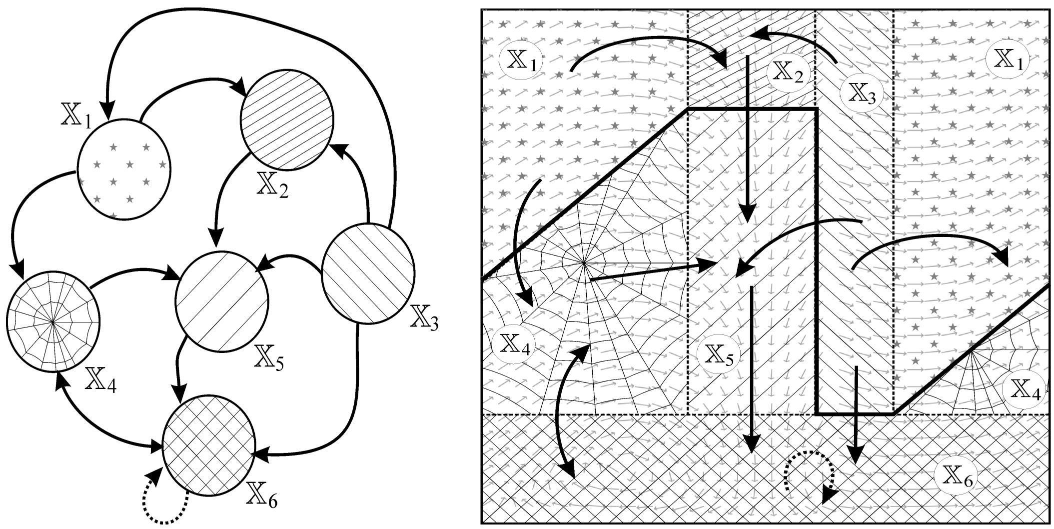
The attractor of the graph is given by the entries equal to 1 in the diagonal of the matrix.

It corresponds to $\mathbb{X}_4 \cup \mathbb{X}_5 \cup \mathbb{X}_6$.

The attractor satisfies

$$\mathbb{A} \subset \left(\mathbb{X}_4 \cup \mathbb{X}_5 \cup \mathbb{X}_6\right).$$

Thus, our robot will be trapped inside the disk with center 0 and radius $d = 7.6$.



$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_6$
$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_1$
$\mathbb{X}_4$
$\mathbb{X}_5$
$\mathbb{X}_4$
$\mathbb{X}_6$

Challenge. Find an accurate inner and outer approximation of $\mathbb{A}$.

Step 1. Find one box in $\mathbb{A}$.