Using interval methods and guaranteed integration of ODEs for the verification of embedded software.

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Motivation

Context of this work: static analysis of safety critical softwares.

Behaviour of the program:
+ collects inputs from a sensor;
+ computes the response to that input;
+ sends orders to actuators.

Abstract interpretation based static analysis:
+ computes invariants, i.e. properties that are true for all possible executions.
+ usually overapproximates the inputs by a constant interval.

Starting point of this work
Embedded programs are a part of a hybrid system and verification techniques don’t take that information into account.
Example of an execution.

```c
int main() {
    while (true) {
        sens.y?X
        X' = update(X);
        if (X' < 0)
            act.cmd!1
        if (X' > 15)
            act.cmd!0
        wait 0.01
    }
}
```
Example of an execution.

```c
int main() {
    while(true) {
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}
```
Problems for the analysis of embedded software

Detection of the switching regions.

Guaranteed integration of the ODEs.

Conclusion.

Example of an execution.

```c
int main() {
    while(true) {
        if (sens.y?X
            X' = update(X);
            if (X'<0)
                act.cmd!1
            if (X' > 15)
                act.cmd!0
            wait 0.01
    }
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Example of an execution.

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    while(true) {
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}
```
Problems for the analysis of embedded software

Detected switching regions.

Guaranteed integration of the ODEs.

Conclusion.

---

Example of an analysis.

```c
int main() {
    while (true) {
        sens.y ? X
        X' = update(X);
        if (X' < 0)
            act.cmd!1
        if (X' > 15)
            act.cmd!0
        wait 0.01
    }
}
```

- Overapproximation of the continuous inputs.
- Forget about the actuators.
- Standard abstract interpretation analysis.
- **Problem**: overapproximation of the results because we lose the relation between two consecutives `sens`.

Actuators

[-10,10]
Our method for the analysis.

1. Analyse the program to detect the “switching regions”

```c
int main() {
    while(true) {
        X = update(X);
        if (X < 0) { act.cmd!1; }
        if (X > 15) { act.cmd!0; }
        wait 0.01
    }
}
```

\[ \{ X : \text{update}(X) < 0 \} \]

\[ \{ X : \text{update}(X) > 15 \} \]
Our method for the analysis.

1. Analyse the program to detect the “switching regions”

2. Use guaranteed integration to overapproximate the trajectories.
Detecting the switching regions: abstracting a boolean function.

int main() {
while(true) {
    sens.y?X
    X'=update(X);
    if (X'<0)
        act.cmd!1
    if (X'>15)
        act.cmd!0
    wait 0.01
}
}

+ A switch occur when an act statement is executed.
+ We need to test if, given a value \( x \) for the sensor inputs, the statement on line \( n \) will be executed.
+ Boolean function \( \varphi \):

\[
given an input \ X, \ is \ the \ statement \ on \ line \ n \ executed?
\]

+ We need to construct the zones \{X : \varphi(X) = 0\} and \{X : \varphi(X) = 1\}, i.e. build a representation of \( \varphi \).

Problems:

+ floating point intervals and not integer functions;
+ representation of boolean functions;
+ computing this representation.
**Representation of a boolean function: quad trees**

Quad tree:
- representation of a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$;
- two kinds of nodes:
  - non-terminal nodes with a box as attribute and $2^n$ children;
  - terminal node with a value $v \in \{0, 1\}$ as attribute.
Representation of a boolean function: quad trees

Quad tree:
- representation of a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$;
- two kinds of nodes:
  - non-terminal nodes with a box as attribute and $2^n$ children;
  - terminal node with a value $v \in \{0, 1\}$ as attribute.
- abstract quad trees: terminal value $v \in \{0, 1, ?\}$.
- the precision depends on the depth.
A simple branch and bound algorithm.

**Problem**

Given a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$ and a precision $N$, construct the corresponding abstract quad tree.

**Example**:

$$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \lor d((x, y), (7, 1.2)) \leq 1$$

+ $x = [0, 10]$ and $y = [0, 10]$
+ $d((x, y), (3.5, 6)) = [0, 78.25]$
+ $d((x, y), (3.5, 6)) \leq 1 = ?$
+ $d((x, y), (7, 1.2)) \leq 1 = ?$
+ $f(x, y) = ?$

**Iterate 1**

$$f([0, 10], [0, 10]) = ?$$
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Given a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$ and a precision $N$, construct the corresponding abstract quad tree.

Example:

$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \lor d((x, y), (7, 1.2)) \leq 1$

- $x = [0, 5]$ and $y = [0, 5]$
- $d((x, y), (7, 1.2)) \subseteq [4, \infty[$
- $d((x, y), (7, 1.2)) \leq 1 = 0$
- $d((x, y), (3.5, 6)) = [1, 48.25]$
- $d((x, y), (3.5, 6)) \leq 1 = ?$
- $f(x, y) = ?$

Iterate 2
A simple branch and bound algorithm.

Problem

Given a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$ and a precision $N$, construct the corresponding abstract quad tree.

Example:

$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \lor d((x, y), (7, 1.2)) \leq 1$
### Problem

Given a function $f : \mathbb{R}^n \rightarrow \mathbb{B}$ and a precision $N$, construct the corresponding abstract quad tree.

### Example:

$$f(x, y) = d((x, y), (3.5, 6)) \leq 1 \lor d((x, y), (7, 1.2)) \leq 1$$

---

Iterate 8
A simple branch and bound algorithm.

Problem

Given a function \( f : \mathbb{R}^n \rightarrow \mathbb{B} \) and a precision \( N \), construct the corresponding abstract quad tree.

Example:
\[
(f(x, y) = d((x, y), (3.5, 6)) \leq 1 \lor d((x, y), (7, 1.2)) \leq 1
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A simple branch and bound algorithm.

Problem

Given a function \( f : \mathbb{R}^n \rightarrow \mathbb{B} \) and a precision \( N \), construct the corresponding abstract quad tree.

Example:

\[
   f(x, y) = d((x, y), (3.5, 6)) \leq 1 \lor d((x, y), (7, 1.2)) \leq 1
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Experimental results.

For the reachability problem.

The boolean function is a simple interval based interpreter that tests if a given line of code is executed.

```c
#include <stdio.h>

double x, y;
/* !npx x = [0,10] */
/* !npx y = [0,10] */

double distance (double x, double y, double a, double b) {
    double res;
    res = (x-a)*(x-a) + (y-b)*(y-b);
    return res;
}

void main() {
    double res, res2;
    int v;

    res = distance (x, y, 3.5, 6);
    res2 = distance (x, y, 7, 1.2);
    if ((res <=1) || (res2 <=1)) {
        v=0;
    }
}
```

Experimental results.

For the reachability problem.

The boolean function is a simple interval based interpreter that tests if a given line of code is executed.

```c
/* npk x = [-2,4] */
/* npk y = [-2,4] */
double x,y;

int update (double x,double y) {
    double a,b,xt,yt,d;
    int i = 0;
    a = 0.32; b = 0.43; d = 0;
    while (d<10) {
        xt = x*x-y*y+a;
        yt = 2*x*y + b;
        x = xt;   y = yt;
        d=x*x+y*y;
        if (i>=10)
            d = 30;
        i++;  
    }
    return i;
}

void main() {
    int v = 0 , i = 0;
    i = update (x,y);
    if (i>=11)
        v = 1;
}
```
Guaranteed integration of ODEs.

Result of the first analysis:

+ we have a *spatial* criteria to detect the mode switchings;
+ we know the sampling *times*, i.e. instants when the sensors return a value.

+ The evolution of the continuous variables is given by ODEs.
+ Given an ODE \( \dot{y} = f(y) \) and a time \( t_n \), we want to compute a box \( y_n \) such that \( y(t_n) \in y_n \).
+ Existing validated methods: based on Taylor series decomposition.
+ Existing non validated numerical methods: Euler, Runge-Kutta, Heun, . . .
A new approach based on non validated methods.

How to turn a numerical method into a validated one:

+ A numerical method is 
  \[ y_{n+1} = \Phi(y_n, h_n). \]

+ Three sources of errors:
  - if \( y_n = y(t_n) \) is exact, \( y_{n+1} \) differs from \( y(t_{n+1}) \);
  - if \( y_n \) has an error \( \epsilon_n \), how is it propagated into the next step;
  - floating point computations.

First implementation: with RK4

\[
\begin{align*}
  k_1 &= f(y_n) \\
  k_2 &= f \left( y_n + \frac{h}{2} k_1 \right) \\
  k_3 &= f \left( y_n + \frac{h}{2} k_2 \right) \\
  k_4 &= f \left( y_n + h k_3 \right) \\
  y_{n+1} &= y_n + \frac{h}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right)
\end{align*}
\]
GRKLlib: three sources of errors.

1. One step error:
   - the function that gives $y_{n+1}$ differs from $y(t)$.
   - $y_{n+1} = \varphi(t_{n+1})$
   - $\forall i \in [0, 4]$, $\frac{d^i y}{dt^i}(t_n) = \frac{d^i \varphi_n}{dt^i}(t_n)$
   - $\exists t' \in [t_n, t_{n+1}]$ such that
     $$y(t_{n+1}) - \varphi_n(t_{n+1}) = \frac{d^5 (y - \varphi_n)}{dt^5}(t')$$

   $$\epsilon_{n+1} = \frac{d^4 f}{dt^4}(\tilde{y}) - \frac{d^5 \varphi_n}{dt^5}([t_n, t_{n+1}])$$
GRKLib: three sources of errors.

2. Error propagation:

- starting point is $y_n$ instead of $y(t_n)$,
- the propagated error is the difference between $\psi(y_n)$ and $\psi(y(t_n))$
- $\exists y \in [y_n, y(t_n)]$ such that $\chi_{n+1} = J(\psi, y)(y_n - y(t_n))$
  $$\chi_n \in J(\psi, [y_n, y_n + \epsilon_n]).\epsilon_n$$
- We use QR-factorization technique to reduce the wrapping effect.

$$\epsilon_{n+1} = \frac{d^4 f}{dt^4} (\tilde{y}) - \frac{d^5 \varphi_n}{dt^5} ([t_n, t_{n+1}]) + J(\psi, [y_n, y_n + \epsilon_n]).\epsilon_n$$
GRKLib: three sources of errors.

3. Computation error:
   + we use floating point numbers and not real numbers to compute \( y_{n+1} \);
   + we must overapproximate the computation errors;
   + Solution: global error arithmetic
     \[
     \langle a \rangle_E = f_a + e_a \epsilon_e \quad \text{and} \quad \langle b \rangle_E = f_b + e_b \epsilon_e
     \]
     \[
     \langle a + b \rangle_E = \uparrow \sim (f_a + f_b) + (e_a + e_b + \downarrow \sim (f_a + f_b)) \epsilon_e
     \]
   + We use this arithmetic to compute \( y_{n+1} \), and thus get an interval \( E_{n+1} \) that contains all the computation errors.

\[
\epsilon_{n+1} = \frac{d^4 f}{dt^4} (\tilde{y}) - \frac{d^5 \psi}{dt^5} ([t_n, t_{n+1}]) + J(\psi, [y_n, y_n + \epsilon_n]) \cdot \epsilon_n + E_{n+1}
\]
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Benchmarks: CPU time VS $T_{\text{end}}$

**Lorenz equations**

\[
\begin{align*}
\dot{y}_1 &= 10(y_2 - y_1) \\
\dot{y}_2 &= y_1(28 - y_3) - y_2 \\
\dot{y}_3 &= y_1 \times y_2 - \frac{8}{3} y_3
\end{align*}
\]
Benchmarks: Enclosure width VS $T_{\text{end}}$

**Rotation problem**

$$\dot{Y} = \begin{pmatrix} 0 & -0.707 & -0.5 \\ 0.707 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{pmatrix} Y$$
Conclusion

Formal verification of embedded softwares

- considering the program alone showed its limits;
- we must take the physical environment into account.

Interval methods:

- allow to abstract the program and thus simplify the analysis;
- allow to compute a safe overapproximation of the continuous dynamics.

Future work:

- unify both analysis into a common tool;
- improve GRKLib by adding higher order methods and by using other domains (zonotopes).