Reachability of uncertain nonlinear systems using a nonlinear hybridization

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SWIM2008. Montpellier, June 19-20

Papers

- N.Ramdani, N.Meslem, Y.Candau.
  Reachability of uncertain nonlinear systems using a nonlinear hybridization.
  In : M. Egerstedt and B. Mishra (Eds.) :
  HSCC 2008, LNCS 4981, pp. 415–428, 2008.

- N.Ramdani, N.Meslem, Y.Candau.
  Reachability analysis of uncertain nonlinear systems using guaranteed set integration.
  IFAC World Congress 2008, Seoul.

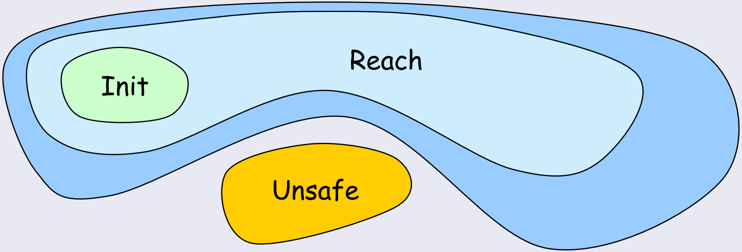
Plan

Reachability computation

Uncertain nonlinear dynamical system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \subseteq \mathbb{D} \subseteq \mathbb{R}^n, \quad \mathbf{p} \in \mathbb{P}$$

Reachable space



$$\mathbb{R}([t_0,\ t]\,;\mathbb{X}_0) = \left\{ \begin{array}{l} \mathbf{x}(\tau),\ \ t_0 \leq \tau \leq t \mid \\ \dot{\mathbf{x}}(\tau) = f(\mathbf{x}, \mathbf{p}, \tau)\ \wedge\ \mathbf{x}(t_0) \in \mathbb{X}_0\ \wedge\ \mathbf{p} \in \mathbb{P} \end{array} \right\}$$

Reachability computation : State-of-the-art

Affine systems

- time discretization
- set integration
- computational geometry : polytopes, zonotopes, ellipsoids ...

Nonlinear systems

- linear hybridization : simplified dynamics ...
- set integration via interval analysis (HyperTech ...)

Our new method : Nonlinear hybridization

Tools

- Set computation via interval analysis
- Set integration via interval Taylor models
- Set integration via Müller's existence theorem
- Hybrid automata

Results

1. No linearization. Truly non linear
2. Integration time step can be varying
3. Analytical solution for reachable space boundaries

Plan

1. Introduction

2. Guaranteed set integration and reachable space
   - Set integration with interval Taylor models
   - Set integration with Müller's theorem
   - Reachable space via set integration

3. Nonlinear hybridization
   - Hybrid automata as bounding systems
   - Example
   - Conclusion

Interval analysis

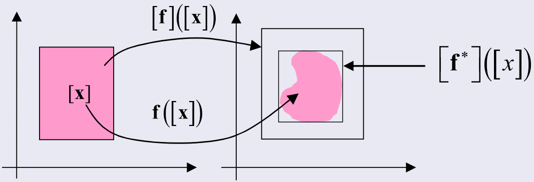
(Dwyer,51) (Warmus,56) (Sunaga,58) (Moore,59)

Extension of real arithmetics to intervals

$$\odot \in \{+,-,\cdot,/\}, [x] \odot [y] = [\{x \odot y \quad | \quad x \in [x], y \in [y]\}]$$

Inclusion function

$$\forall [\mathbf{x}] \subseteq \mathbb{D}, \mathbf{f}([\mathbf{x}]) \subseteq [\mathbf{f}]([\mathbf{x}])$$



Verified numerical implementation

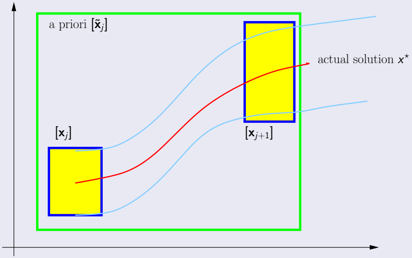
Directed rounding $\longrightarrow$ Outward rounding

Guaranteed set integration with Taylor models

(Moore,66) (Eijgenraam,81) (Lohner,88) (Rihm,94) (Berz,98) (Nedialkov,99)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \, \mathbf{x}(t_0) \in [\mathbf{x}_0]\,, \, \mathbf{p} \in [\mathbf{p}]$$

Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$



- Proof of existence
- Yield *a priori* solution $[\tilde{\mathbf{x}}_j]$: $\forall \tau \in [t_j,\ t_{j+1}] \quad x(\tau) \in [\tilde{\mathbf{x}}_j]$

Guaranteed set integration with Taylor models

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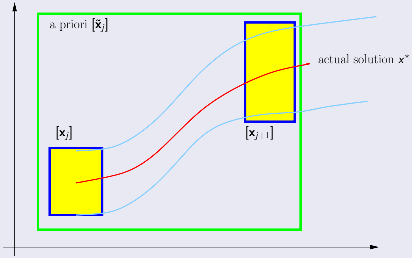
Time grid $\rightarrow$ $\quad t_0 < t_1 < t_2 < \cdots < t_N$



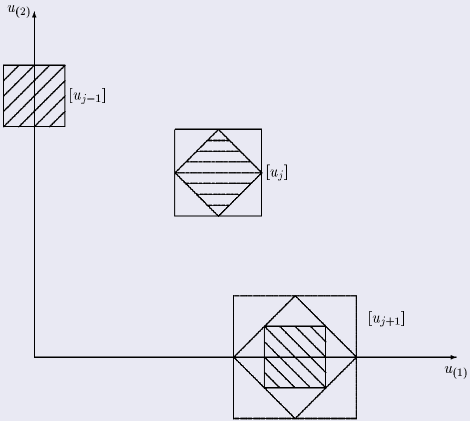
- Compute tight enclosure $[\mathbf{x}_{j+1}] \ni \mathbf{x}(t_{j+1})$

$$[\mathbf{x}_{j+1}] = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t_{j+1} - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (t_{j+1} - t_j)^k \mathbf{f}^{[k]}([\tilde{\mathbf{x}}_j], [\mathbf{p}])$$

Guaranteed set integration with Taylor models

(Moore,66) (Eijgenraam,81) (Lohner,88) (Rihm,94) (Berz,98) (Nedialkov,99)

Need to control wrapping effect



Guaranteed set integration with Taylor models

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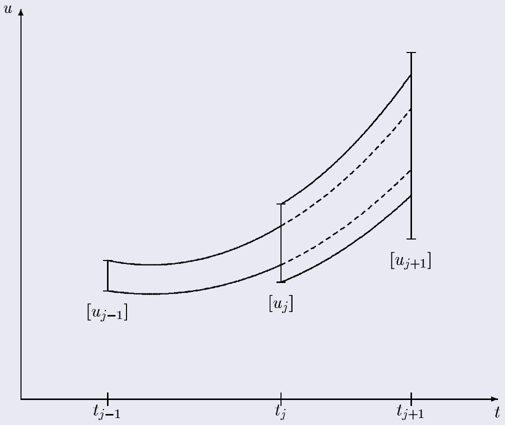

Guaranteed set integration with Taylor models

(Moore,66) (Eijgenraam,81) (Lohner,88) (Rihm,94) (Berz,98) (Nedialkov,99)

Need to control wrapping effect

- Mean value forms
- Matrice preconditioning
- Linear transforms

$\Rightarrow$ Fail when set size is large !

Guaranteed set integration with Müller's theorem

(Müller, 27) (Walter, 97)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \le t \le t_N,\ \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \mathbf{p} \in [\mathbf{p}]$$

Existence Müller's theorem

$$\text{if } \begin{cases} \forall t \in [t_0,\ t_N],\ \forall \mathbf{x} \in \mathbb{D},\ \forall \mathbf{p} \in \mathbb{P} \\ \forall i\ \min_{x_i=\omega_i} f_i(\mathbf{x}, \mathbf{p}, t) \ge D^{\pm}\omega_i(t) \\ \forall i\ \max_{x_i=\Omega_i} f_i(\mathbf{x}, \mathbf{p}, t) \le D^{\pm}\Omega_i(t) \\ \omega(t_0) \le \mathbf{x}(t_0) \le \Omega(t_0) \end{cases} \Rightarrow \begin{cases} \text{solution exists} \\ \\ \forall t \in [t_0,\ t_N], \\ \omega(t) \le \mathbf{x}(t) \le \Omega(t) \end{cases}$$

Solve coupled system $\{\omega, \Omega\}$ with interval Taylor models

$$\rightarrow \begin{cases} [\mathbf{x}](t) = [\mathrm{Inf}(\omega(t)), \mathrm{Sup}(\Omega(t))] \\ [\tilde{\mathbf{x}}](t) = \left[\mathrm{Inf}(\tilde{\omega}(t)), \mathrm{Sup}(\tilde{\Omega}(t))\right] \end{cases}$$

Bracketing functions

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \le t \le t_N,\ \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \mathbf{p} \in [\mathbf{p}]$$

Formal expression of $\overline{f}_i$

Study monotonicity → Analyze the signs of partial derivatives of f_i

for $l \neq i$, **if** $\frac{\partial f_i}{\partial x_l} \geq 0$ **then** replace $x_l \leftarrow \Omega_l$, in f_i
else replace $x_l \leftarrow \omega_l$

for $k = 1, \ldots, n_p$, **if** $\frac{\partial f_i}{\partial p_k} \geq 0$ **then** replace $p_k \leftarrow \overline{p}_k$
else replace $p_k \leftarrow \underline{p}_k$

Repeat for $i = 1, \ldots, n$, $\Rightarrow \begin{cases} \dot{\omega}(t) = \underline{f}(\omega, \Omega, \underline{\mathbf{p}}, \overline{\mathbf{p}}, t), & \omega(t_0) = \underline{\mathbf{x}}_0 \\ \dot{\Omega}(t) = \overline{f}(\omega, \Omega, \underline{\mathbf{p}}, \overline{\mathbf{p}}, t), & \Omega(t_0) = \overline{\mathbf{x}}_0 \end{cases}$

Reachable space via set integration

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N,\ \mathbf{x}(t_0) \in [\mathbf{x}_0],\ \mathbf{p} \in [\mathbf{p}]$$

$$\text{Time grid} \rightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N$$

$\longrightarrow$ Analyse partial derivatives $\longrightarrow$ Use Müller's theorem

$\longrightarrow$ $[\mathbf{x}](t) = [\mathrm{Inf}(\omega(t)), \mathrm{Sup}(\Omega(t))]$

Analytical formulas for reachable space boundaries

$$\forall \tau \in [t_j,\ t_j + h_j] \quad \mathbf{x}(\tau) \in [\mathbf{x}(\tau)]$$

$$[\mathbf{x}(\tau)] = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (\tau - t_j)^i f^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (\tau - t_j)^k f^{[k]}([\tilde{\mathbf{x}}_j], [\mathbf{p}])$$

$$\mathbb{R}([t_0,\ t]\,; [\mathbf{x}_0]) \subseteq \cup_{\tau \in \{t_0,\, t\}} [\mathbf{x}(\tau)] \subseteq \cup_{j \in \{0,\, t\}} [\tilde{\mathbf{x}}_j]$$

Reachable space via set integration

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \leq t \leq t_N, \ \mathbf{x}(t_0) \in [\mathbf{x}_0], \ \mathbf{p} \in [\mathbf{p}]$$

Time grid $\longrightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N$

$\longrightarrow$ Analyse partial derivatives $\longrightarrow$ Use Müller's theorem

$\longrightarrow [\mathbf{x}](t) = [\mathrm{Inf}(\omega(t)), \mathrm{Sup}(\Omega(t))]$

Bracketing functions in the general case

Signs of partial derivatives change with integration time

$\longrightarrow$ Hybridization : Hybrid automata as bounding systems

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3. Nonlinear hybridization
   - Hybrid automata as bounding systems
   - Example
   - Conclusion

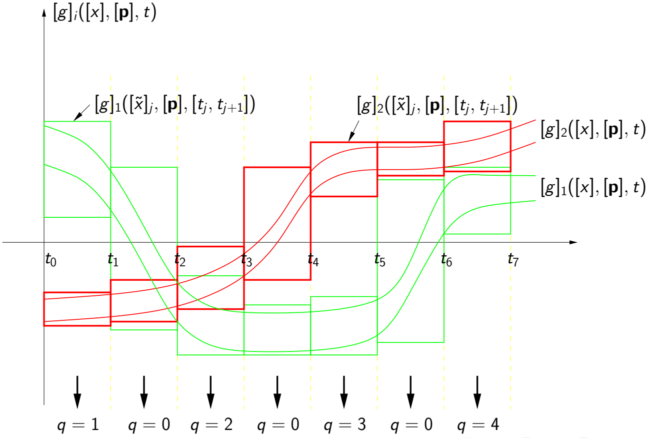
Nonlinear hybridization

Illustrative example : $\dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}, \quad p_i \in [\underline{p}_i, \overline{p}_i]$

Nonlinear hybridization

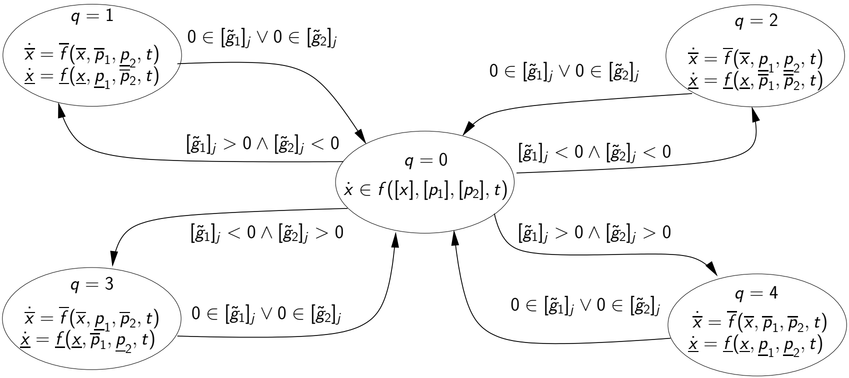
Illustrative example : $\dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}, \quad p_i \in [\underline{p}_i, \overline{p}_i]$

$g_i(.) = \frac{\partial f}{\partial p_i}(.)$



Nonlinear hybridization

Illustrative example : $\dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}, \quad p_i \in [\underline{p}_i, \overline{p}_i]$



$$g_i(.) = \frac{\partial f}{\partial p_i}(.), \quad [\tilde{g}_i]_j = g_i([\tilde{x}_j], [p_1], [p_2], [t_j, t_{j+1}])$$

Nonlinear hybridization

Illustrative example : $\dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}, \quad p_i \in [\underline{p}_i, \overline{p}_i]$

Mode switching

- *Full interval mode* $q = 0 \rightarrow$ *Bracketing systems* mode $q \neq 0$

  $\rightarrow$ switch and carry on
- *Bracketing systems* mode $q \neq 0 \rightarrow$ *Full interval* mode $q = 0$

  $\rightarrow$ switch and re-do time step calculation

Nonlinear hybridization

Illustrative example : $\dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}, \quad p_i \in [\underline{p}_i, \overline{p}_i]$

Time grid $\rightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N$

Hybrid Bounding algorithm

1 Initialize (select bounding systems)
2 Do loop
3 Integrate one step ahead $\rightarrow [\tilde{\mathbf{x}}_j],\ [\mathbf{x}_{j+1}]$
4 Check Switching $\leftarrow$ $\text{sign}(\frac{\partial f}{\partial x_l}(.))$, $\text{sign}(\frac{\partial f}{\partial p_k}(.))$, $[\tilde{\mathbf{x}}_j]$, $[t_j, t_{j+1}]$
5 Switch mode if necessary (change bounding systems)
end Do

Uncertain nonlinear system from bio-reactors

Haldane model. Biotechnological process in a stirred reactor

$$\begin{cases} \dot{x} &= f_x(x,s) &= (\mu_0 \frac{s}{s+k_s+s^2/k_i} - \alpha d)x \\ \dot{s} &= f_s(x,s) &= -k\mu_0 \frac{s}{s+k_s+s^2/k_i} x + (s_{in} - s)d \end{cases}$$

Biomass density : x,

Substrate concentration : s,

Concentration of input substrate : $s_{in}(t) = s_{in}^0 + 15\cos(1/5t)$,

Uncertain parameters : $\mu_0 = 0.75 \pm 1\%$, $s_{in}^0 = 65 \pm 1.5\%$.

Initial state : $x(t_0) \times s(t_0) = [9.5, 10.5] \times [36, 44]$.

Coefficients : $k = 42.14$, $k_s = 9.28 mmol/l$, $k_i = 256 mmol/l$, $\alpha = 0.5$, $d = 2$.

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$$\begin{cases} \dot{x} &= f_x(x,s) &= (\mu_0 \frac{s}{s+k_s+s^2/k_i} - \alpha d)x \\ \dot{s} &= f_s(x,s) &= -k\mu_0 \frac{s}{s+k_s+s^2/k_i} x + (s_{in} - s)d \end{cases}$$

Signs of partial derivatives

$\forall t > t_0,$

$\partial f_x / \partial \mu_0 > 0 \wedge \partial f_s / \partial x < 0 \wedge \partial f_s / \partial \mu_0 < 0 \wedge \partial f_s / \partial s_{in}^0 > 0$

$\text{sign}(\partial f_x / \partial s) = \text{sign}(k_s k_i - s^2)$

Uncertain nonlinear system from bio-reactors

$q = 1,\ s > \sqrt{k_s k_2},\ \partial f_x / \partial s < 0$

$$\begin{cases} \dot{\underline{x}} &= \underline{\mu}_0 \frac{\overline{s}}{\overline{s}+k_s+\overline{s}^2/k_i}\underline{x} - \alpha d \underline{x} \\ \dot{\underline{s}} &= -k\overline{\mu}_0 \frac{\underline{s}}{\underline{s}+k_s+\underline{s}^2/k_i}\overline{x} + d(\underline{s}_{in} - \underline{s}) \\ \dot{\overline{x}} &= \overline{\mu}_0 \frac{\underline{s}}{\underline{s}+k_s+\underline{s}^2/k_i}\overline{x} - \alpha d \overline{x} \\ \dot{\overline{s}} &= -k\underline{\mu}_0 \frac{\overline{s}}{\overline{s}+k_s+\overline{s}^2/k_i}\underline{x} + d(\overline{s}_{in} - \overline{s}) \end{cases}$$

$q = 2,\ s < \sqrt{k_s k_2},\ \partial f_x / \partial s > 0$

$$\begin{cases} \dot{\underline{x}} &= \underline{\mu}_0 \frac{\underline{s}}{\underline{s}+k_s+\underline{s}^2/k_i}\underline{x} - \alpha d \underline{x} \\ \dot{\underline{s}} &= -k\overline{\mu}_0 \frac{\overline{s}}{\overline{s}+k_s+\overline{s}^2/k_i}\overline{x} + d(\underline{s}_{in} - \underline{s}) \\ \dot{\overline{x}} &= \overline{\mu}_0 \frac{\overline{s}}{\overline{s}+k_s+\overline{s}^2/k_i}\overline{x} - \alpha d \overline{x} \\ \dot{\overline{s}} &= -k\underline{\mu}_0 \frac{\underline{s}}{\underline{s}+k_s+\underline{s}^2/k_i}\underline{x} + d(\overline{s}_{in} - \overline{s}) \end{cases}$$

$q = 0$. Original uncertain model

$$\begin{cases} \dot{x} &= f_x(x,s) &= (\mu_0 \frac{s}{s+k_s+s^2/k_i} - \alpha d)x \\ \dot{s} &= f_s(x,s) &= -k\mu_0 \frac{s}{s+k_s+s^2/k_i}x + (s_{in} - s)d \end{cases}$$

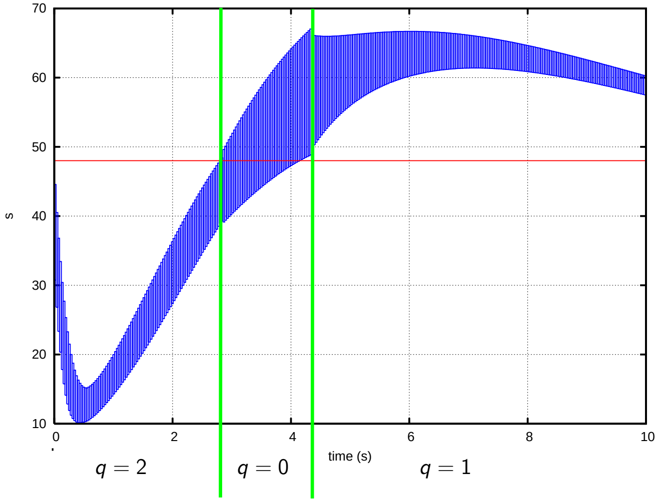
Uncertain nonlinear system from bio-reactors

Numerical implementation

- C++ Class Libraries
- Interval computation → `Profil/BIAS`
- Taylor coefficients and differentiation → `FADBAD++`

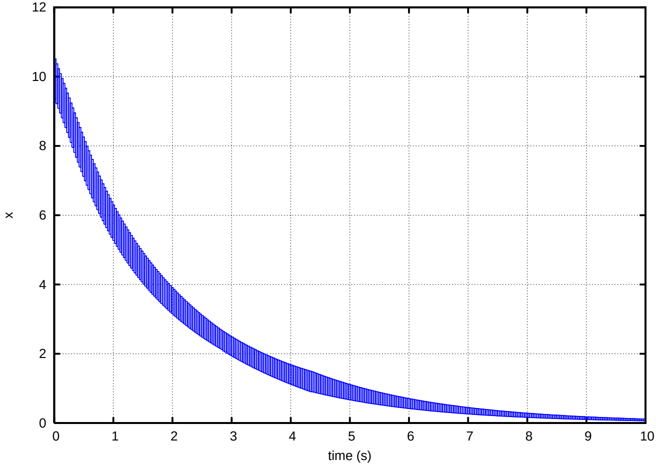
Uncertain nonlinear system from bio-reactors

Time history of s component



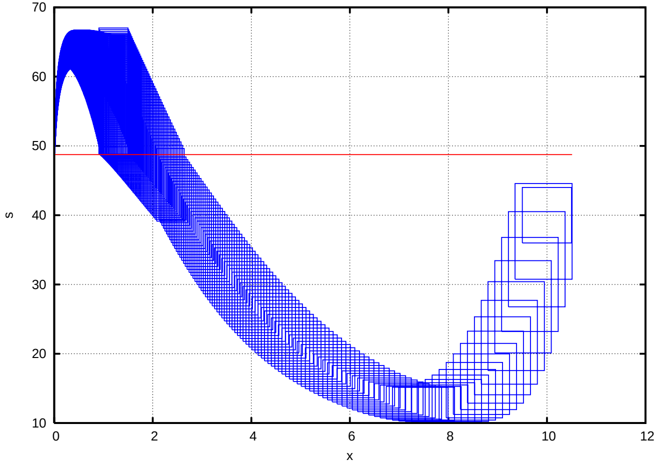
Uncertain nonlinear system from bio-reactors

Time history of x component



Uncertain nonlinear system from bio-reactors

Reachable space



Concluding remarks

Conclusion

- Nonlinear hybrid automata as bounding systems

  $\longrightarrow$ promising tool for nonlinear reachability in presence of uncertainty

Further work

- Event detection $\longrightarrow$ guard crossing
- Convergence issue
- Use with state of the art verification tools