Level Sets and Controls in a Two Pursuers One Evader Differential Game

S.A. Ganebny, S.S. Kumkov, S. Le Ménec, and V.S. Patsko

Abstract This study deals with a problem of pursuit-evasion with two pursuers and one evader having linear dynamics. The pursuers try to minimize the final miss (an ideal situation is to get exact capture), the evader counteracts them. Results of numerical construction of level sets (Lebesgue sets) of the value function are given. A feedback method for producing optimal control is suggested. The paper includes also numerical simulations of optimal motions of the objects in different situations.

1 Introduction

Nowadays, group pursuit-evasion games (several pursuers and/or several evaders) are studied intensively: [2, 5, 6, 10, 16]. From a general point of view, often, a group pursuit-evasion game (without any hierarchy among players) can be treated as an antagonistic differential game, where all pursuers are joined into a player, whose objective is to minimize some functional, and, similarly, all evaders are joined into another player, who is the opponent to the first one. The theory of differential games gives an existence theorem for the value function of such a game. But, usually, any more concrete results (for example, concerning effective constructing the value function) cannot be obtained. This is due to high dimension of the state vector of the corresponding game. Just these reasons can explain why group pursuit-evasion games are very difficult and are investigated usually by means of specific methods and under very strict assumptions.

In this study, we investigate a pursuit-evasion game with two pursuers and one evader. Such a model formulation arises during analysis of a problem, where two aircrafts (or missiles) intercept another one in the horizontal plane. The peculiarity of the game explored in this work is that solvability sets (the

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sets wherefrom the interception can be guaranteed with miss, which is not greater than some given value) and optimal feedback controls can be build numerically in a one-to-one antagonistic game, where the pursuers are joined into one useful control. Such an investigation is the aim of this study.

2 Formulation of Problem

We consider a game in the plane. Let us assume that initial closing velocities are parallel and quite large and control accelerations affect only lateral components of object velocities. Thus, one can suppose that instants of passages of the evader by each of the pursuers are fixed. Below, we call them termination instants and denote by T_{f1} and T_{f2}, respectively. We consider both the cases of equal and different termination instants. The players' controls define the lateral deviations of the evader from the first and second pursuers at the termination instants. Minimum of absolute values of these deviations is called *the resulting miss*. The objective of the pursuers is minimization of the resulting miss, the evader maximizes it. The pursuers generate their controls by a coordinated effort (from one control center).

In Fig. 1, one can see one possible initial location of the pursuers and evader, when they move towards each other. Also, the evader can move from both pursuers, or from one of them, but towards another one. Below, we consider lateral motions only, so all these cases are studied uniformly.

In the relative linearized system, the dynamics is the following (see [11, 12]):

$$\begin{array}{ll} \ddot{y}_1 = -a_{P1} + a_E, & \ddot{y}_2 = -a_{P2} + a_E, \\ \dot{a}_{P1} = (A_{P1}u_1 - a_{P1})/l_{P1}, & \dot{a}_{P2} = (A_{P2}u_2 - a_{P2})/l_{P2}, \\ \dot{a}_E = (A_E v - a_E)/l_E. & \end{array} \tag{1}$$

Here, y_1 and y_2 are the current lateral deviations of the evader from the first and second pursuers; a_{P1}, a_{P2}, a_E are the lateral accelerations of the pursuers and evader; u_1, u_2, v are the players' controls; A_{P1}, A_{P2}, A_E are the maximal values of the accelerations; l_{P1}, l_{P2}, l_E are the time constants describing the inertiality of servomechanisms. So, a_{P1}, a_{P2}, a_E are the physical lateral accelerations, and u_1, u_2, v are respective command controls.

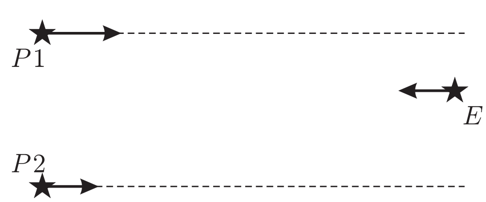

Fig. 1 Schematic initial positions of the pursuers and evader

The controls have bounded absolute values:

$$|u_1| \le 1, \quad |u_2| \le 1, \quad |v| \le 1. \tag{2}$$

The linearized dynamics of the objects in the problem under consideration is typical (see, for example, [15]).

Consider new coordinates x_1 and x_2, which are the values of y_1 and y_2 forecasted to the corresponding termination instants T_{f1} and T_{f2} under zero players' controls. One has

$$x_i = y_i + \dot{y}_i\tau_i - a_{Pi}l_{Pi}^2 h(\tau_i/l_{Pi}) + a_E l_E^2 h(\tau_i/l_E), \quad i = 1, 2. \tag{3}$$

Here, x_i and y_i depend on t, and

$$\tau_i = T_{fi} - t, \quad h(\alpha) = e^{-\alpha} + \alpha - 1.$$

We have $x_i(T_{fi}) = y_i(T_{fi})$.

Passing to a new dynamics in "equivalent" coordinates x_1 and x_2 (see [11, 12]), we obtain:

$$\begin{aligned} \dot{x}_1 &= -A_{P1}l_{P1}h(\tau_1/l_{P1})u_1 + A_E l_E h(\tau_1/l_E)v, \\ \dot{x}_2 &= -A_{P2}l_{P2}h(\tau_2/l_{P2})u_2 + A_E l_E h(\tau_2/l_E)v. \end{aligned} \tag{4}$$

Join both pursuers $P1$ and $P2$ into one player, which will be called the *first player*. The evader E is the *second player*. The first player governs the controls u_1 and u_2; the second one governs the control v. We introduce the following payoff functional:

$$\varphi\big(x_1(T_{f1}), x_2(T_{f2})\big) = \min\big(\big|x_1(T_{f1})\big|, \big|x_2(T_{f2})\big|\big), \tag{5}$$

which is minimized by the first player and maximized by the second one. Thus, we get a standard antagonistic game with dynamics (4) and payoff functional (5). This game has the value function $V(t, x)$, where $x = (x_1, x_2)$. Each level set

$$W_c = \big\{(t, x) : V(t, x) \le c\big\}$$

of the value function coincides with the maximal stable bridge (see [8, 9]); also called capture zone (see [7]); or capture bassin (see [1, 3]) built from the target set

$$M_c = \big\{(t, x) : t = T_{f1}, |x_1| \le c;\ t = T_{f2}, |x_2| \le c\big\}.$$

The set W_c can be treated as the solvability set for the pursuit-evasion game with the result c.

When $c = 0$, we have the situation of the exact capture. The exact capture implies equality to zero of at least one of y_i at the instant T_{fi}, $i = 1, 2$.

The works [11, 12] consider only cases with exact capture and pursuers "stronger" than the evader. The latter means that the parameters A_{Pi}, A_E

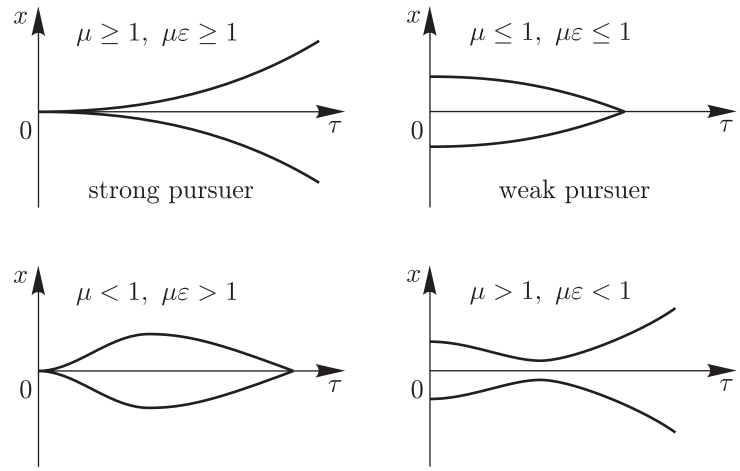

Fig. 2 Different variants of the stable bridges evolution in an individual game

and l_{Pi}, l_E ($i = 1, 2$) are such that the maximal stable bridges in the individual games ($P1$ vs. E and $P2$ vs. E) grow monotonically in the backward time.

Considering individual games of each pursuer vs. the evader, one can introduce parameters [14] $\mu_i = A_{Pi}/A_E$ and $\varepsilon_i = l_E/l_{Pi}$. They and only they define the structure of the maximal stable bridges in the individual games. Namely, depending on values of μ_i and $\mu_i\varepsilon_i$, there are 4 cases of the bridge evolution (see Fig. 2):

- expansion in the backward time (a strong pursuer);
- contraction in the backward time (a weaker pursuer);
- expansion of the bridge until some backward time instant and further contraction;
- contraction of the bridge until some backward time instant and further expansion (if the bridge still has not broken).

Respectively, given combinations of pursuers' capabilities and individual games durations (equal/different), there are significant number of variants for the problem with two pursuers and one evader. The case of two strong pursuers is considered below. The solutions of the other cases are presented in [4].

The main objective of this study is to construct the sets W_c for typical cases of the game under consideration. The difficulty of the problem is that *time sections* $W_c(t)$ of these sets are non-convex. Constructions are made by means of an algorithm for constructing maximal stable bridges worked out by the authors for problems with two-dimensional state variable. The algorithm is similar to the one used in [13]. Another objective is to build optimal feedback controls of the first player (that is, of the pursuers $P1$ and $P2$) and the second one (the evader E).

3 Strong Pursuers, Equal Termination Instants

Add dynamics (4) by a "cross-like" target set

$$M_c = \{|x_1| \le c\} \cup \{|x_2| \le c\}, \quad c \ge 0,$$

at the instant $T_f = T_{f1} = T_{f2}$. Then we get a standard linear differential game with fixed termination instant and non-convex target set. The collection $\{W_c\}$ of maximal stable bridges describes the value function of the game (4) with payoff functional (5).

For the considered case of two stronger pursuers, choose the following parameters:

$$\begin{array}{lll} A_{P1} = 2, & A_{P2} = 3, & A_E = 1, \\ l_{P1} = 1/2, & l_{P2} = 1/0.857, & l_E = 1, \\ T_{f1} = T_{f2} = 6. & & \end{array}$$

1. Structure of maximal stable bridges. Fig. 3 shows results of constructing the set $W = W_0$ (that is, with $c = 0$). In the figure, one can see several time sections $W(t)$ of this set. The bridge has a quite simple structure. At the initial instant $\tau = 0$ of the backward time (when $t = 6$), its section coincides with the target set M_0, which is the union of two coordinate axes. Further, at the instants $t = 4, 2, 0$, the cross thickens, and two triangles are added to it. The widths of the vertical and horizontal parts of the cross correspond to sizes of the maximal stable bridges in the individual games with the first and second pursuers. These triangles are located in the II and IV quadrants (where the signs of x_1 and x_2 are different, in other words, when the evader is between the pursuers) give the zone where the capture is possible only under collective actions of both pursuers (trying to avoid one of the pursuer, the evader is captured by another one).

These additional triangles have a simple explanation from the point of view of problem (1). Their hypotenuses have slope equal to 45°, that is, are described by the equation $|x_1| + |x_2| = \text{const}$. The instant τ, when the hypotenuse reaches a point (x_1, x_2), corresponds to the instant, when the pursuers cover together the distance $|x_1(0)| + |x_2(0)|$, which is between them at the initial instant $t = 0$. Therefore, at this instant, both pursuers come to the same point. Since the evader was initially between the pursuers, it is captured at this instant.

The set W built in the coordinates of system (4) coincides with the description of the solvability set obtained analytically in [11, 12]. The solvability set for system (1) is defined as follows: if in the current position of system (1) at the instant t, the forecasted coordinates x_1, x_2 are inside the time section $W(t)$, then under the controls u_1, u_2 the motion is guided to the target set M_0; otherwise, if the forecasted coordinates are outside the set $W(t)$, then

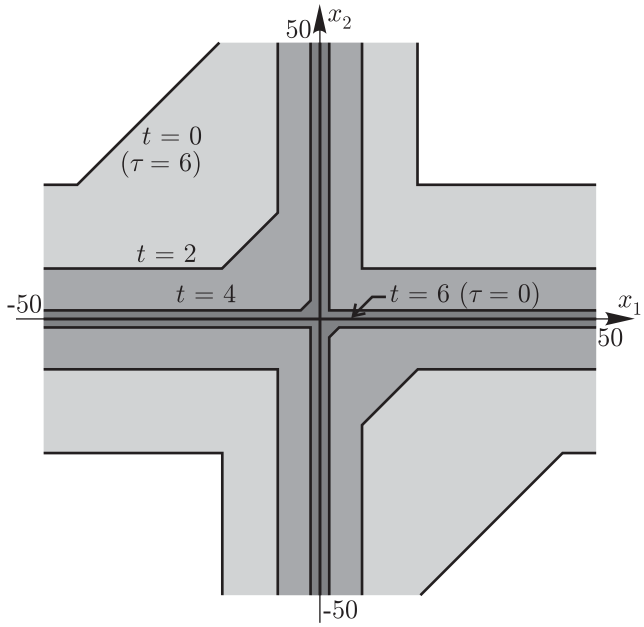

Fig. 3 Two strong pursuers, equal termination instants: time sections of the bridge W

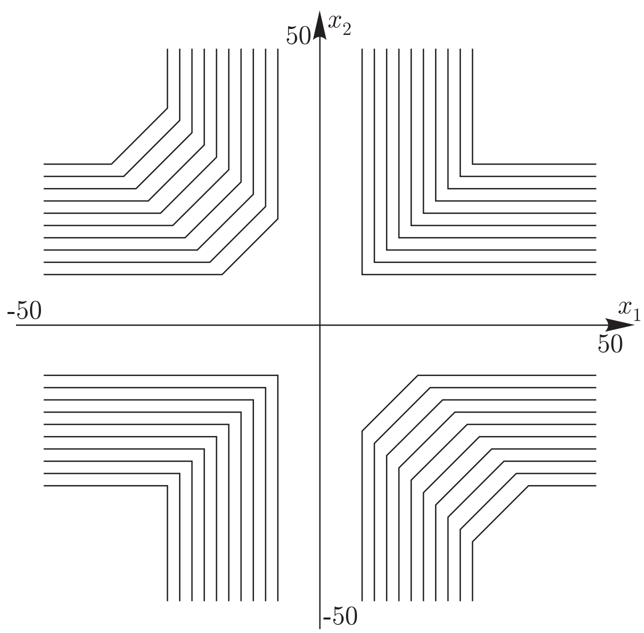

Fig. 4 Two strong pursuers, equal termination instants: level sets of the value function, $t = 2$

there is an evader's control v, which deviates system (4) from the target set, therefore, there is no exact capture in original system (1).

Time sections $W_c(t)$ of other bridges W_c, $c > 0$, have shape similar to $W(t)$. In Fig. 4, one can see the sections $W_c(t)$ at $t = 2$ ($\tau = 4$) for a collection $\{W_c\}$ corresponding to some serie of values of the parameter c. For other instants t, the structure of the sections $W_c(t)$ is similar. The sets $W_c(t)$ describe the value function $x \to V(t, x)$.

2. Feedback control of the first player. The first player governs two controls u_1 and u_2. Velocity component of system (4) depending on u_1 is horizontal, and the component depending on u_2 is vertical. If to analyze the structure of sections $W_c(t)$ at some instant t, one can conclude that at any horizontal line, a minimum of the value function $x \to V(t, x)$ is attained at some interval including $x_1 = 0$. It follows from this that for optimal feedback control it is necessary to take $u_1^0(t, x) = 1$ if $x_1 > 0$, and $u_1^0(t, x) = -1$ if $x_1 < 0$. Thus, the vertical axis is a *switching line* for the control u_1. In the axis, the optimal control can be taken arbitrary under constraint $|u_1| \leq 1$. In the same way, at any vertical line, the minimum of the function $x \to V(t, x)$ is attained in some segment including $x_2 = 0$. Take $u_2^0(t, x) = 1$ if $x_2 > 0$, and $u_2^0(t, x) = -1$ if $x_2 < 0$. The switching line for the control u_2 is the horizontal axis. In the axis, the choice of the control is also arbitrary under condition $|u_2| \leq 1$.

The switching lines (the coordinate axes) at any t divide the plane x_1, x_2 into 4 cells. In each of these cells, the optimal control of the first player is constant.

The vector control $\big(u_1^0(t, x), u_2^0(t, x)\big)$ is applied in a discrete scheme (see [8, 9]) with some time step Δ: a chosen control is kept constant during a time step Δ. Then, on the basis of the new position, a new control is chosen, etc. When $\Delta \to 0$, this control guarantees to the first player a result not greater than $V(t_0, x_0)$ for any initial position (t_0, x_0).

3. Feedback control of the second player. Now, let us describe the optimal control of the second player. The vectogram of the second player in system (4) is a segment parallel to the diagonal of I and III quadrants. Using the sets $W_c(t)$ at some instant t, let us analyze the change of the function $x \to V(t, x)$ along the lines parallel to this diagonal. Consider some of these line such that it passes through the II quadrant. One can see that local minima are attained at points, where the line crosses the axes Ox_1 and Ox_2, and a local maximum is in the segment, where the line coincides with the boundary of some level set of the value function. The situation is similar for lines passing through the IV quadrant.

As the switching lines for the second player's control v, let us take three lines: the axes Ox_1 and Ox_2, and a slope line $\Pi(t)$, which consists of two semi-lines passing through middles of the diagonal parts of the level sets boundaries in the II and IV quadrants. In the considered case in the switching line, the control v can take arbitrary values such that $|v| \leq 1$. Inside each of 6 cells, to which the plane is separated by the switching lines, the control is taken either $v = +1$, or $v = -1$ that one pulls the system towards the points of maximum. Applying this control in a discrete scheme with time step Δ, the second player guarantees with $\Delta \to 0$ the result not less than $V(t_0, x_0)$ for any initial position (t_0, x_0).

Note. Since $W(t) \neq \varnothing$, then the global minimum of the function $x \to V(t,x)$ is attained at any $x \in W(t)$ and equal 0. Thus, when the position (t,x) of the system is such that $x \in W(t)$, the players can choose, generally speaking, any controls under their constraints. If $x \notin W(t)$, the choices should be made according to the described above rules based on the switching lines.

4. Optimal motions. In Fig. 5, one can see results of optimal motion simulations. This figure contains time sections $W(t)$ (thin solid lines; the same sections as in Fig. 3), switching lines $\Pi(0)$ at the initial instant and $\Pi(6)$ at the termination instant of the direct time (dotted lines), and two trajectories for two different initial positions: $\xi_{\mathrm{I}}(t)$ (thick solid line) and $\xi_{\mathrm{II}}(t)$ (dashed line). The motion $\xi_{\mathrm{I}}(t)$ starts from the point $x_1^0 = 40$, $x_2^0 = -25$ (marked by a round), which is inside the initial section $W(0)$ of the set W. So, the evader is captured: the endpoint of the motion (also marked by a round) is at the origin. The initial point of the motion $\xi_{\mathrm{II}}(t)$ has coordinates $x_1^0 = 25$, $x_2^0 = -50$ (marked by a star). This position is outside the section $W(0)$, and

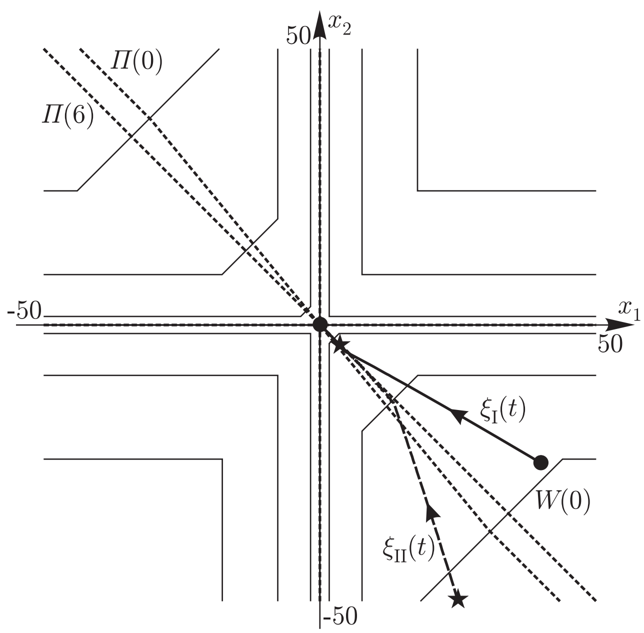

Fig. 5 Two strong pursuers, equal termination instants: result of optimal motion simulation

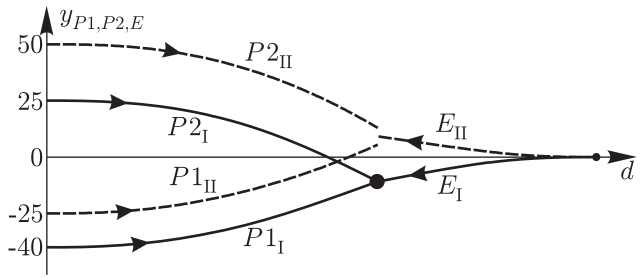

Fig. 6 Two strong pursuers, equal termination instants: trajectories in the original space

the evader escapes from the exact capture: the endpoint of the motion (also marked by a star) has non-zero coordinates.

Fig. 6 gives the trajectories of the objects in the original space. Values of longitudinal components of the velocities are taken such that the evader moves towards the pursuers. For all simulations here and below, we take

$$y_1^0 = -x_1^0, \quad y_2^0 = -x_2^0, \quad \dot{y}_1^0 = \dot{y}_2^0 = 0, \quad a_{P1}^0 = a_{P2}^0 = a_E^0 = 0.$$

Solid lines correspond to the first case, when the evader is successfully captured (at the termination instant, the positions of both pursuers are the same as the position of the evader). Dashed lines show the case, when the evader escapes: at the termination instant no one of the pursuers superposes with the evader. In this case, one can see as the evader aims itself to the middle between the terminal positions of the pursuers (this guarantees the maximum of the payoff functional φ).

4 Conclusion

Presence of two pursuers acting together and minimizing the miss from the evader leads to non-convexity of time sections of the value function, when the situation is considered as a standard antagonistic differential game, where both pursuers are joined into one player. In the paper, results of numerical study of this problem are given for some variants of the parameters. Complementary results and more variants of the parameters are described in [4]. The structure of the solution depends on the presence or absence of dynamic advantage of one or both pursuers over the evader. Optimal feedback control methods of the pursuers and evader are built by preliminary construction and processing of level (Lebesgue) sets of the value function (maximal stable bridges) for some quite fine grid of values of the payoff. Switching lines obtained for each scalar component of controls depend on time, and only they, not the level sets, are used for generating controls. Optimal controls are produced at any current instant depending on the location of the state point respectively to the switching lines at this instant. Accurate proof of the suggested optimal control method needs for some additional study.

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