Attraction domain and nonlinear dynamical system using interval analysis.

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SWIM08 : a Small Workshop on Interval Methods

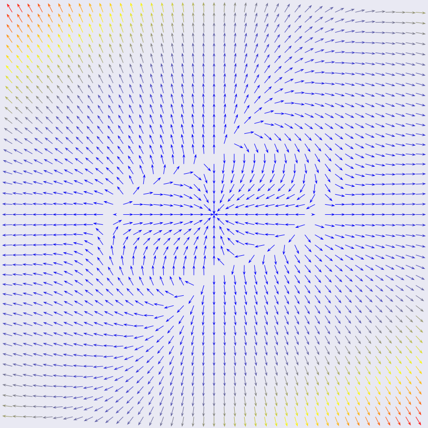
Montpellier 2008

Outline

1. Introduction - Dynamical system
   - Equilibrium state - Stability
   - Attraction Domain
2. Lyapunov theory
   - Positivity
   - Lyapunov Theory
   - Algorithm A
3. Discretization
   - Algorithm B

Compute the attraction domain of an asymptotically stable point

$$\begin{cases} \dot{x} = f(x) \\ x \in \mathbb{R}^n \end{cases}, f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n).$$



Let us denote by $\{\varphi^t : \mathbb{R}^n \longrightarrow \mathbb{R}^n\}_{t\in\mathbb{R}}$ the flow, i.e.

$$\left.\frac{d}{dt}\varphi^t x\right|_{t=0} = f(x) \text{ and } \varphi^0 = Id \qquad (1)$$

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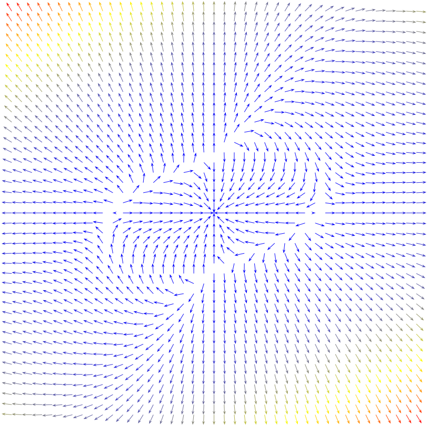
The function $t \mapsto \varphi^t x$ is the solution of $\dot{x} = f(x)$ satisfying $x(0) = x$.

Definition

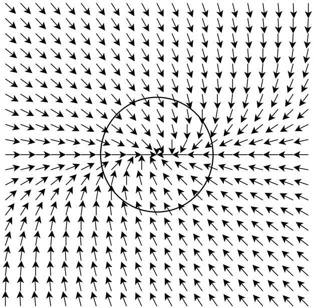
A point $x \in \mathbb{R}^n$, x is an *equilibrium state* if $f(x) = 0$ *i.e.* $\varphi^t(x) = x, \forall t \in \mathbb{R}$.



Definition

A set D is *stable* if :

$$\varphi^{\mathbb{R}^+}(D) \subset D$$



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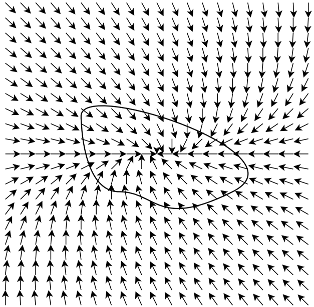
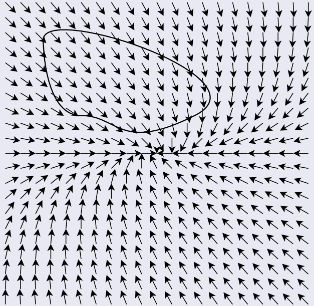
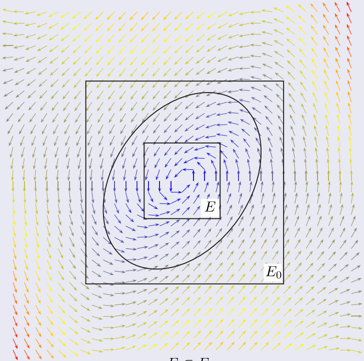


Example of a non stable set



Definition

An equilibrum state x_∞ is *asymptotically* (E, E_0)*-stable* if



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Definition

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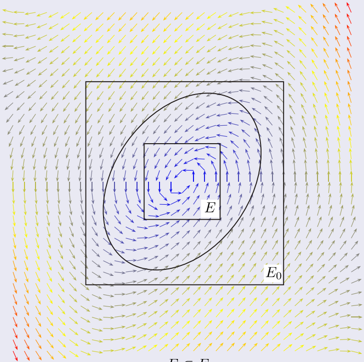
- $\varphi^{\mathbb{R}^+}(E) \subset E_0$



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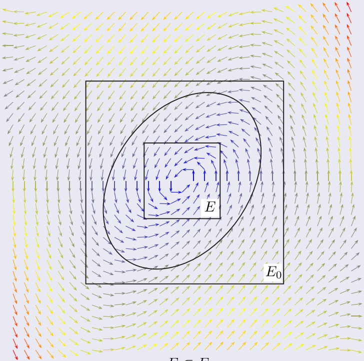
An equilibrum state x_∞ is *asymptotically* (E, E_0)*-stable* if

- $\varphi^{\mathbb{R}^+}(E) \subset E_0$
- $\varphi^{\infty}(E) = \{x_\infty\}$



$E \subset E_0$

Definition

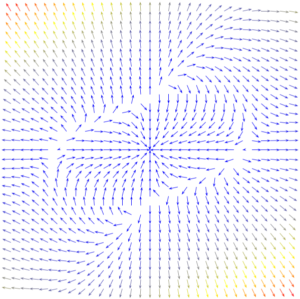
The attraction domain of x_∞ is the set

$$A_{x_\infty} = \{x \in D \mid \varphi^\infty(x) = x_\infty\}.$$

Solver.

Compute the attraction domain A_{x_∞}.

1.
2.
3.



Compute the attraction domain A_{x_∞}.

1. Show that there exists an unique equilibrium point x_∞,
2.
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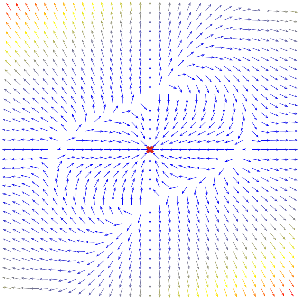

Compute the attraction domain A_{x_∞}.

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2. Prove that x_∞ is asymptotically stable and compute a neighborhood of x_∞ included in the attraction domain. (Alg. A)
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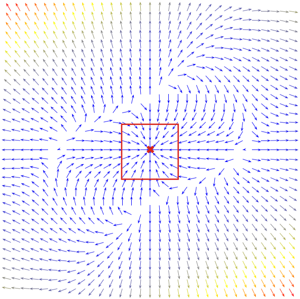

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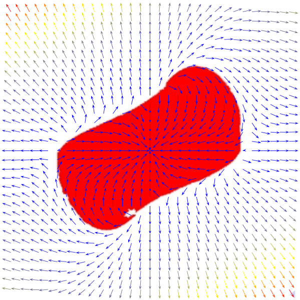
2. Prove that x_∞ is asymptotically stable and compute a neighborhood of x_∞ included in the attraction domain.
3. Discretize the vector field to compute a sequence A_n such that $A_n \rightarrow_{n\rightarrow\infty} A_{x_\infty}$ where A_{x_∞} is the attraction domain of x_∞. (Alg. B)



Positivity

A proof that $\forall x \in [x], f(x) \geq 0$.

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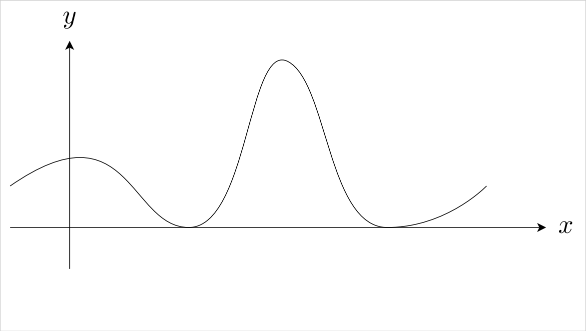
- $\forall x \in [x], f(x) = 0$: algebra calculus.
- In other cases ?


y
x

Algebra calculus is not enought . . .

In the cases where function are not polynomials.

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In the general case, one only has :

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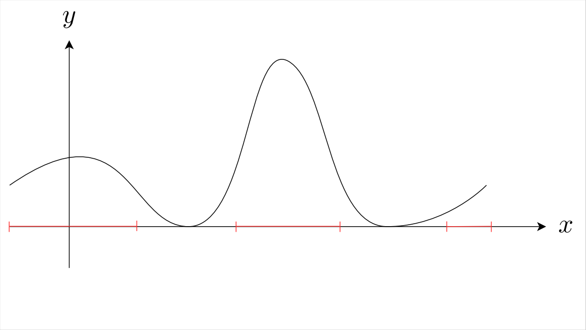
In the general case, one only has :

$$f([x]) \subsetneq [f]([x]).$$

- multiple occurrence of variables.
- outward rounding.


y
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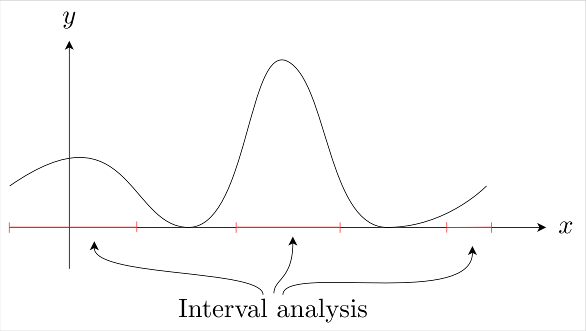
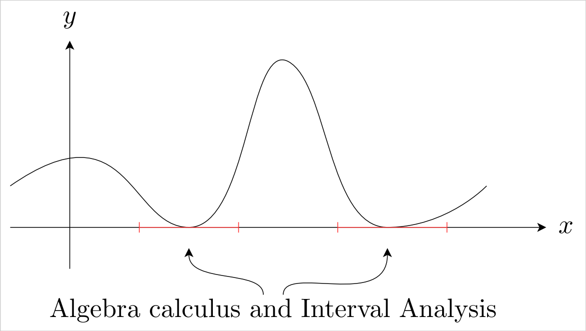

y
x
Interval analysis


y
x
Algebra calculus and Interval Analysis

Theorem

Let $x_0 \in E$ where E is a convex set of $\mathbb{R}^n$, and $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$. One has :

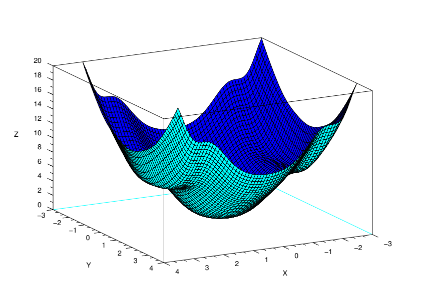
If

1. $\exists x_0$ such that $f(x_0) = 0$ and $Df(x_0) = 0$.
2. $\forall x \in E, D^2 f(x) > 0$.

then $\forall x \in E, f(x) \geq 0$.

Example

To prove that $f(x) \geq 0, \forall x \in [-1/2, 1/2]^2$
where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by
$f(x, y) = -\cos(x^2 + \sqrt{2}\sin^2 y) + x^2 + y^2 + 1.$



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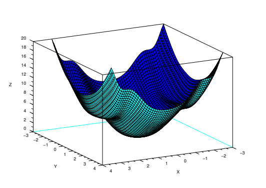
Let $f:\mathbb{R}^2 \to \mathbb{R}$ the function defined by :
$f(x,y) = -\cos(x^2 + \sqrt{2}\sin^2 y) + x^2 + y^2 + 1.$

1. One has : $f(0,0) = 0$ and $\nabla f(0,0) = 0$

$$\nabla f(x,y) = \begin{pmatrix} 2x(\sin(x^2+\sqrt{2}\sin^2 y)+1) \\ 2\sqrt{2}\cos y \sin y \sin\left(\sqrt{2}\sin^2 y + x^2\right) + 2y \end{pmatrix}.$$



$$\nabla^2 f = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$\begin{aligned} a_{1,1} &= 2\sin\left(\sqrt{2}\sin^2 y + x^2\right) \\ &\quad +4x^2\cos\left(\sqrt{2}\sin^2 y + x^2\right) + 2. \\ a_{2,2} &= -2\sqrt{2}\sin^2 y\sin(\sqrt{2}\sin^2 y + x^2) \\ &\quad +2\sqrt{2}\cos^2 y\sin(\sqrt{2}\sin^2 y + x^2) \\ &\quad +8\cos^2 y\sin^2 y\cos(\sqrt{2}\sin^2 y + x^2) + 2. \\ a_{1,2} = a_{2,1} &= = 4\sqrt{2}x\cos y\sin y \\ &\quad \cos\left(\sqrt{2}\ \sin y^2 + x^2\right). \end{aligned}$$

Evaluation with interval analysis gives : $\forall x \in [-1/2, 1/2]^2$, $\nabla^2 f(x) \subset [A]$

$$[A] = \begin{pmatrix} [1.9, 4.1] & [-1.3, 1.4] \\ [-1.3, 1.4] & [1.9, 5.4] \end{pmatrix}.$$

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Definition

A symmetric matrix A is positive definite if

$$\forall x \in \mathbb{R}^n - \{0\}, x^T A x > 0$$

S^{n+} denite the set of $n \times n$ symmetric positive definite matrices.

Definition

A set of symmetric matrices $[A]$ is an interval of symmetric matrices if :

$$[A] = \{(a_{ij})_{ij}, a_{ij} = a_{ji}, a_{ij} \in [a]_{ij}\}$$

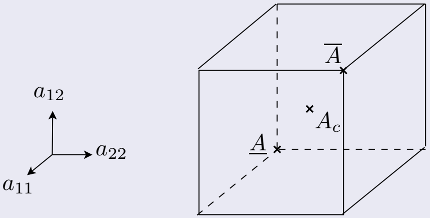
i.e.

$$[\underline{A}, \overline{A}] = \left\{A \text{ symmetric}, \underline{A} \leq A \leq \overline{A}\right\}.$$

Example

Using $\mathbb{R}^2$ is a symmetric matrix A

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}$$

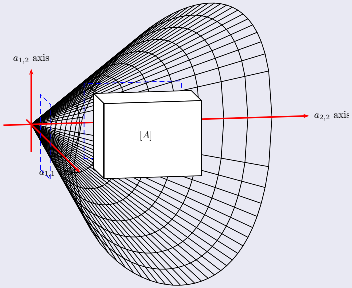


Remark - Rohn

Let $V([A])$ finite set of corners of $[A]$. Since S^{n+} and $[A]$ are convex subset of S^n :

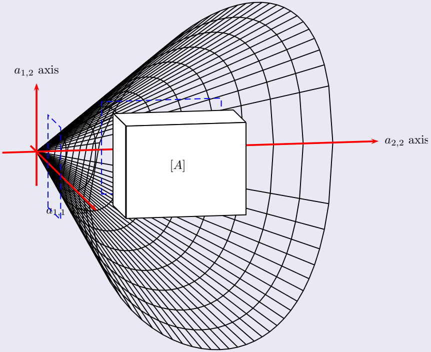
$$[A] \subset S^{n+} \Leftrightarrow V([A]) \subset S^{n+}$$

S^n is a vector space of dimension $\frac{n(n+1)}{2}$. $\#V([A]) = 2^{\frac{n(n+1)}{2}}$.



Theorem- Adefeld

Let $[A]$ a symmetric interval matrix
and $C = \{z \in \mathbb{R}^n \text{ tel que } |z_i| = 1\}$
If $\forall z \in C$, $A_z = A_c + \mathrm{Diag}(z)\Delta\mathrm{Diag}(z)$ is positive definite
then $[A]$ is positive definite.



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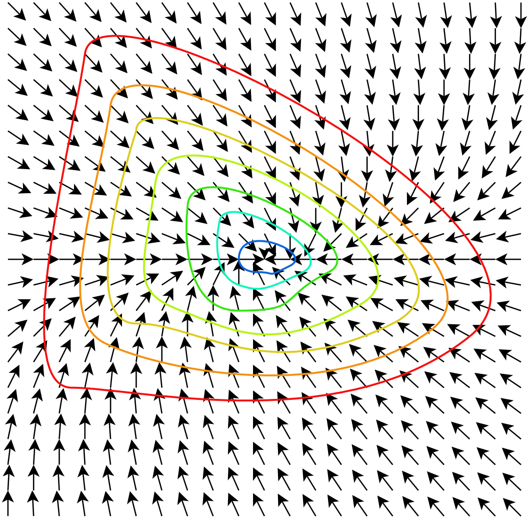
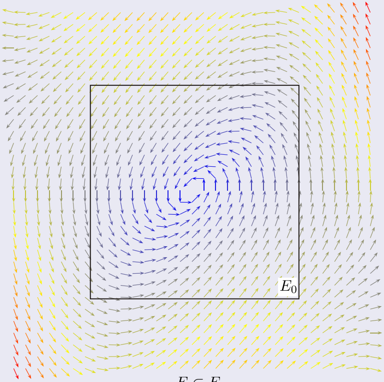
3. $\langle \nabla L(x), f(x) \rangle < 0, \ \forall x \in E - \{x_\infty\}$



Lyapunov theorem

Let E_0 a compact subset of $\mathbb{R}^n$ and $x_\infty \in E_0$.
If $L : E_0 \to \mathbb{R}$ is of Lyapunov ($\dot{x} = f(x)$) then there exists a subset $E \subset E_0$ such that x_∞ is asymptotically (E, E_0)-stable.



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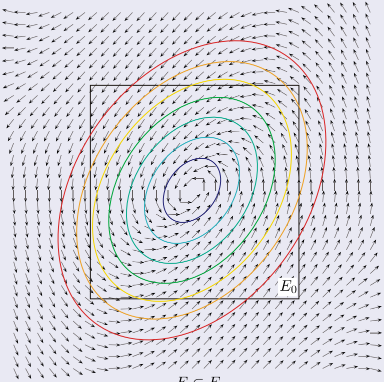

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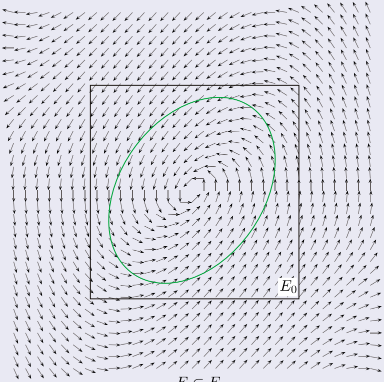

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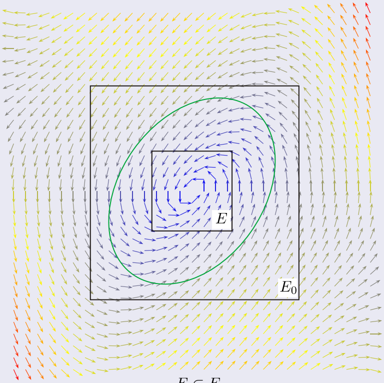

$E \subset E_0$

In the linear case :

$$\dot{x} = Ax \quad (2)$$

With $L = x^T W x$, $W \in S^n$
one has $\langle \nabla L(x), f(x) \rangle = x^T (A^T W + WA) x$.

Lyapunov conditions

S^n is the set of symmetric matrices.
S^{n+} is the set of positive definite symmetric matrices.

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Lyapunov conditions

1. $W \in S^{n+}$.
2. $-(A^T W + WA) \in S^{n+}$.

S^n is the set of symmetric matrices.
S^{n+} is the set of positive definite symmetric matrices.

Theorem

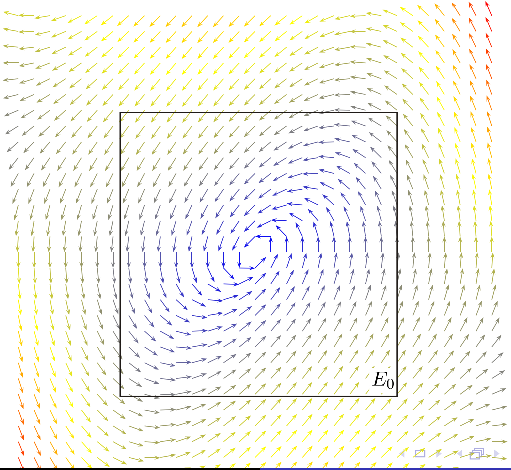
Let $\dot{x} = Ax$, O is asymptotically stable if and only if the solution W of the equation

$$A^T W + WA = -I$$

is positive definite.

Algorithm A

Step 1. Prove that E_0 contains a unique equilibrium state.
Step 2. Find $[x_\infty] \subset E_0$ such that $x_\infty \in [x_\infty]$.



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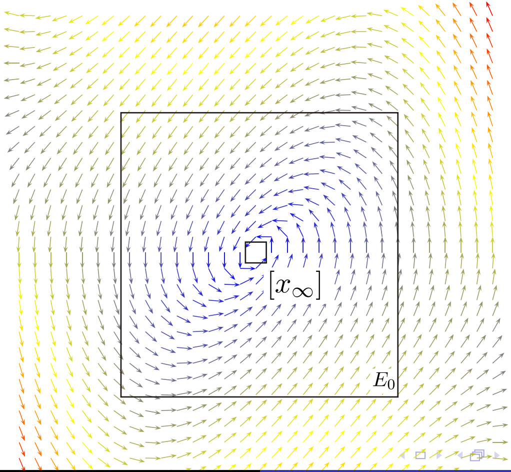
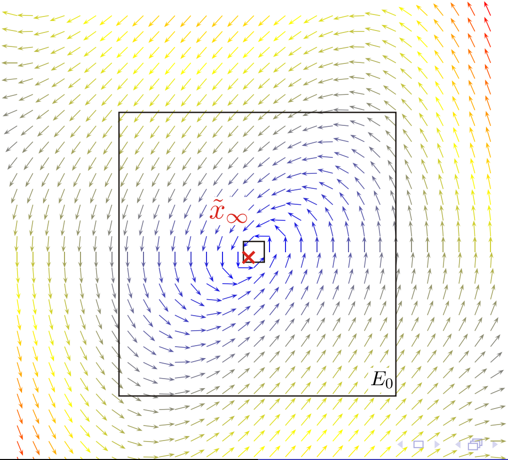


Algorithm A

Step 3. Linearize around x_∞ with $\tilde{x}_\infty$ ($\tilde{x}_\infty \in [x_\infty]$) :

$$\overline{(x - x_\infty)}^{\,\cdot} = Df(\tilde{x}_\infty)(x - x_\infty).$$



Algorithm A

Step 3. Linearize around x_∞ with $\tilde{x}_\infty$ ($\tilde{x}_\infty \in [x_\infty]$) :
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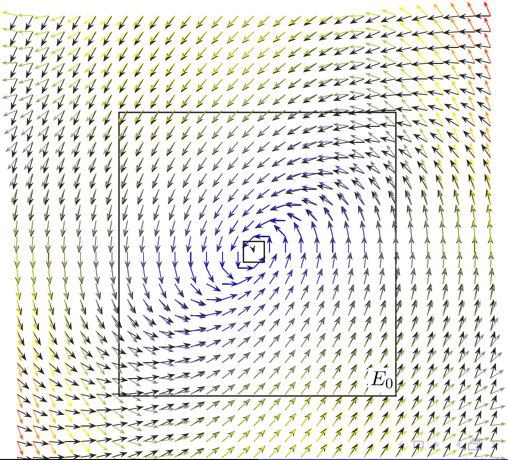
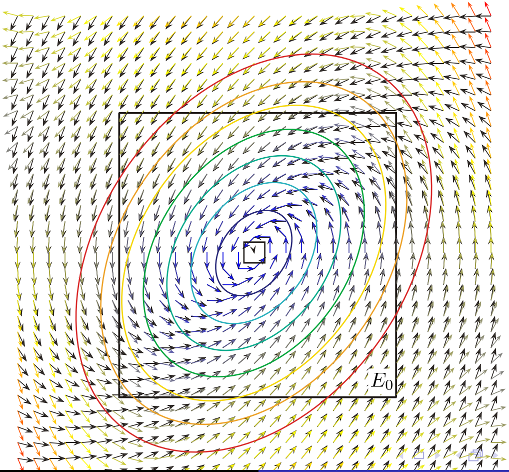


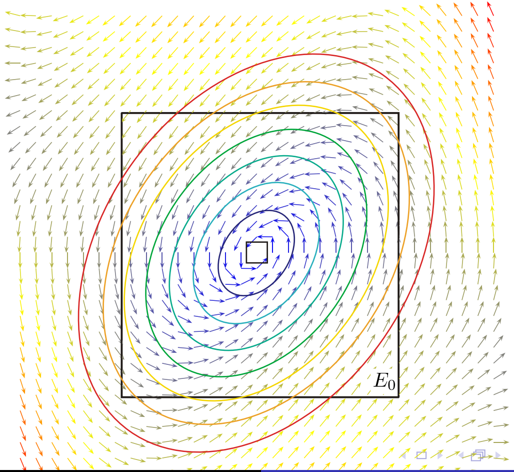
Algorithm A

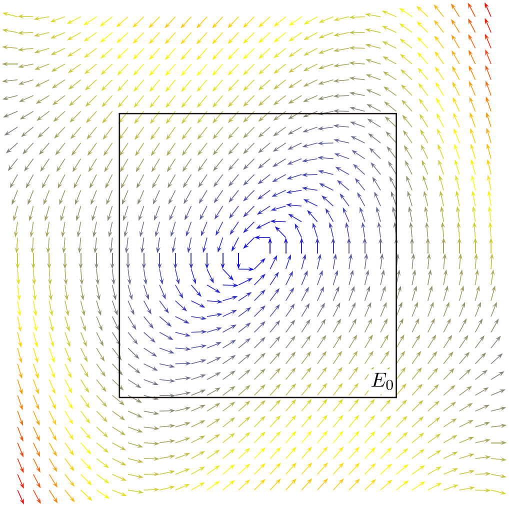
Step 4. Find a Lyapunov function L_{x_∞} for the linear system $Df(\tilde{x}_\infty)$.

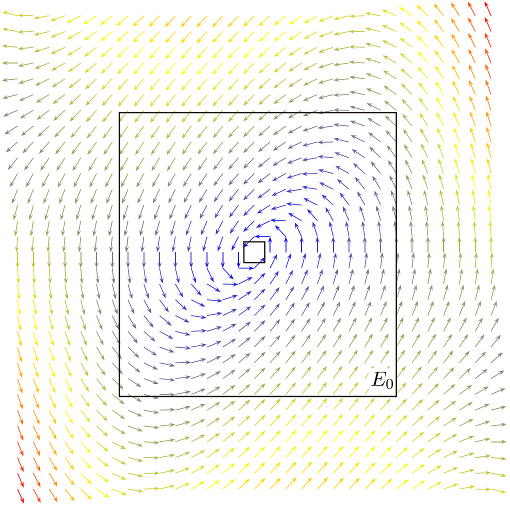


Algorithm A

Step 5. Check that L_{x_∞} is of Lyapunov for the non linear system $\dot{x} = f(x)$.




E_0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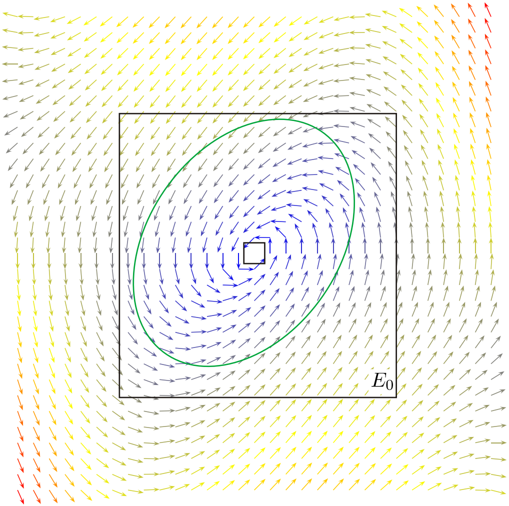
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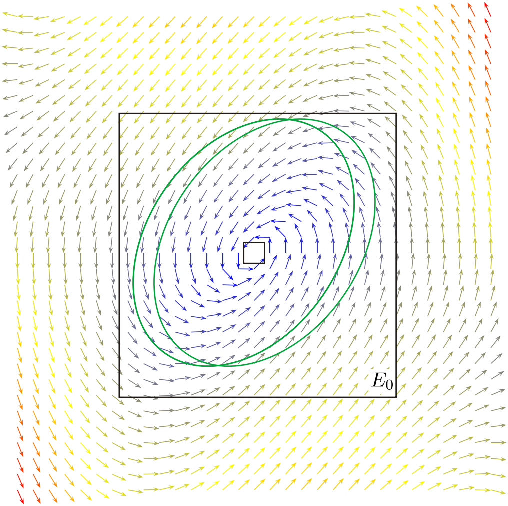
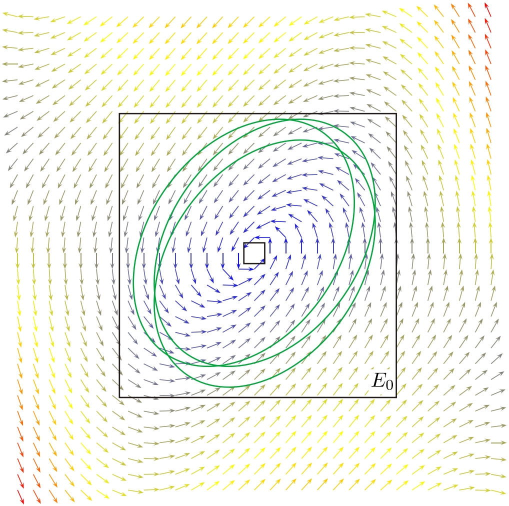

E_0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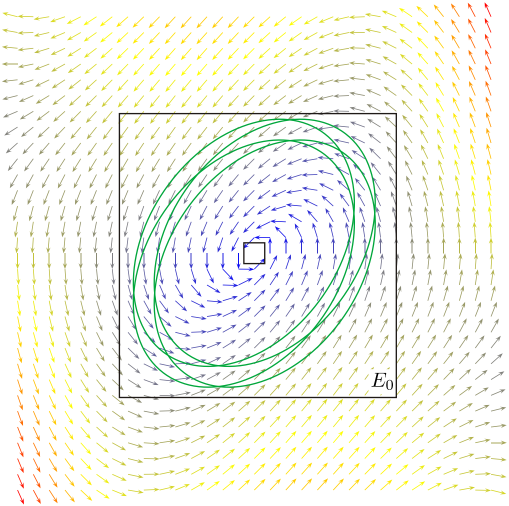
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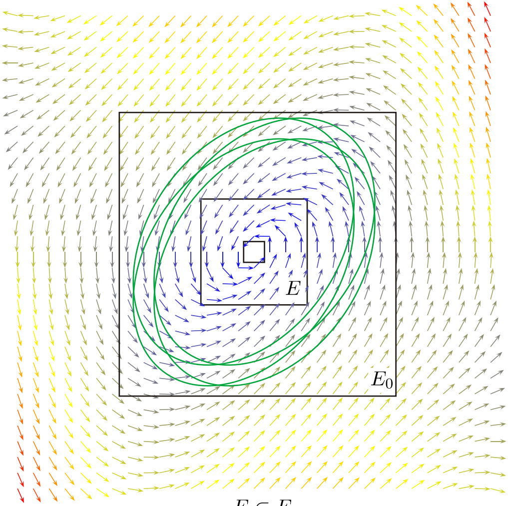
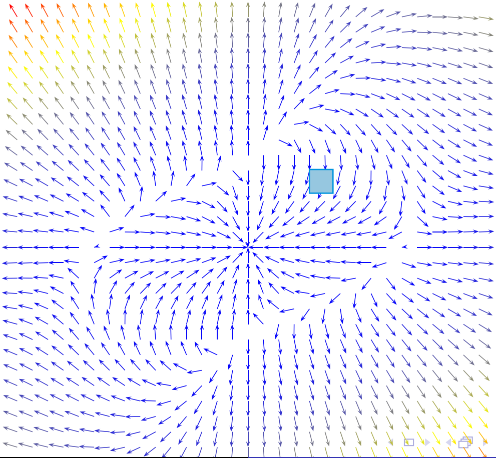


$E \subset E_0$

Picard-Linderlöf

Let $\dot{x} = f(x)$ and $t \in \mathbb{R}$, there exists a guaranteed method able to compute an inclusion function for $\varphi^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$.



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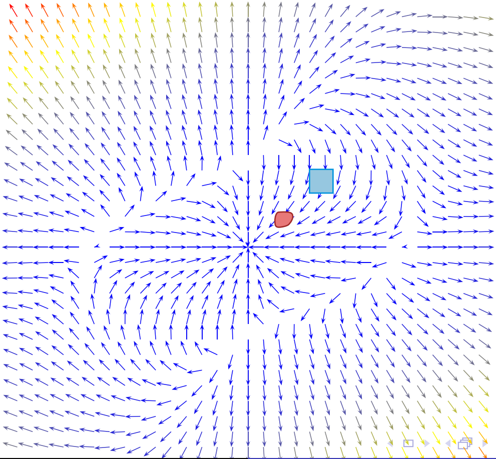

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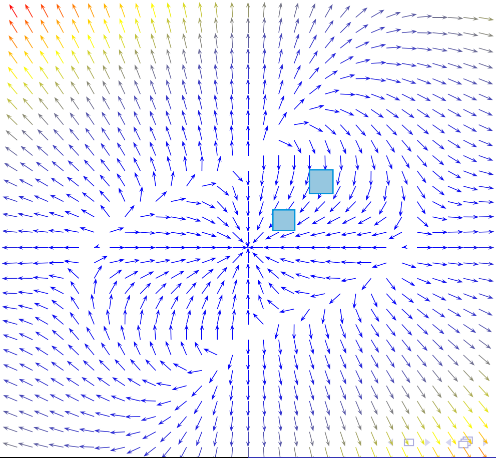
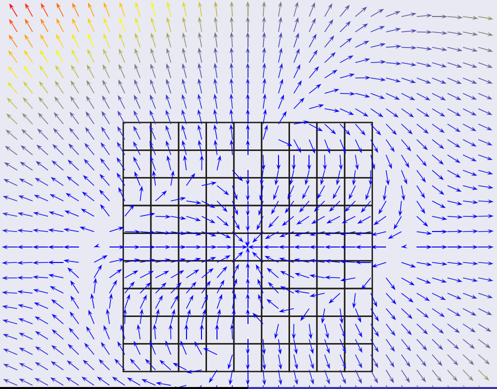


Definition

Let $t \in \mathbb{R}$, and $\{\mathbb{S}_i\}_{i \in I}$ a paving of $\mathbb{S}$, let us denote by $\mathcal{R}$ the relation on I defined by :

$$i\mathcal{R}j \;\Leftrightarrow \varphi^t(\mathbb{S}_i) \cap \mathbb{S}_j \neq \emptyset.$$



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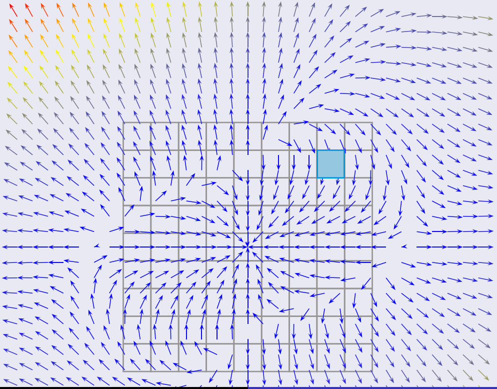

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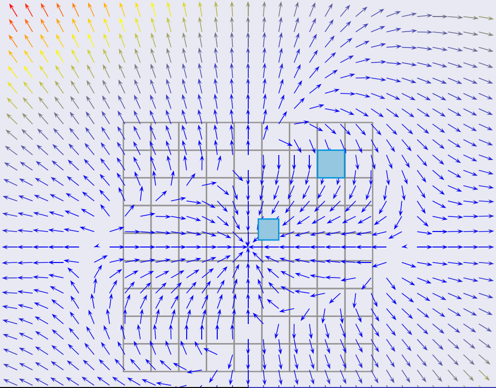

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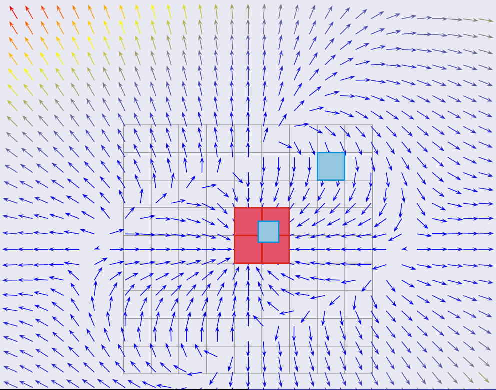

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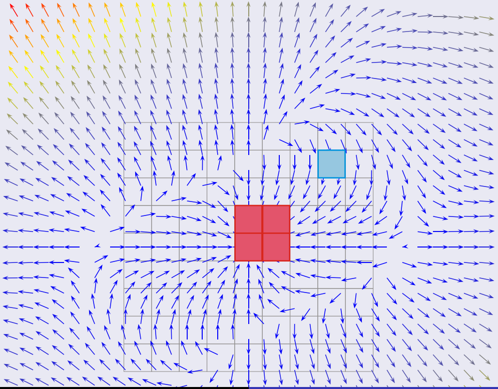
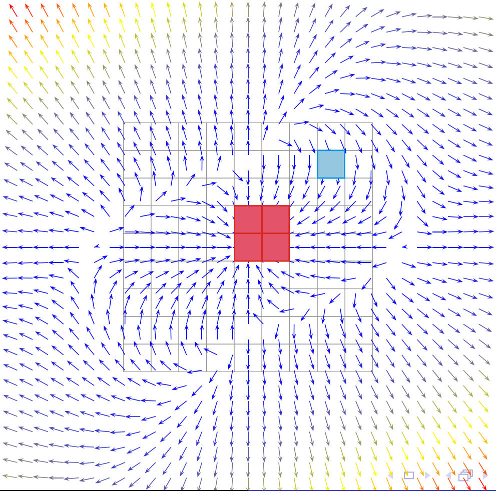


Proposition

If $\forall j \in I, i\mathcal{R}j \Rightarrow \mathbb{S}_j \subset A_{x_\infty}$ then $\mathbb{S}_i \subset A_{x_\infty}$.



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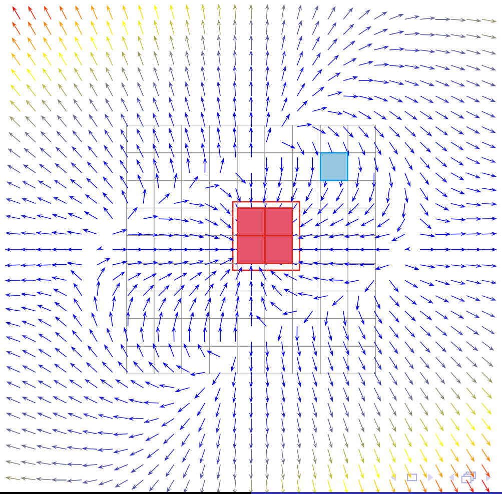

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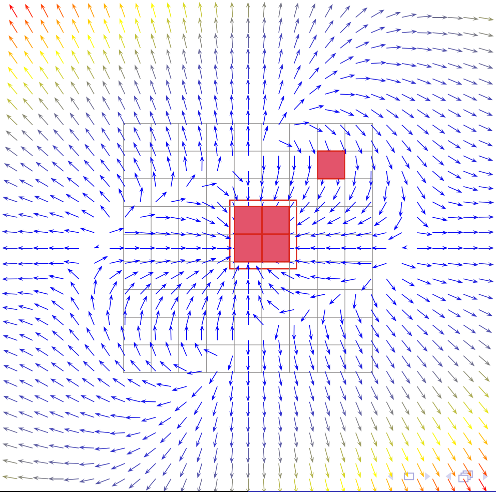

Algorithm B

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- $\dot{x} = f(x)$,
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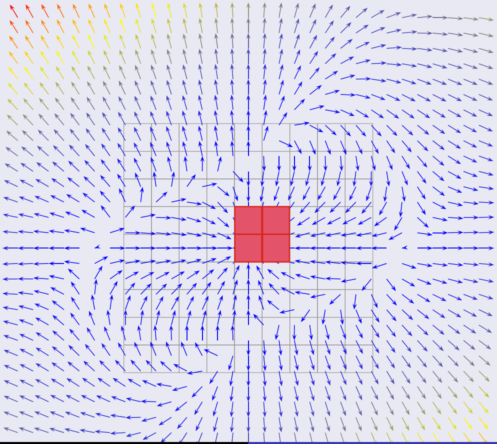
2. Compute the relation $\mathcal{R}$.
3. For each i of I, if

$$\forall j \in I, i\mathcal{R}j \Rightarrow \mathbb{S}_j \subset A$$

then $A := A \cup \mathbb{S}_i$.

Example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(x^2 - xy + 3y^2 - 1) \\ y(x^2 - 4yx + 3y^2 - 1) \end{pmatrix}$$



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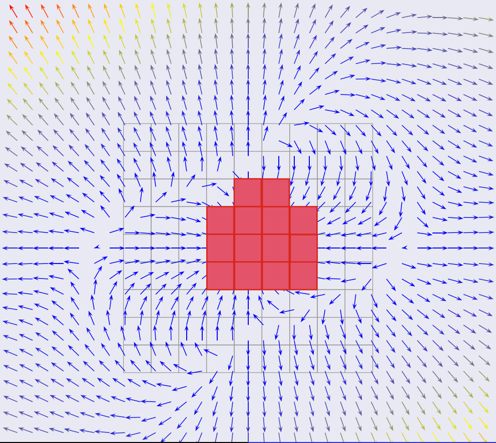

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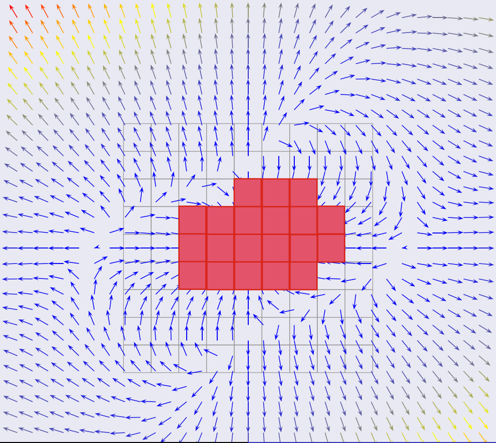

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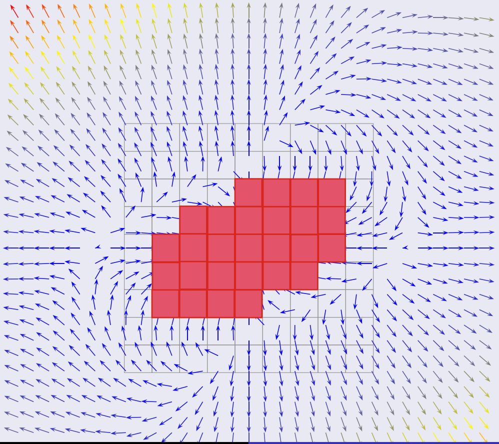

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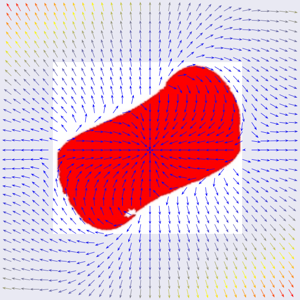

- Thank you for your attention !