Using interval methods and guaranteed integration of ODEs for the verification of embedded software.

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Motivation

Context of this work : static analysis of safety critical softwares.

Behaviour of the program:

- collects inputs from a sensor;
- computes the response to that input;
- sends orders to actuators.

Abstract interpretation based static analysis:

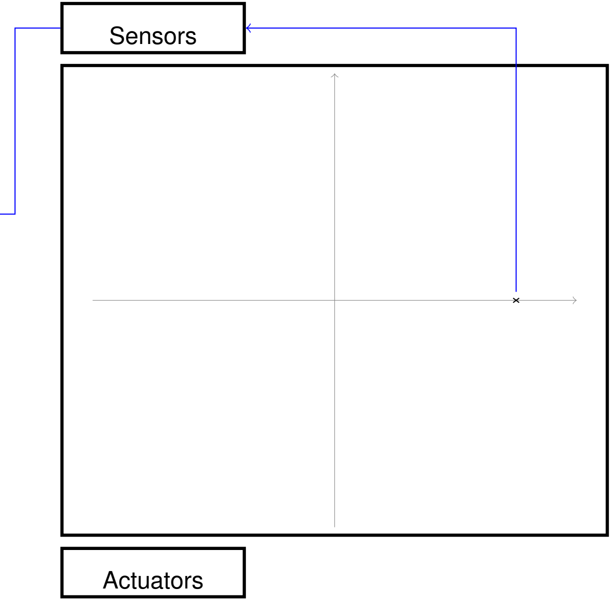
- computes invariants, i.e. properties that are true for all possible executions.
- usualy overapproximates the inputs by a constant interval.

Starting point of this work

Embedded programs are a part of a hybrid system and verification techniques don't take that information into account.

Example of an execution.

```
int main() {
 while(true) {
   sens.y?X
   X'=update(X);
   if (X'<0)
    act.cmd!1
   if (X'>15)
    act.cmd!0
   wait 0.01
 }
}
```



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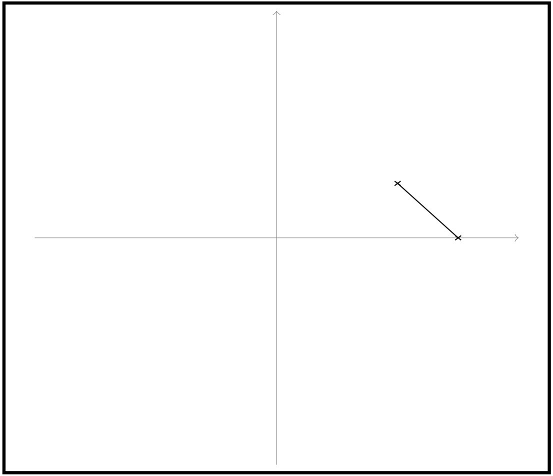

Actuators

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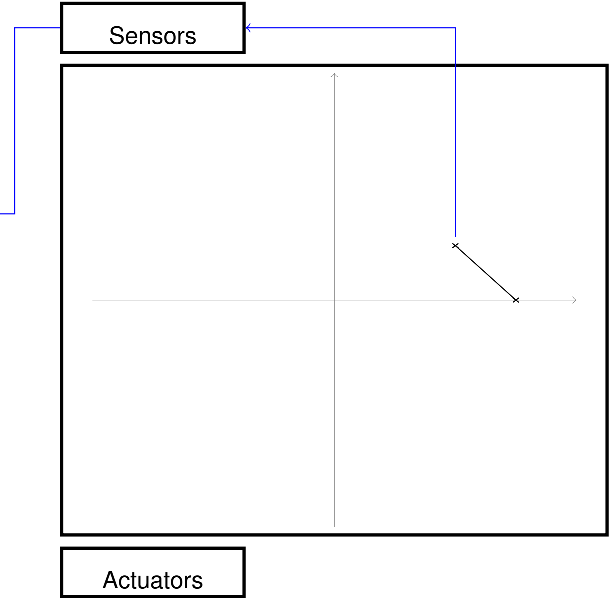

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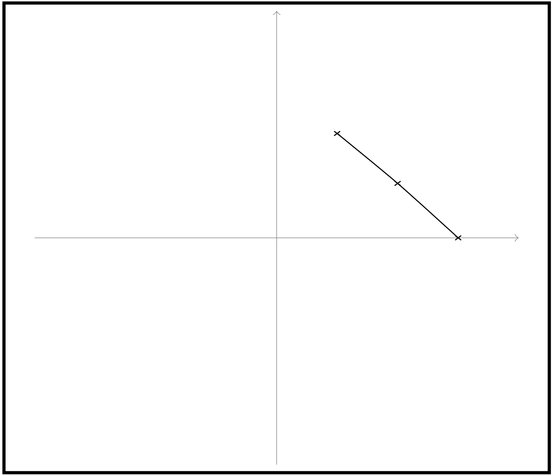

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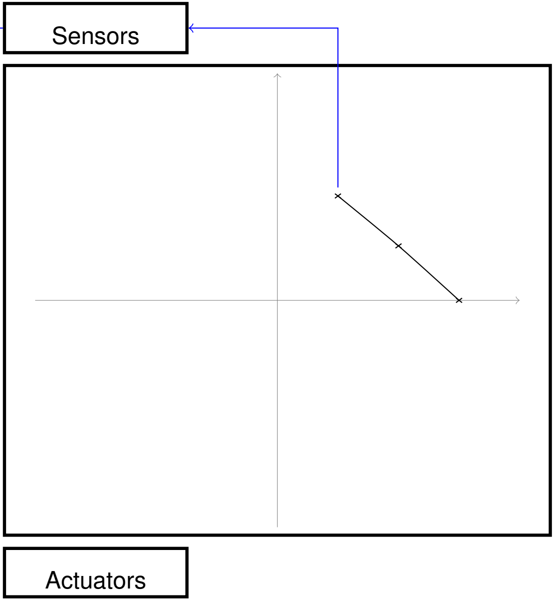

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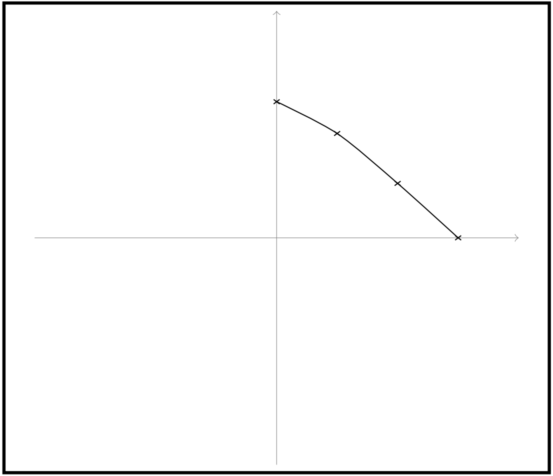

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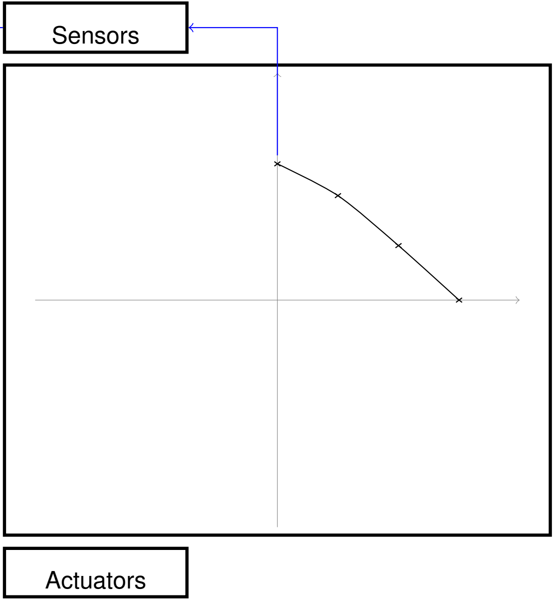

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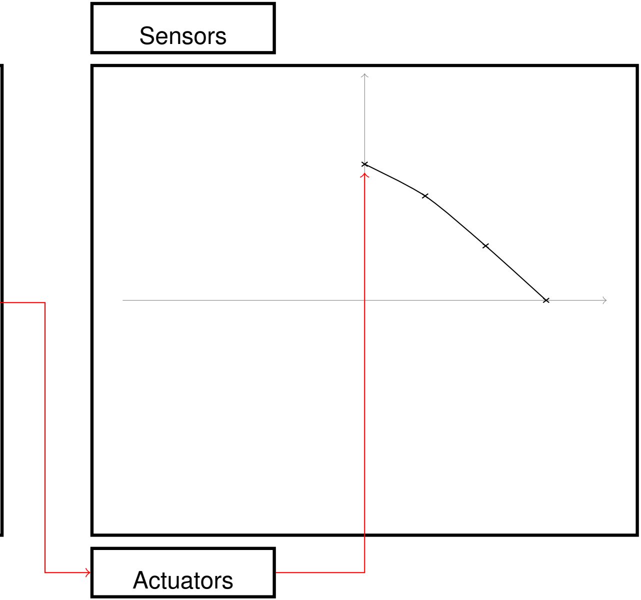

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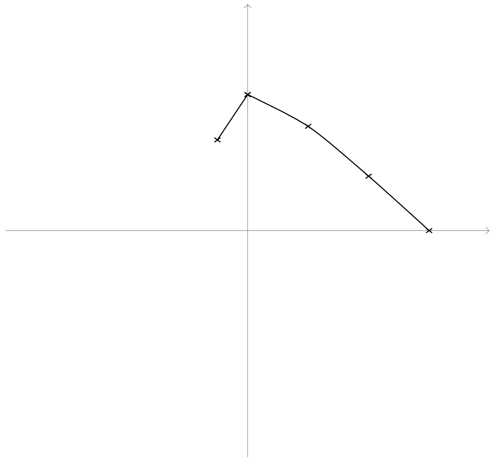

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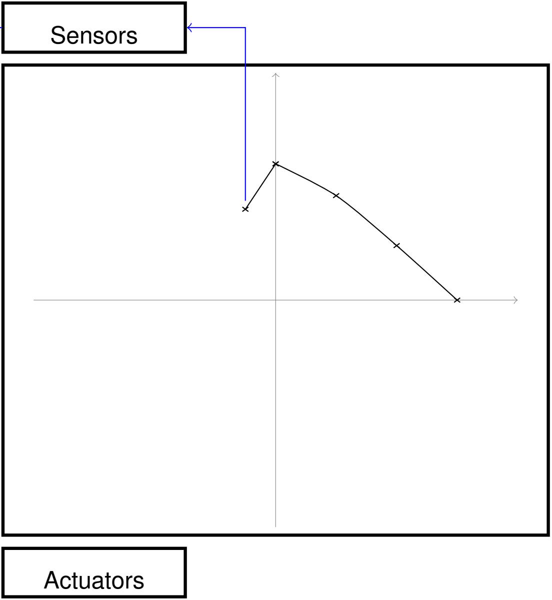

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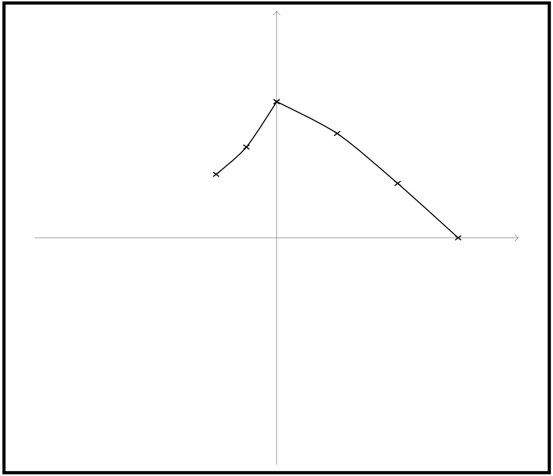

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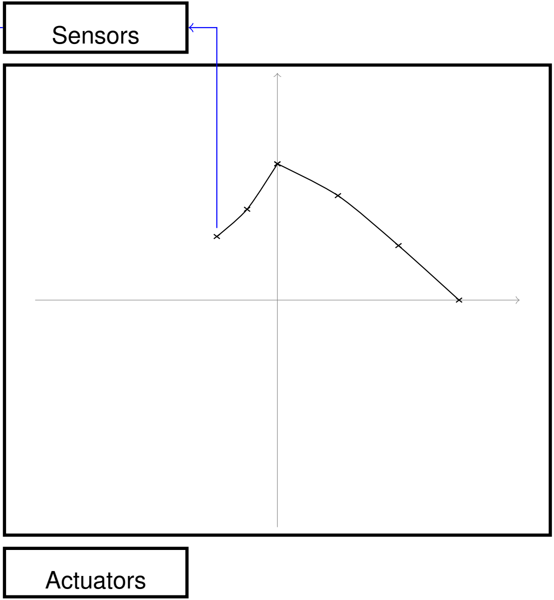

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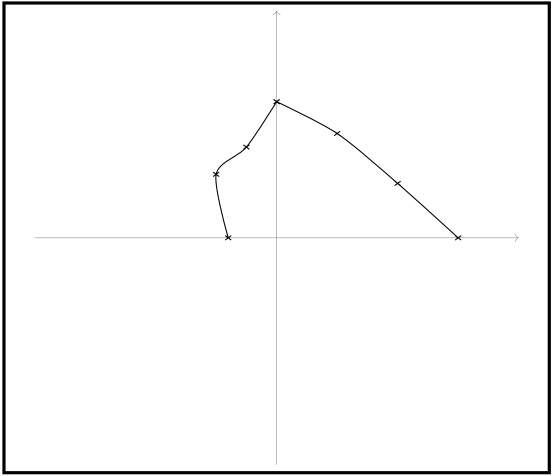

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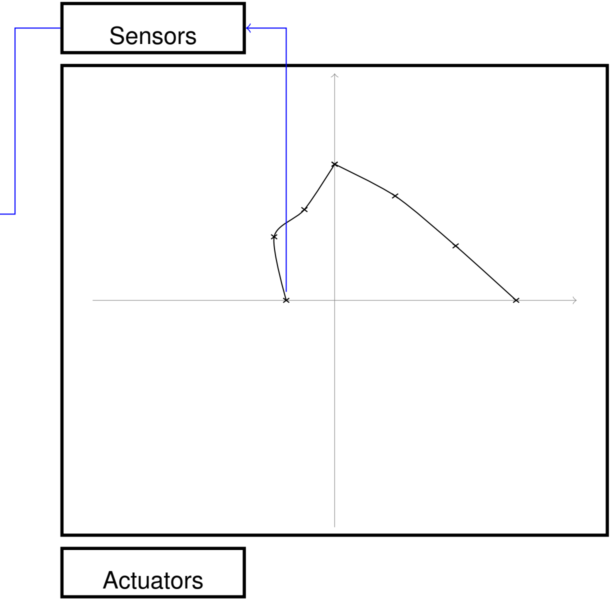

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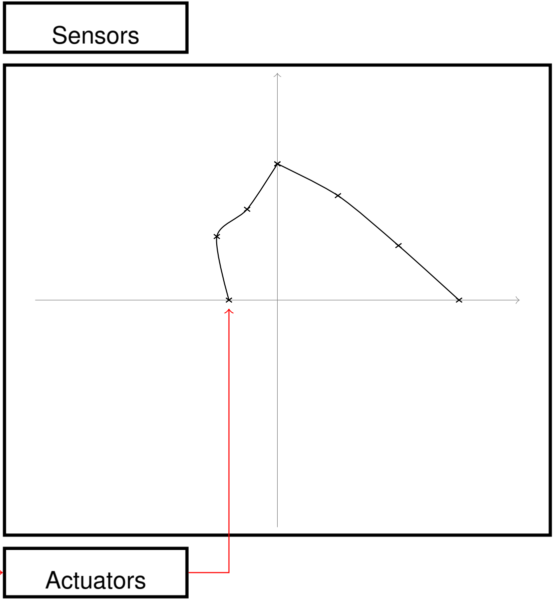

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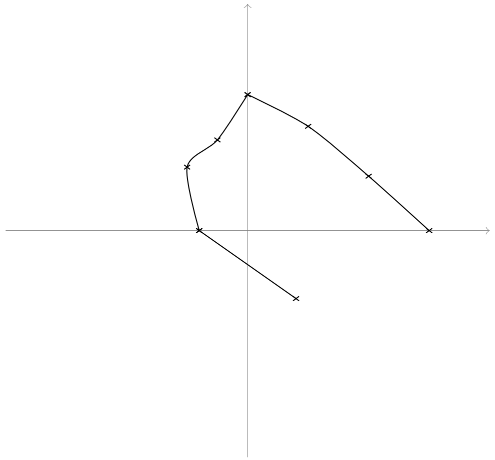

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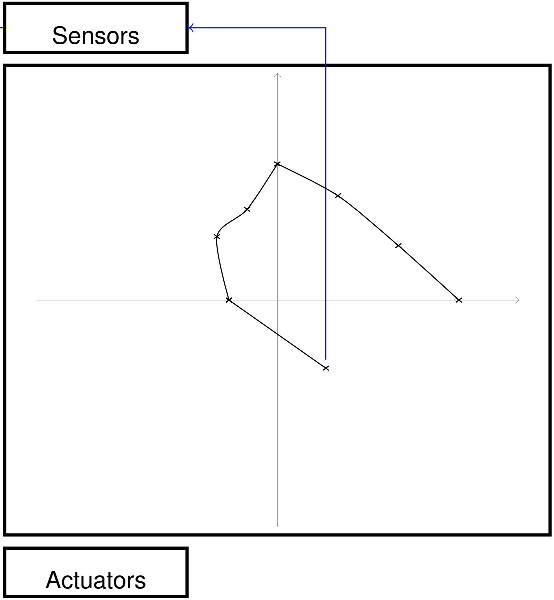

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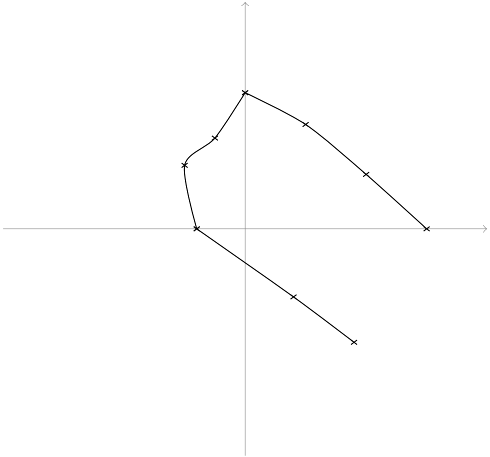

Actuators

Example of an analysis.

[-10,10]

```
int main() {
 while(true) {
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   act.cmd!1
  if (X'>15)
   act.cmd!0
  wait 0.01
 }
}
```

- Overapproximation of the continuous inputs.
- Forget about the actuators.
- Standard abstract interpretation analysis.
- Problem : overapproximation of the results because we lose the relation between two consecutives `sens`.

Actuators

Our method for the analysis.

1. Analyse the program to detect the "switching regions"

```
int main() {
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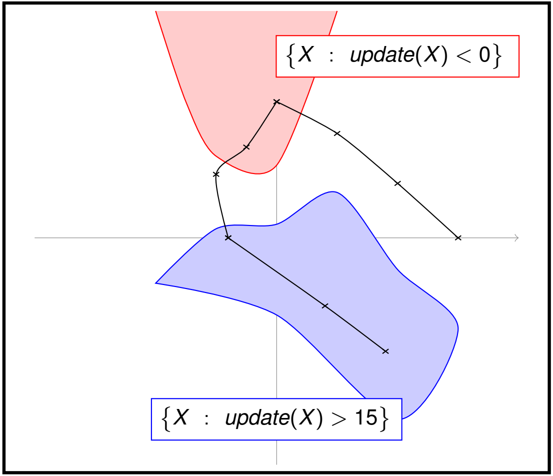
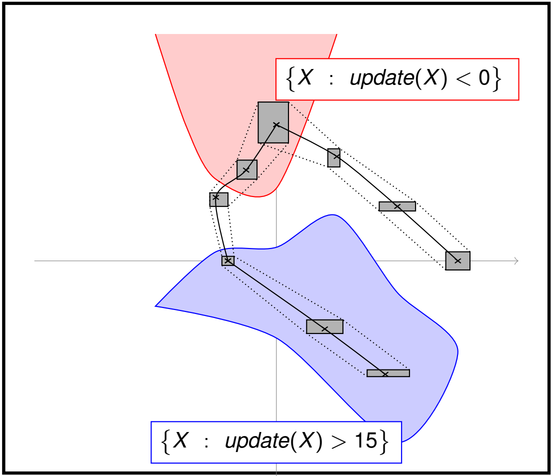


Our method for the analysis.

1. Analyse the program to detect the “switching regions”

2. Use guaranteed integration to overapproximate the trajectories.



Detecting the switching regions: abstracting a boolean function.

```
int main() {
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  wait 0.01
 }
}
```

- A switch occur when an `act` statement is executed.
- We need to test if, given a value `X` for the sensor inputs, the statement on line n will be executed.
- Boolean function φ:

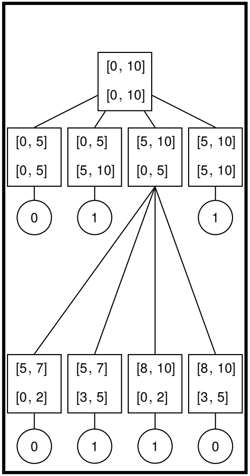
  given an input X, is the statement on line n executed?

- We need to construct the zones $\{X : \varphi(X) = 0\}$ and $\{X : \varphi(X) = 1\}$, i.e. build a representation of φ.

Problems:

- floating point intervals and not integer functions;
- representation of boolean functions;
- computing this representation.

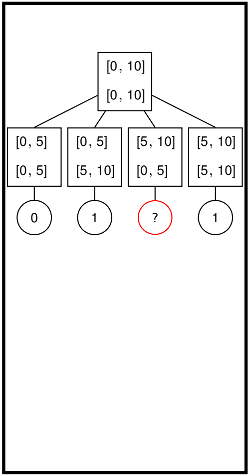
Representation of a boolean function : quad trees



Quad tree:

+ representation of a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$;
+ two kinds of nodes :
  - non-terminal nodes with a box as attribute and 2^n children;
  - terminal node with a value $v \in \{0, 1\}$ as attribute.

Representation of a boolean function : quad trees



Quad tree:

+ representation of a function $f : \mathbb{R}^n \to \mathbb{B}$;
+ two kinds of nodes :
  - non-terminal nodes with a box as attribute and 2^n children;
  - terminal node with a value $v \in \{0, 1\}$ as attribute.
+ abstract quad trees: terminal value $v \in \{0, 1, ?\}$.
+ the precision depends on the depth.

A simple branch and bound algorithm.

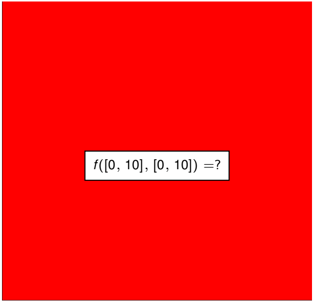
Problem

Given a function $f : \mathbb{R}^n \to \mathbb{B}$ and a precision N, construct the corresponding abstract quad tree.

Example :

$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \vee d((x, y), (7, 1.2)) \leq 1$

- $x = [0, 10]$ and $y = [0, 10]$
- $d((x, y), (3.5, 6)) = [0, 78.25]$
- $d\big((x, y), (3.5, 6)\big) \leq 1 =?$
- $d\big((x, y), (7, 1.2)\big) \leq 1 =?$
- $f(x, y) =?$



Iterate 1

A simple branch and bound algorithm.

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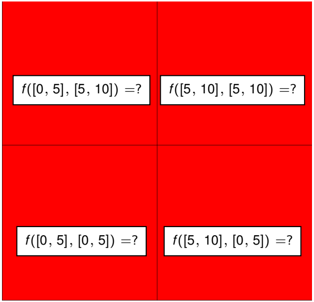
$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \vee d((x, y), (7, 1.2)) \leq 1$

- $x = [0, 5]$ and $y = [0, 5]$
- $d((x, y), (7, 1.2)) \subseteq [4, \infty[$
- $d\big((x, y), (7, 1.2)\big) \leq 1 = 0$
- $d((x, y), (3.5, 6)) = [1, 48.25]$
- $d\big((x, y), (3.5, 6)\big) \leq 1 = ?$
- $f(x, y) = ?$



Iterate 2

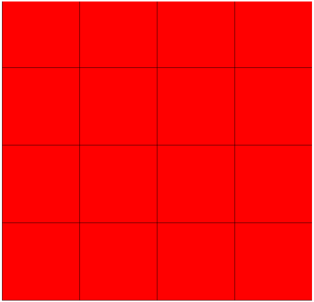
A simple branch and bound algorithm.

Problem

Given a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$ and a precision N, construct the corresponding abstract quad tree.

Example :

$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \vee d((x, y), (7, 1.2)) \leq 1$



Iterate 4

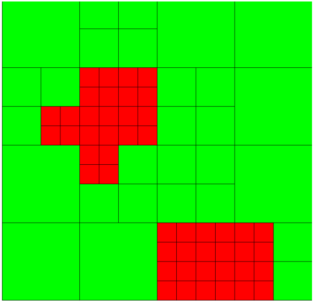
A simple branch and bound algorithm.

Problem

Given a function $f : \mathbb{R}^n \to \mathbb{B}$ and a precision N, construct the corresponding abstract quad tree.

Example :

$f(x,y) = d((x,y),(3.5,6)) \leq 1 \vee d((x,y),(7,1.2)) \leq 1$



Iterate 8

A simple branch and bound algorithm.

Problem

Given a function $f : \mathbb{R}^n \to \mathbb{B}$ and a precision N, construct the corresponding abstract quad tree.

Example :

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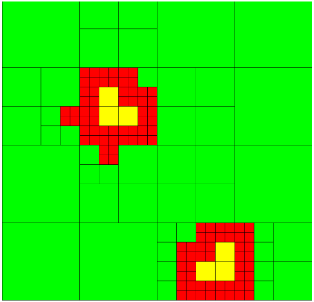

Iterate 10

A simple branch and bound algorithm.

Problem

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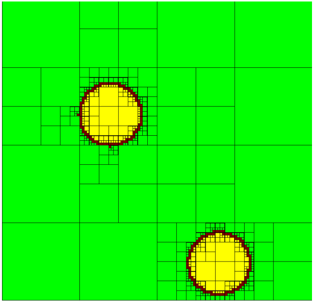
Example :

$$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \vee d((x, y), (7, 1.2)) \leq 1$$



Iterate 16

Experimental results.

For the reachability problem.

The boolean function is a simple interval based interpreter that tests if a given line of code is executed.

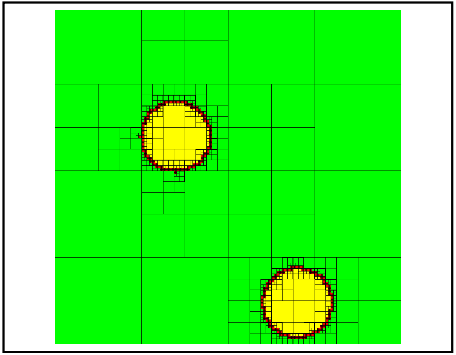
```
double x,y;
/*!npk x = [0,10] */
/*!npk y = [0,10] */


double distance (double x,double y,
                 double a,double b) {
  double res;
  res = (x-a)*(x-a) + (y-b)*(y-b);
  return res;
}


void main() {
  double res,res2;
  int v;

  res = distance (x,y,3.5,6);
  res2 = distance (x,y,7,1.2);
  if ((res<=1)||(res2<=1)) {
    v=0;
  }

}
```



Experimental results.

For the reachability problem.

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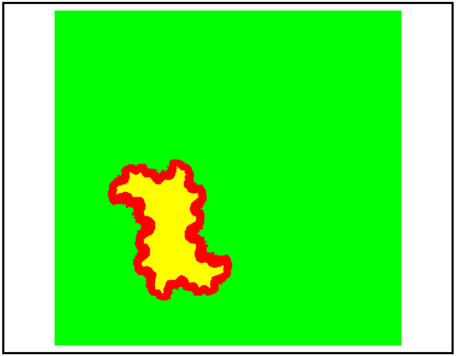
```
/* !npk x = [-2,4] */
/* !npk y = [-2,4] */
double x,y;

int update (double x,double y) {
  double a,b,xt,yt,d;
  int i = 0;
  a = 0.32; b = 0.43; d = 0;
  while (d<10) {
    xt = x*x-y*y+a;
    yt = 2*x*y + b;
    x = xt;    y = yt;
    d=x*x+y*y;
    if (i>=10)
      d = 30;
    i++;
  }
  return i;
}

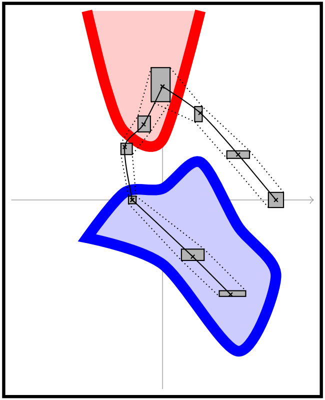
void main() {
  int v = 0 , i = 0;
  i = update (x,y);
  if (i>=11)
    v = 1;
}
```



Guaranteed integration of ODEs.

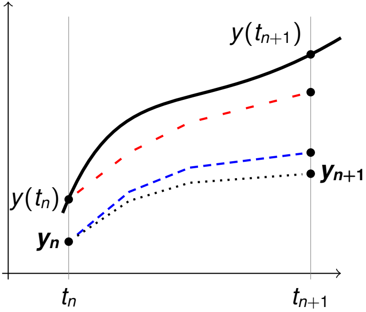
Result of the first analysis:

- we have a *spatial* criteria to detect the mode switchings;
- we know the sampling *times*, i.e. instants when the sensors return a value.



- The evolution of the continuous variables is given by ODEs.
- Given an ODE $\dot{y} = f(y)$ and a time t_n, we want to compute a box $\boldsymbol{y_n}$ such that $y(t_n) \in \boldsymbol{y_n}$.
- Existing validated methods: based on Taylor series decomposition.
- Existing non validated numerical methods: Euler, Runge-Kutta, Heun, ...

A new approach based on non validated methods.



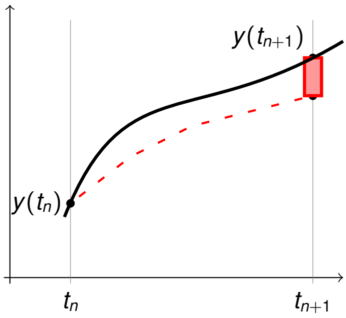
How to turn a numerical method into a validated one :

- A numerical method is $y_{n+1} = \Phi(y_n, h_n)$.
- Three sources of errors :
  - if $y_n = y(t_n)$ is exact, y_{n+1} differs from $y(t_{n+1})$;
  - if y_n has an error ϵ_n, how is it propagated into the next step;
  - floating point computations.

First implementation: with RK4

$$k_1 = f(y_n) \qquad k_2 = f\left(y_n + \frac{h}{2}k_1\right)$$
$$k_3 = f\left(y_n + \frac{h}{2}k_2\right) \qquad k_4 = f(y_n + hk_3)$$
$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

GRKLib : three sources of errors.



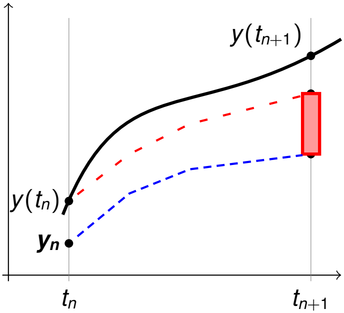
1. One step error:

- the function that gives y_{n+1} differs from $y(t)$.
- $y_{n+1} = \varphi(t_{n+1})$
- $\forall i \in [0,4],\ \frac{d^i y}{dt^i}(t_n) = \frac{d^i \varphi_n}{dt^i}(t_n)$
- $\exists t' \in [t_n, t_{n+1}]$ such that $y(t_{n+1}) - \varphi_n(t_{n+1}) = \frac{d^5(y-\varphi_n)}{dt^5}(t')$

$$y(t_{n+1}) - \varphi_n(t_{n+1}) \in \frac{d^4 f}{dt^4}(\tilde{y}) - \frac{d^5 \varphi_n}{dt^5}([t_n, t_{n+1}])$$

$$\epsilon_{n+1} = \frac{d^4 f}{dt^4}(\tilde{y}) - \frac{d^5 \varphi_n}{dt^5}([t_n, t_{n+1}])$$

GRKLib : three sources of errors.



2. Error propagation:

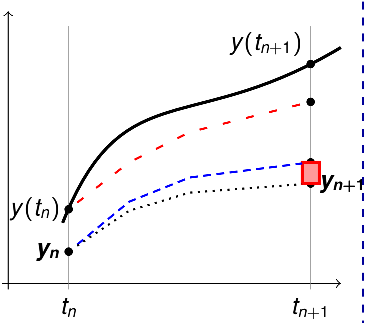
- starting point is $\boldsymbol{y_n}$ instead of $y(t_n)$,
- the propagated error is the difference between $\psi(y_n)$ and $\psi(y(t_n))$
- $\exists y \in [y_n, y(t_n)]$ such that $\chi_{n+1} = J(\psi, y).(y_n - y(t_n))$

$$\chi_n \in J(\psi, [y_n, y_n + \epsilon_n]).\epsilon_n$$

- We use QR-factorization technique to reduce the wrapping effect.

$$\epsilon_{n+1} = \frac{d^4 f}{dt^4}(\tilde{y}) - \frac{d^5 \varphi_n}{dt^5}([t_n, t_{n+1}]) + J(\psi, [y_n, y_n + \epsilon_n]).\epsilon_n$$

GRKLib : three sources of errors.



3. Computation error:

- we use floating point numbers and not real numbers to compute y_{n+1};
- we must overapproximate the computation erros;
- Solution: *global error arithmetic*

$$[\![a]\!]_{\mathbb{E}} = f_a + e_a\overrightarrow{\varepsilon_e} \;\; and \;\; [\![b]\!]_{\mathbb{E}} = f_b + e_b\overrightarrow{\varepsilon_e}$$

$$[\![a+b]\!]_{\mathbb{E}} = \uparrow_{\sim} (f_a + f_b) + (e_a + e_b + \downarrow_{\sim} (f_a + f_b))\overrightarrow{\varepsilon_e}$$

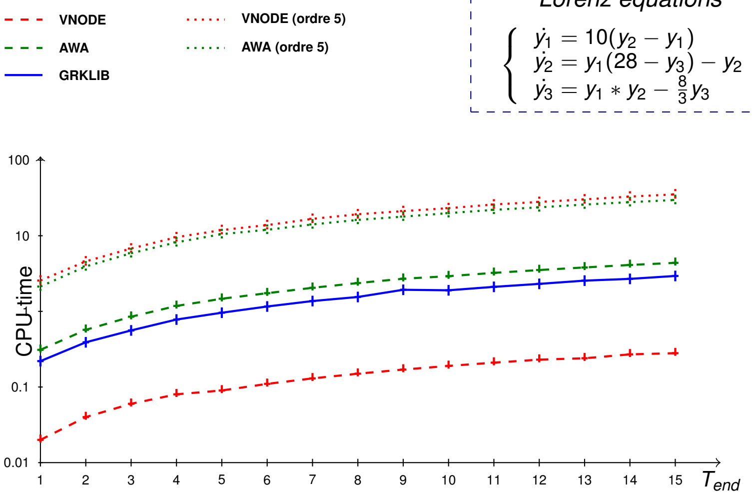
- We use this arithmetic to compute y_{n+1}, and thus get an interval E_{n+1} that contains all the computation erros.

$$\epsilon_{n+1} = \frac{d^4 f}{dt^4}(\tilde{y}) - \frac{d^5 \varphi_n}{dt^5}([t_n, t_{n+1}]) + J(\psi, [y_n, y_n + \epsilon_n]).\epsilon_n + E_{n+1}$$

Benchmarks: CPU time VS T_{end}

Lorenz equations

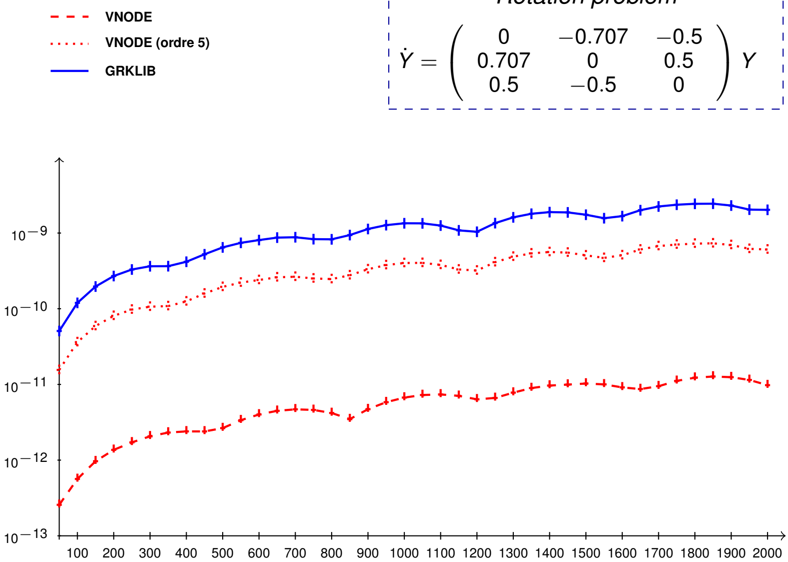
$$\begin{cases} \dot{y}_1 = 10(y_2 - y_1) \\ \dot{y}_2 = y_1(28 - y_3) - y_2 \\ \dot{y}_3 = y_1 * y_2 - \frac{8}{3} y_3 \end{cases}$$



Benchmarks: Enclosure width VS T_{end}

Rotation problem

$$\dot{Y} = \begin{pmatrix} 0 & -0.707 & -0.5 \\ 0.707 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{pmatrix} Y$$



Conclusion

Formal verification of embedded softwares

- considering the program alone showed its limits;
- we must take the physical environment into account.

Interval methods:

- allow to abstract the program and thus simplify the analysis;
- allow to compute a safe overapproximation of the continuous dynamics.

Future work:

- unify both analysis into a common tool;
- improve GRKLib by adding higher order methods and by using other domains (zonotopes).