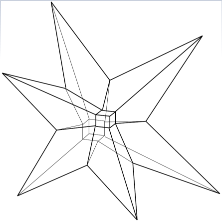


Improving Newton Existence Test
SWIM 2013

Clément Aubry

clement.aubry@ecole-navale.fr

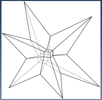
June 5-7, 2013

Naval Academy Research Institute, CC600, 29240 Brest

Armées France.

Outline

Outline



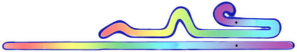
Brouwer Fixed-Point Theorem

Theorem (Euclidean space)

Every continuous function from a closed ball of an Euclidean space to itself has a fixed point.

Explanations – Anecdote

- 1D piece of string
- 2D sheet of paper
- 3D cup of cofee



Outline

Definition

Consider a smooth function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $[\mathbf{x}] \in \mathbb{R}^n$.
Denote by $\mathbf{J_f}$ is Jacobian Matrix.

Definition (Newton test, Moore [Moore, 1979])

The Newton operator is defined by

$$\mathcal{N}\left(\mathbf{f}, [\mathbf{J_f}], [\mathbf{x}]\right) = \widehat{\mathbf{x}} - [\mathbf{J_f}]^{-1}\left([\mathbf{x}]\right).\mathbf{f}\left(\widehat{\mathbf{x}}\right) \tag{1}$$

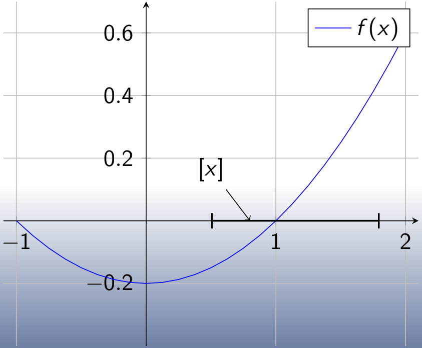
If $\mathcal{N}([\mathbf{x}]) \subset [\mathbf{x}]$ then $[\mathbf{x}]$ contains a unique zero $\mathbf{x}^*$ of $\mathbf{f}$. It is also in $\mathcal{N}([\mathbf{x}])$.

Moore, R. E. (1979).
Methods and Applications of Interval Analysis.
SIAM, Philadelphia, PA.

Explanations

Take a function $f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 0.2(x^2-1) \end{array} \right.$. We want to prove existence and uniqueness of $f(x) = 0, \forall x \in [0.5, 1.8]$.

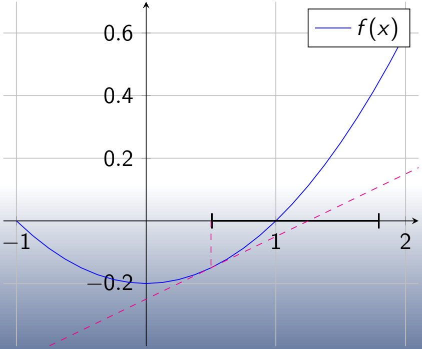
- Choose an initial domain for $[x]$



Explanations

Take a function $f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 0.2(x^2-1) \end{array} \right.$. We want to prove existence and uniqueness of $f(x) = 0, \forall x \in [0.5, 1.8]$.

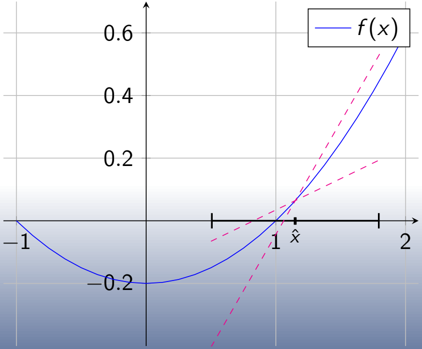
- Choose an initial domain for $[x]$
- Evaluate slopes over $[x]$



Explanations

Take a function $f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 0.2(x^2 - 1) \end{array} \right.$. We want to prove existence and uniqueness of $f(x) = 0, \forall x \in [0.5, 1.8]$.

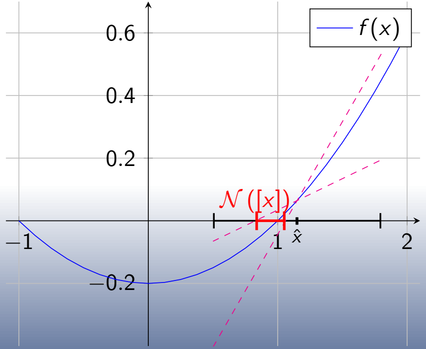
- Choose an initial domain for $[x]$
- Evaluate slopes over $[x]$
- Center slopes on $\hat{x}$



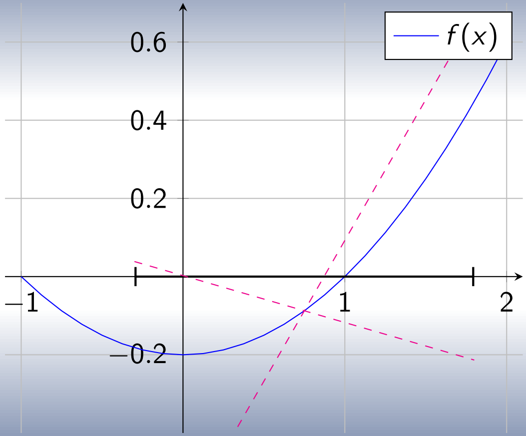
Explanations

Take a function $f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 0.2(x^2 - 1) \end{array} \right.$. We want to prove existence and uniqueness of $f(x) = 0, \forall x \in [0.5, 1.8]$.

- Choose an initial domain for $[x]$
- Evaluate slopes over $[x]$
- Center slopes on $\hat{x}$
- Intersection gives us $\mathcal{N}([x])$



Problem 1 – Jacobian invertibility



Outline

Definitions

An uncertain function $\mathbf{f}$ could be described in several ways:

1. As a tube: $\mathbf{f} \in [\mathbf{f}]$ with $[\mathbf{f}] \triangleq [\mathbf{f}^-, \mathbf{f}^+]$
2. As a parametrized function: $\mathbf{f} \in [\mathbf{f}]$ with $[\mathbf{f}] = \{\varphi(\mathbf{p}, \mathbf{x}), \mathbf{p} \in [\mathbf{p}]\}$

If $J_f \in [J_f]$, we define:

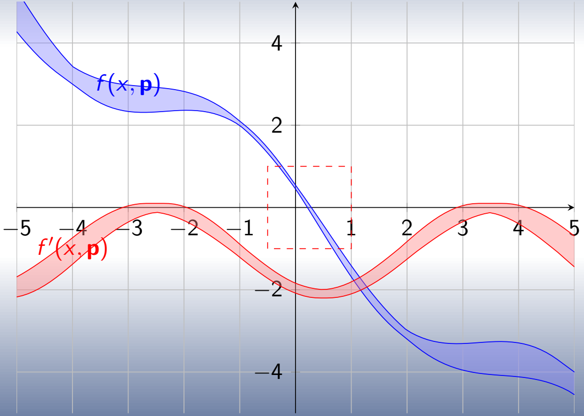
Definition (Interval Newton Test)

$$\mathcal{N}([\mathbf{f}], [\mathbf{J_f}], [\mathbf{x}]) = \widehat{\mathbf{x}} - [\mathbf{J_f}]^{-1}([\mathbf{x}]) \cdot [\mathbf{f}](\widehat{\mathbf{x}}) \tag{2}$$

$$\mathcal{N}([\mathbf{x}]) = \widehat{\mathbf{x}} - \Sigma([\mathbf{J_f}]([\mathbf{x}]), [\mathbf{f}](\widehat{\mathbf{x}})) \tag{3}$$

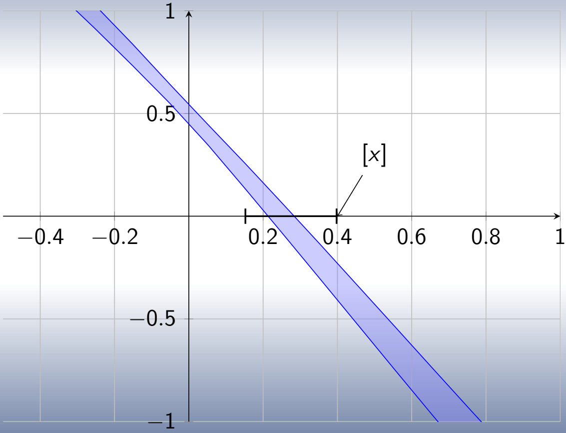
Explanations

Take a function $f(x, [\mathbf{p}]) = \cos(p_1.x + p_2) - p_3.x$ with $[\mathbf{p}] = [1, 1.1]^3$.



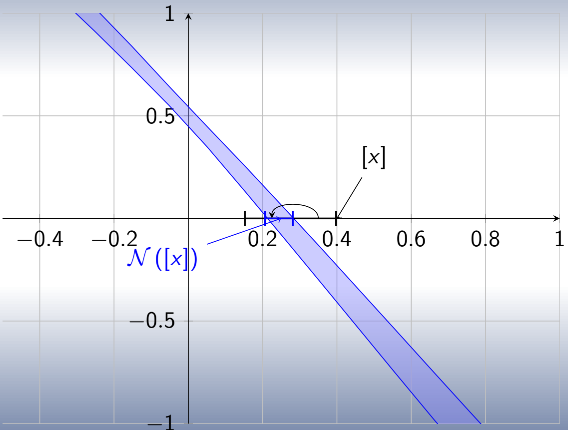
Example

We want to prove: $\forall \mathbf{p} \in [\mathbf{p}], \exists! x, f(x, \mathbf{p}) = 0$



Example

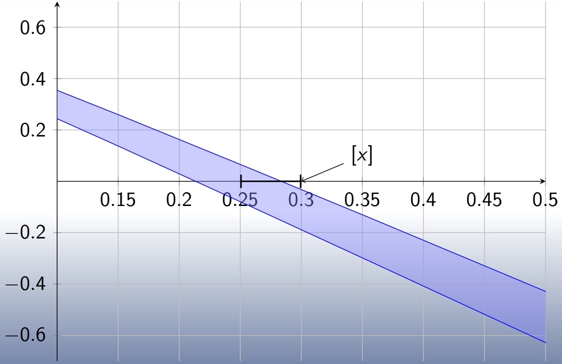
We want to prove: $\forall \mathbf{p} \in [\mathbf{p}], \exists! x, f(x, \mathbf{p}) = 0$



Problem 2 - Bad initial box

When $[x]$ is overlapping $X_0 = \{x | 0 \in [f](x)\}$ and $X_0 \not\subset [x]$, we must start an iteration process.

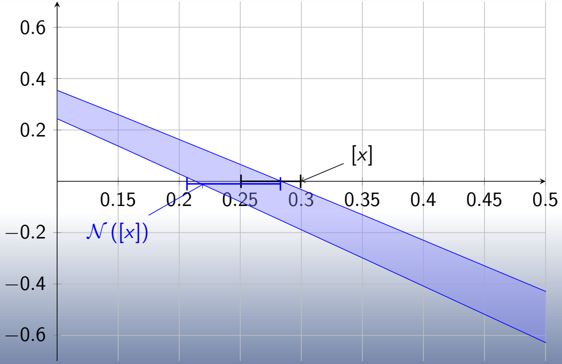
We want to prove: $\forall \mathbf{p} \in [\mathbf{p}], \exists! x \in [0.25, 0.3], f(x, \mathbf{p}) = 0$

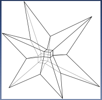


Problem 2 – Bad initial box

When $[x]$ is overlapping $X_0 = \{x | 0 \in [f](x)\}$ and $X_0 \not\subset [x]$, we must start an iteration process.

We want to prove: $\forall \mathbf{p} \in [\mathbf{p}], \exists! x \in [0.25, 0.3], f(x, \mathbf{p}) = 0$

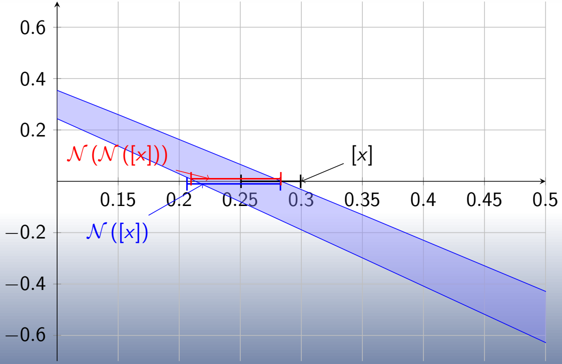




Problem 2 – Bad initial box

When $[x]$ is overlapping $X_0 = \{x | 0 \in [f](x)\}$ and $X_0 \not\subset [x]$, we must start an iteration process.

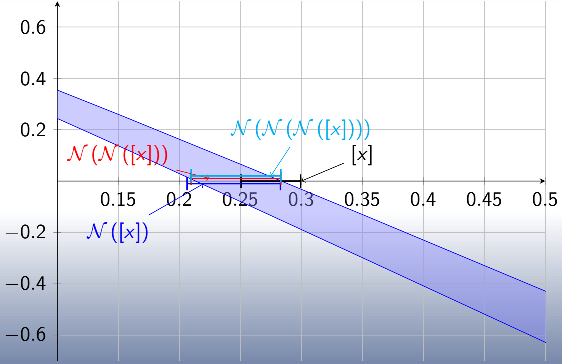
We want to prove: $\forall \mathbf{p} \in [\mathbf{p}], \exists! x \in [0.25, 0.3], f(x, \mathbf{p}) = 0$



Problem 2 – Bad initial box

When $[x]$ is overlapping $X_0 = \{x | 0 \in [f](x)\}$ and $X_0 \not\subset [x]$, we must start an iteration process.

We want to prove: $\forall \mathbf{p} \in [\mathbf{p}], \exists! x \in [0.25, 0.3], f(x, \mathbf{p}) = 0$



Outline

$\mathcal{E}$-inflation

Original definition of $\mathcal{E}$-inflation by Rump [Rump, 1980].

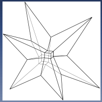
Definition

For a real interval $[x]$

$$[x] \circ \mathcal{E} = \begin{cases} [x] + w([x]) \cdot [-\mathcal{E}, +\mathcal{E}] & \text{for } w([x]) \neq 0 \\ [x] + [-\eta, +\eta] & \text{otherwise} \end{cases} \tag{4}$$

where η represents the smallest representable machine number and $\mathcal{E}$ the inflation coefficient.

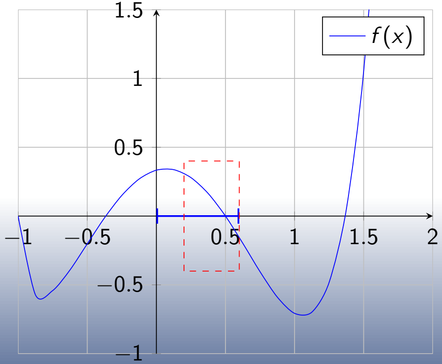
Rump, S. M. (1980).
Kleine fehlerschranken bei matrixproblemen.
Dissertation,Universität Karlsruhe.



N-inflation

Idea: Replace initial box $[x]$ by a small sized interval in $[x]$.
Iterate with $\mathcal{E}$-inflated input, $\mathcal{E} = \eta$.

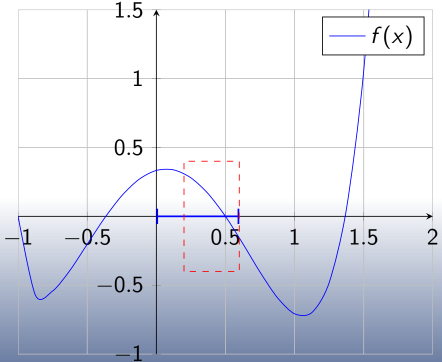
- $f(x) = \frac{(4x^3-6x+1)(\sqrt{x+1})}{3-x}$



N-inflation

Idea: Replace initial box $[x]$ by a small sized interval in $[x]$.
Iterate with $\mathcal{E}$-inflated input, $\mathcal{E} = \eta$.

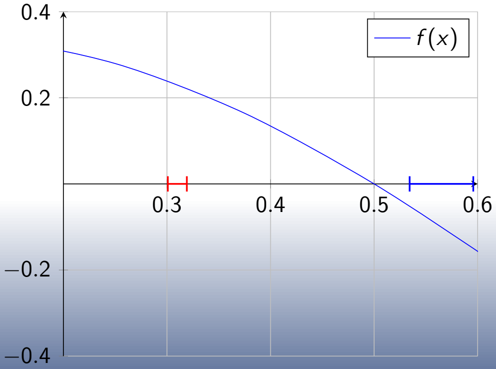
- $f(x) = \frac{(4x^3 - 6x + 1)(\sqrt{x+1})}{3-x}$
- $\mathcal{N}([0; 0.6]) = [-\infty, +\infty]$



N-inflation

Idea: Replace initial box $[x]$ by a small sized interval in $[x]$.
Iterate with $\mathcal{E}$-inflated input, $\mathcal{E} = \eta$.

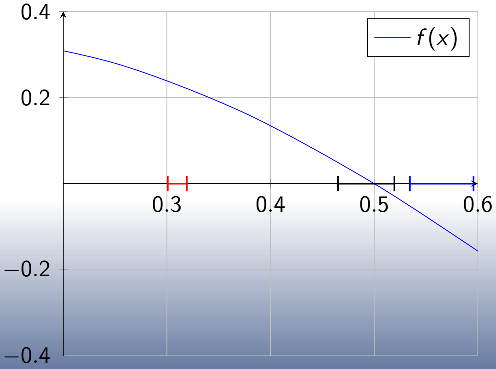
- $f(x) = \frac{(4x^3-6x+1)(\sqrt{x+1})}{3-x}$
- $\mathcal{N}([0; 0.6]) = [-\infty, +\infty]$
- $\mathcal{N}([0.3; 0.32]) = [0.5335; 0.5966]$

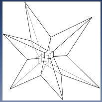


N-inflation

Idea: Replace initial box $[x]$ by a small sized interval in $[x]$.
Iterate with $\mathcal{E}$-inflated input, $\mathcal{E} = \eta$.

- $f(x) = \frac{(4x^3-6x+1)(\sqrt{x+1})}{3-x}$
- $\mathcal{N}([0; 0.6]) = [-\infty, +\infty]$
- $\mathcal{N}([0.3; 0.32]) = [0.5335; 0.5966]$
- $\mathcal{N}^2([0.3; 0.32]) = [0.4642; 0.5203]$

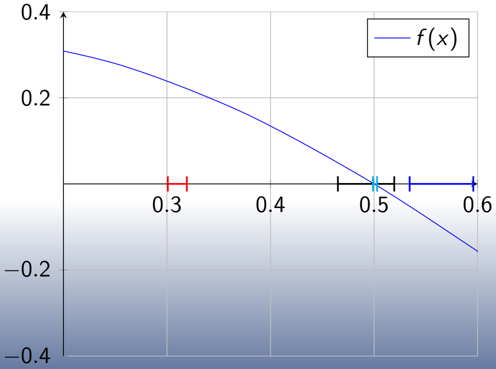




N-inflation

Idea: Replace initial box $[x]$ by a small sized interval in $[x]$.
Iterate with $\mathcal{E}$-inflated input, $\mathcal{E} = \eta$.

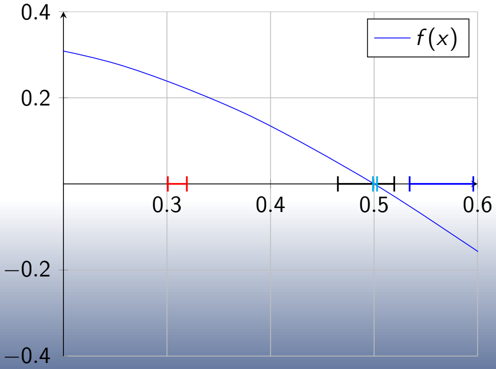
- $f(x) = \frac{(4x^3 - 6x + 1)(\sqrt{x+1})}{3-x}$
- $\mathcal{N}([0; 0.6]) = [-\infty, +\infty]$
- $\mathcal{N}([0.3; 0.32]) = [0.5335; 0.5966]$
- $\mathcal{N}^2([0.3; 0.32]) = [0.4642; 0.5203]$
- $\mathcal{N}^3([0.3; 0.32]) = [0.4981; 0.5037]$



N-inflation

Idea: Replace initial box $[x]$ by a small sized interval in $[x]$.
Iterate with $\mathcal{E}$-inflated input, $\mathcal{E} = \eta$.

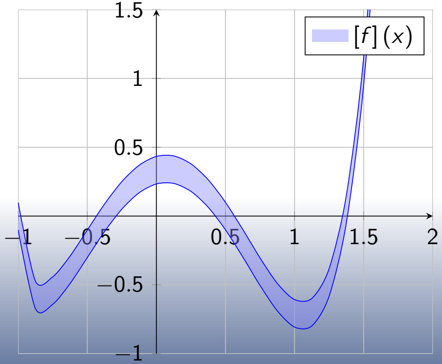
- $f(x) = \frac{(4x^3-6x+1)(\sqrt{x+1})}{3-x}$
- $\mathcal{N}([0; 0.6]) = [-\infty, +\infty]$
- $\mathcal{N}([0.3; 0.32]) = [0.5335; 0.5966]$
- $\mathcal{N}^2([0.3; 0.32]) = [0.4642; 0.5203]$
- $\mathcal{N}^3([0.3; 0.32]) = [0.4981; 0.5037]$
- $\underline{\mathcal{N}^3([0.3; 0.32]) \subset \mathcal{N}^2([0.3; 0.32])}$



N-inflation

Idea: Replace initial box $[x]$ by a small sized interval in $[x]$.
Iterate with $\mathcal{E}$-inflated input, $\mathcal{E} = \eta$.

- $f(x) = \frac{(4x^3-6x+1)(\sqrt{x+1})}{3-x}$



- Similar when $f \in [f], \dot{f} \in [\dot{f}]$

Outline

Definition

Definition (Uniqueness Operator)

$\mathcal{N} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is a **uniqueness operator** if it verifies:

$$\mathcal{N}([\mathbf{x}]) \subset [\mathbf{x}] \Rightarrow \exists ! \mathbf{x}^* \in [\mathbf{x}], \mathcal{N}(\{\mathbf{x}^*\}) = \{\mathbf{x}^*\} \tag{5}$$

Proposition ([Delanoue, 2006])

If an operator $\mathcal{N}$ is a uniqueness operator. So does $\mathcal{N}^k$, with $\mathcal{N}^k = \mathcal{N} \circ \mathcal{N} \circ \ldots \circ \mathcal{N}$.

Delanoue, N. (2006).
Algorithmes numériques pour l'analyse topologique.
PhD dissertation, Université d'Angers, Angers, France.

Proposition

Definition (N-Kleene)

$$I \cap \mathcal{N} \cap \mathcal{N}^2 \cap \ldots \cap \mathcal{N}^\infty = \mathcal{N}^* \tag{6}$$

- If $\mathcal{N}$ is a uniqueness operator for a box $[x]$, so does $\mathcal{N}^*$.
- $\mathcal{N}^*$ is more efficient than $\mathcal{N}$.

Outline

Results - Loop Detection

Problem Statement (Loop detection [Aubry et al., 2013])

We search:

$$\mathbf{f}(\mathbf{t}) = 0, \mathbf{t} = (t_1, t_2) \text{ a t-pair} \tag{7}$$

With

$$\begin{aligned} \mathbf{f}(\mathbf{t}) &= \int_{t_1}^{t_2} \mathbf{v}(\tau) d\tau \\ \mathbf{v} \in [\mathbf{v}] &\Rightarrow \mathbf{f} \in [\mathbf{f}] \\ \mathbf{J_f} &= (-\mathbf{v}(t_1)\ \mathbf{v}(t_2)) \in (-[\mathbf{v}](t_1)\ \ [\mathbf{v}](t_2)) \end{aligned}$$

2 equations, 2 unknown, parameter in infinite dimension.

Aubry, C., Desmare, R., and Jaulin, L. (2013).
Loop detection of mobile robots using interval analysis.
Automatica, 49(2):463–470.

Demo Loop Detection and N-inflation.

Conclusion

Results

- Decrease probability of Jacobian non-invertibility.
- Strategy: local research of a uniqueness box.
- Better automation of uniqueness tests.

To go further

- Perform benchmarking.
- Test with different solver in Newton operator definition.

Thanks for your attention.
Questions?