Survey of Proposals for the Standardization of Interval Arithmetic

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Definition of interval arithmetic:

► definition of interval arithmetic by Moore: 1966
► modal arithmetic by Gardenes et al.: 1985
► extended interval arithmetic by Ratz: 1996
► definition based on a set point of view: Jaulin et al., 2001
► implementation using floating-point arithmetic: Hickey, Ju and van Emden, 2001
► definition based on limits: cset theory: Walster, Hansen and Pryce, 2002
Definition of interval arithmetic:

- taking into account the existence of complex results: Verdonk et al., 2005
- Fortran: in the 90s
- C++: Brönnimann, Melquiond and Pion, 2006
- hardware support: Kirchner and Kulisch, 2006
- ...
Context: IEEE P1788 for the standardization of IA

Dagstuhl, January 2008: decision to produce a standard for interval arithmetic.

Also: decision to have a standard under the auspices of IEEE.

Spring 2008: under the sponsorship of the IEEE committee for floating-point arithmetic, proposal of a **working group for the standardization of interval arithmetic**, approved by IEEE the 12 June 2008, under the number P1788.
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Do not hesitate to join!
Outline of this talk

Context

Historically...

Moore 1966

Extensions

Two points of view: forward and backward

Which set of numbers?

Reals, extended reals, complex numbers?

Link with FP arithmetic

Other mathematical models

Miscellaneous

Conclusion and future work
Initial definition: Moore 1966

Initial definition by Moore (1962, published in 1966):

- \([a, b] + [c, d] = [a + c, b + d]\);
- \([a, b] - [c, d] = [a - d, b - c]\);
- \([a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]\);
- \(1 / [c, d] = [1/d, 1/c]\) if \(0 \not\in [c, d]\);
- \([a, b] / [c, d] = [a, b] \times (1/[c, d])\) if \(0 \not\in [c, d]\);
- \(f([a, b]) = \text{convex hull} (\{f(x) : x \in [a, b]\})\):
  formulas using only the endpoints when \(f\) is monotonous,
  more complicated otherwise.
Unsatisfying definition

Division is not total: \([1, 2]/[-1, 2]??\)

The system is not closed.
It is desirable that every possible combination of \(<\) operator, operands \(>\) yields a result within the system.
Extended interval arithmetic

Ratz 1996

(or maybe Kahan or Hanson in 1968)
Let $x$ and $y$ be two intervals.

$$x/y = \{z : y \cdot z = x, x \in x, y \in y\}.$$
Extended interval arithmetic

Division by an interval containing 0

Main concern: Newton iteration to solve $f(x) = 0$ without losing any solution.

Proposals:

- Jaulin et al.: $1/[-2, 2] = (-\infty, +\infty)$ but $[3, 4]/[0, 0] = \emptyset$;
- $[0, 1]/[0, 1] = [0, +\infty)$ since only nonnegative terms can be produced (Ratschek & Rokne 1988);
- $[1, 2]/[0, 1] = \{-\infty\} \cup [1, +\infty]$ (cset theory);
- $[0, 1]/[0, 1] = (-\infty, +\infty)$ (Ratz)
Remark: arguments outside the domain

More generally, how should $f(x)$ be handled when $x$ is not included in the domain of $f$?

- return NaI (Not an Interval)? i.e. handle exceptional values such as NaI and infinities?
- intersect $x$ with the domain of $f$ prior to the computation, silently?
- intersect $x$ with the domain of $f$ prior to the computation and raise a flag?
- return the set of every possible limits $\lim_{y\to x} f(y)$ for every possible $x$ in the domain of $f$ (but not necessarily $y$)?
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Forward and backward

Wording inspired from constraint programming and forward-backward propagation.
Forward and Backward

Forward:
it corresponds to the "natural extension" à la Moore.

\[ f(\mathfrak{x}) = \{ f(x) : x \in \mathfrak{x} \} \]

or

\[ f(\mathfrak{x}) = \{ \lim_{x \to y} f(x) : y \in \mathfrak{x} \} \]

or

\[ f(\mathfrak{x}) = \{ \lim_{x \to y} f(x) : x \in \mathfrak{x}, y \in \mathfrak{x} \} \]
Forward and Backward

**Backward:**

it corresponds to the philosophy of Ratz: one does not want to lose any solution.

\[ f(x) = \text{convex hull}(\{ y : \exists x \in x, f^{-1}(y) = x \} \) \]

Eg.

\[ \sqrt{[1, 2]} = \text{convex hull}([-\sqrt{2}, -1] \cup [1, \sqrt{2}]) = [-\sqrt{2}, \sqrt{2}]. \]

I personally prefer the wording **relations** to **backward operations**, since I would also prefer to keep the two separate parts of the answer and thus \( \sqrt{ \) is no more a function, since it returns two arguments, but it is a relation.
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With or without the infinities?

Should we work with $\mathbb{IR} = (-\infty, +\infty)$ or with $\tilde{\mathbb{IR}} = [-\infty, +\infty]$? Should the infinities be first class citizens or outlaws?
With or without the infinities?

Should we work with \( IR = (\neg \infty, +\infty) \) or with \( \bar{IR} = [-\infty, +\infty] \)?
Should the infinities be first class citizens or outlaws?

If they are first class citizens, \([0, 1]/[0, 1] = \{-\infty\} \cup [0, +\infty]\) (cset theory) becomes natural.
Complex results... 

Verdonk, Vervloet, Cuyp 2005

Proposal: add flags to indicate whether there could also exist complex results, nonzero complex results...
Definition related to floating-point arithmetic

Lozinski 1973, MPFI

Implementation based on IEEE-754 floating-point arithmetic.

Point of view: also based on floating-point arithmetic:

\[ f(x) = \{ f(x) \in IF : x \in x \text{ and } x \in IF \}. \]

Eg. \( \sqrt{[-1, 4]} \supset [0, 2] \cup \{ \text{NaN} \} \) and thus \( \sqrt{[-1, 4]} = \text{Nal} \) (Not an Interval).
Hickey, Ju, van Emden 2001

Definition based on the set of reals \( IR = (-\infty, +\infty) \).
Interval = closed connected set in \( IR \), ie. one of \( \emptyset \), \( (-\infty, b] \), \( [a, +\infty) \) or \( [a, b] \) where \( a \in IR \) and \( b \in IR \).

Clever implementation using IEEE-754 floating-point arithmetic:

- infinities exist and can be handled;
- use of signed zeroes: \([0, 1]\) is represented as \([+0, 1]\)
  and thus \([0, 1]/[0, 1]\) naturally yields \([0, +\infty)\).

Idea: \([0, 1]\) contains only nonnegative numbers and is almost certainly too wide, ie. the exact result may well contain only positive numbers.
No non-standard analysis (with infinitesimally small numbers between 0 and any positive number).
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Wraparound intervals

**Kulisch:** 
\[ [3, 4] / [-2, 1] = (-\infty, -2] \cup [3, +\infty) \]
To return only one result, return \([3, -2]\).

**Markov:** 
\([a, b] + [-a, -b] = [0, 0]\)
Algebraic structure (group instead of simply a monoid) is recovered.
Modal arithmetic

Gardenes, Mielgo and Trepat, 1985

Goldsztein 2005, Shary...

Idea: an improper interval $x$ in an operation is interpreted as
$\{\exists x \in x : \ldots\}$.

Restriction: every $\forall$ quantifier must appear before $\exists$ quantifiers in
the interpretation.
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List of operations

- arithmetic operations and functions: $+, -, \times, /, \sqrt{\cdot}$, power (more tricky), elementary functions, special functions?
- set operations: $\cap$, $\cup$, convex union, $\setminus$
- interval operations: $\text{inf}$, $\text{sup}$, mid, width or radius...
Comparisons

At least three possible definitions:

- **certainly** $<, \leq, >, \geq$: 
  \[ x < y \iff \forall x \in x, \forall y \in y, x < y \]

- **possibly** $<, \leq, >, \geq$: 
  \[ x < y \iff \exists x \in x, \exists y \in y, x < y \]

- **Kulisch** $<, \leq, >, \geq$: 
  \[ x = [x, \bar{x}] < y = [y, \bar{y}] \iff x < y \text{ and } \bar{x} < \bar{y}. \]
Algebraic manipulations of expressions

Should we allow the compiler to manipulate the expressions to optimize the computational time?
Forbidden in pure IEEE-floating point mode, because the usual algebraic rules do not apply to floating-point computations.

Ibid. for interval expressions?
What about algebraic manipulations by the user (yielding different results)?
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Conclusion

Wanted: a standard where

- the system is closed, i.e. any operation between any operands results in an element of the system;
- its implementation, using floating-point arithmetic, is closed;
- everything is mathematically sound:
  - *Thou shalt not lie*: the inclusion property is valid;
- the implementation is easy and efficient (even if hardware implementation is not required, furthermore some points are language-dependent);
- it is easy to implement other mathematical models (wraparound intervals, modal arithmetic...).
Future work

The IEEE committee will have to

► complete this list
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  and you can help us!

- discuss every point, its pro and cons (using counterexamples)
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- agree on the most sensible choice...
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▸ discuss every point, its pro and cons (using counterexamples)
  and you can help us!

▸ agree on the most sensible choice...
  and then you will vote to tell us if we were right!

See you in 4 (or 6, or 8) years time, to introduce you the new standard!
To join IEEE P1788

Send me: Nathalie.Revol@ens-lyon.fr an e-mail with

- your first name and name
- your affiliation
- your complete address
- your e-mail address
- whether you plan to subscribe to the mailing list or to serve on the committee.

Serving on the committee: 3-4 meetings per year, 3 days each, alternately in Europe and North America (very probably).