Polyhedral Relaxations for Constraint Satisfaction Problems

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Problem formulation

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\}.$$

The midpoint and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

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Constraint programming problem

Enclose the set $\mathcal{S}$ described by

$$\begin{aligned} f_i(x_1,\dots,x_n) &= 0, \quad i = 1,\dots,m, && (\, f(x) = 0\,)\\ g_j(x_1,\dots,x_n) &\le 0, \quad j = 1,\dots,\ell, && (\, g(x) \le 0\,)\end{aligned}$$

on a box $\mathbf{x}$.

Linearization

Our approach

- linearize constraints,
- compute new bounds and iterate.

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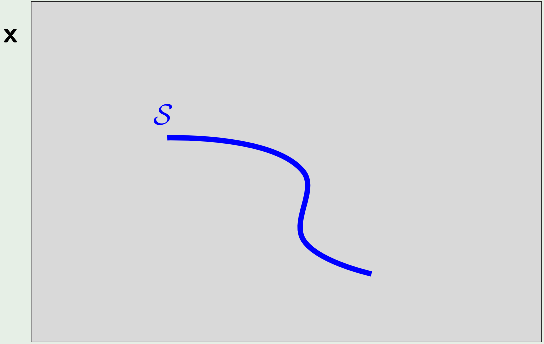

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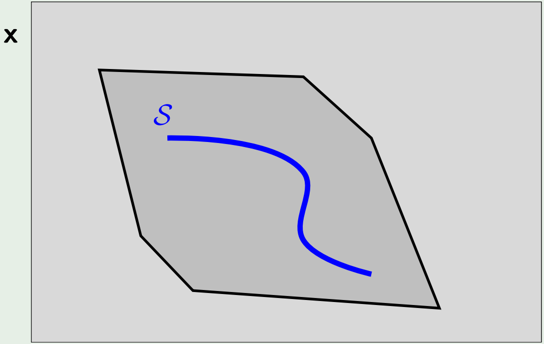

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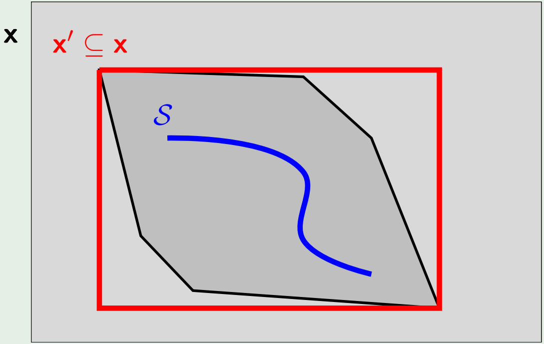

Linearization

Interval linearization

Let $x^0 \in \mathbf{x}$, called the center. Suppose that a function $h : \mathbb{R}^n \mapsto \mathbb{R}^s$ satisfies

$$h(x) \subseteq S_h(\mathbf{x}, x^0)(x - x^0) + h(x^0), \quad \forall x \in \mathbf{x}$$

for a suitable interval-valued function $S_h : \mathbb{IR}^n \times \mathbb{R}^n \mapsto \mathbb{IR}^{s \times n}$.

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Techniques

- mean value form
- slopes
- special structure analysis (McCorming-like linearizations ...)

Linearization

Interval linear programming formulation

Now, the set $\mathcal{S}$ is enclosed by a set described by

$$\begin{aligned} A(x-x^0)+f(x^0)&=0, \quad \text{for some } A\in\mathbf{A},\\ B(x-x^0)+g(x^0)&\leq 0, \quad \text{for some } B\in\mathbf{B}, \end{aligned}$$

for some interval matrices $\mathbf{A}$ and $\mathbf{B}$.

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What remains to do

- Solve the interval linear program
- choose $x^0 \in \mathbf{x}$

Vertex selection of x^0

Case $x^0 := \underline{x}$

Let $x^0 := \underline{x}$. Since $x - \underline{x}$ is non-negative, the solution set to

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is described by

$$\begin{aligned} &\underline{A}x \leq \underline{A}\,\underline{x} - f(\underline{x}), \quad \overline{A}x \geq \overline{A}\underline{x} - f(\underline{x}), \\ &\underline{B}x \leq \underline{B}\,\underline{x} - g(\underline{x}). \end{aligned}$$

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- Similarly if x^0 is any other vertex of $\mathbf{x}$
- Araya, Trombettoni & Neveu (2012) recommend two opposite corners

Non-vertex selection of x^0

General case

Let $x^0 \in \mathbf{x}$. The solution set to

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is described by

$$\begin{aligned} |A_c(x-x^0)+f(x^0)| &\leq A_\Delta|x-x^0|,\\ B_c(x-x^0) &\leq B_\Delta|x-x^0|-g(x^0). \end{aligned}$$

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- Non-linear description due to the absolute values.
- How to get rid of them?

Non-vertex selection of x^0

Solution

Linearize the absolute values.

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Theorem (Beaumont, 1998)

For every $y \in \mathbf{y} \subset \mathbb{R}$ with $\underline{y} < \overline{y}$ one has

$$|y| \leq \alpha y + \beta, \qquad (*)$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}} \quad \text{and} \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

Moreover, if $\underline{y} \geq 0$ or $\overline{y} \leq 0$ then $(*)$ holds as equation.

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Consequences

- For nice functions (linear, convex), non-vertex selection of x^0 makes no progress
- Non-vertex selection of x^0 is more useful more non-convex are f, g

Example

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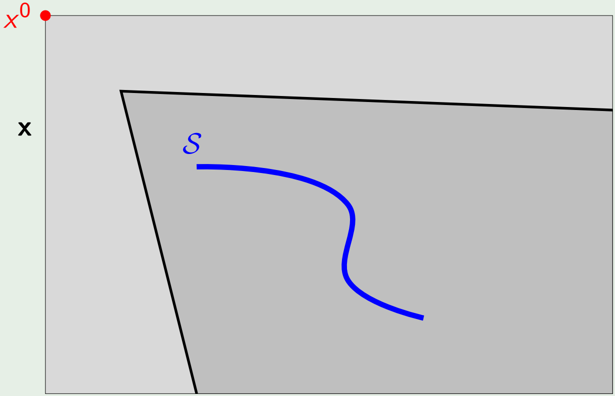
Typical situation when choosing x^0 to be vertex:



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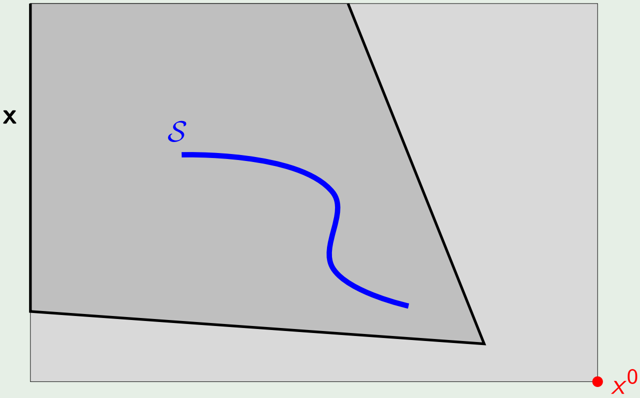
Typical situation when choosing x^0 to be the opposite vertex:



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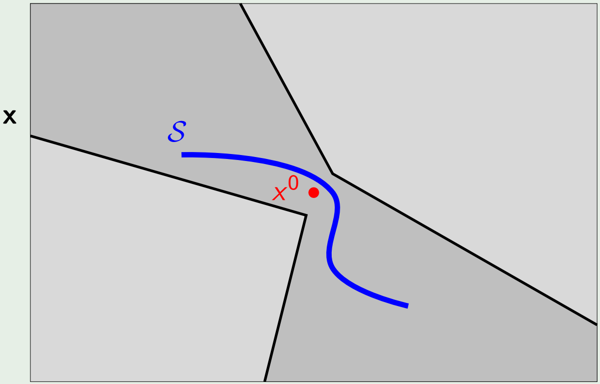

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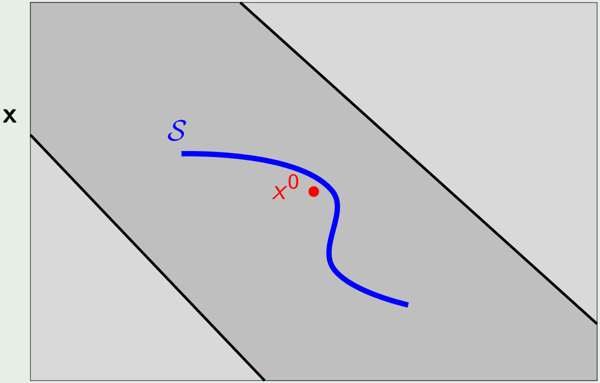
Typical situation when choosing $x^0 = x_c$ (after linearization):



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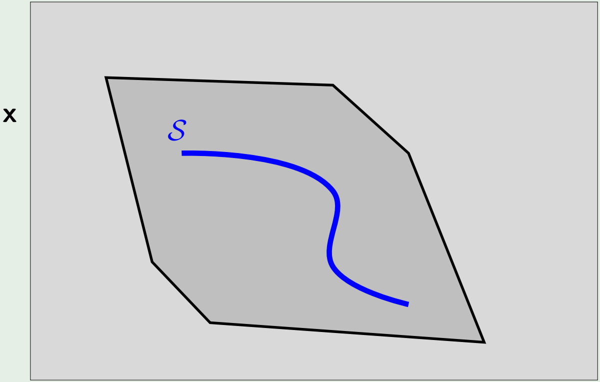
Typical situation when choosing all of them:

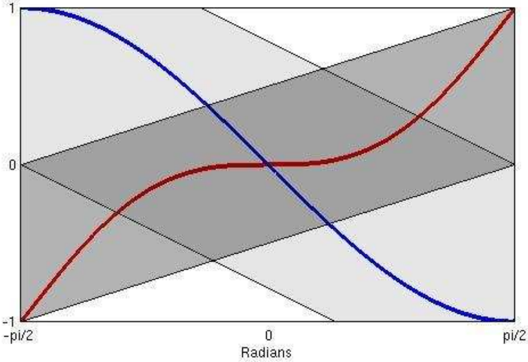


Example II.

Constraints:

$$\pi^2 y - 4x^2 \sin x = 0, \quad y - \cos\left(x + \tfrac{\pi}{2}\right) = 0, \qquad x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}],\ y \in [-1, 1].$$

Center: $x^0 = (0, 0)$



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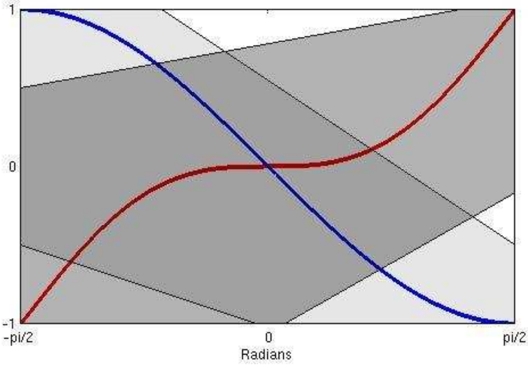
Center: $x^0 = (\frac{\pi}{6}, 0)$



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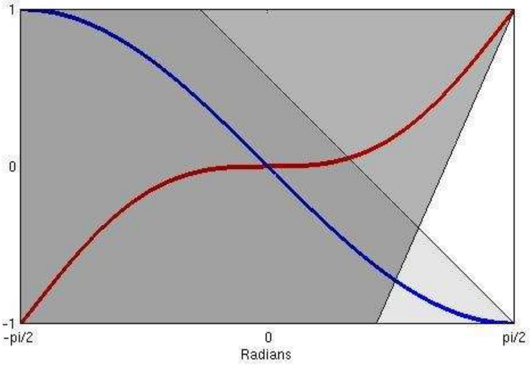

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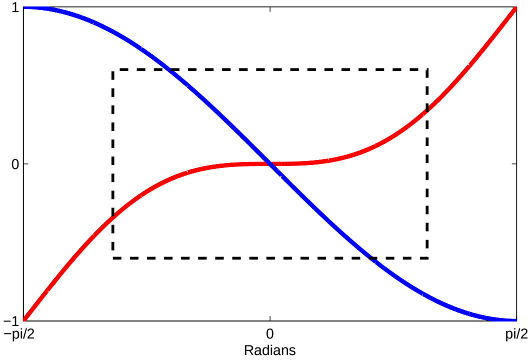
Contraction for centers $x^0 = (0,0), (\frac{\pi}{2}, 0), (-\frac{\pi}{2}, 0)$



Comparison to Parallel Linearization

Suppose that $h : \mathbb{R}^n \mapsto \mathbb{R}^s$ has the following interval linear enclosure on $\mathbf{x}$

$$h(x) \subseteq \mathbf{A}(x - x^0) + h(x^0), \quad \forall x \in \mathbf{x}$$

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Theorem (Jaulin, 2001)

For any $A \in \mathbf{A}$ we have

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Theorem

For any selection of $x^0 \in \mathbf{x}$ and $A \in \mathbf{A}$, the interval linear programming approach yields always as tight enclosures as the parallel linearization.

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References

M. Hladík and J. Horáček.
Interval linear programming techniques in constraint programming and global optimization.
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