Resolution of nonlinear interval problems

using symbolic interval arithmetic

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1 Interval problem

Interval optimization

$$\min_{[\mathbf{x}] \in \mathbb{IR}^n} f([\mathbf{x}])$$

where $f : \mathbb{IR}^n \to \mathbb{R}$ and $\mathbb{IR}^n$ is the set of boxes in $\mathbb{R}^n$.

Interval inequality

Characterize the set

$$\mathbf{S} = \{[\mathbf{x}] \in \mathbb{IR}^n, \mathbf{f}([\mathbf{x}]) \leq \mathbf{0}\},$$

where $\mathbf{f} : \mathbb{IR}^n \to \mathbb{R}^p$.

Interval inclusion

Characterize the set

$$\mathbf{S} = \{[\mathbf{x}] \in \mathbb{IR}^n, [\mathbf{x}] \subset [\mathbf{f}]([\mathbf{x}])\}$$

where $[\mathbf{f}] : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$.

Quantified interval inequalities

Characterize the set

$$\mathbf{S} = \{[\mathbf{x}] \in \mathbb{IR}^n, \exists [\mathbf{y}] \in \mathbb{IR}^p, \mathbf{f}\left([\mathbf{x}], [\mathbf{y}]\right) \leq \mathbf{0}\}$$

where $\mathbf{f} : \mathbb{IR}^n \times \mathbb{IR}^p \rightarrow \mathbb{R}^m$.

2 Boundarification

An interval constraint is a function from $\mathbb{IR}^n$ to $\{0, 1\}$. An example of interval constraint is

$$C\left([\mathbf{x}]\right) \;:\; [x_1] \subset [x_2],$$

where $[\mathbf{x}] = [x_1] \times [x_2]$.

An interval constraint is monotonic if

$$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow (C([\mathbf{x}]) \Rightarrow C([\mathbf{y}])) .$$

For instance $C([x]) \stackrel{\text{def}}{=} (0 \in [x])$ is monotonic.

Define the *intervalization* function i as follows

$$
i : \left\{ \begin{array}{ccl} \mathbb{R}^{2n} & \longrightarrow & \mathbb{IR}^n \\ \begin{pmatrix} x_1^- \\ x_1^+ \\ \vdots \\ x_n^- \\ x_n^+ \end{pmatrix} & \longrightarrow & \begin{array}{l} [\mathbf{x}] = \begin{pmatrix} \left[x_1^-, x_1^+\right] \\ \vdots \\ \left[x_n^-, x_n^+\right] \end{pmatrix} \text{ if } \forall i, x_i^- \leq x_i^+ \\ [\mathbf{x}] = \emptyset \text{ otherwise} \end{array} \end{array} \right.
$$

An interval constraint $C([\mathbf{x}])$ from $\mathbb{IR}^n$ to $\{0,1\}$ is equivalent to a constraint $\underline{\overline{C}}$ on their bounds:

$$\underline{\overline{C}} : \left\{ \begin{array}{ccccc} \mathbb{R}^{2n} & \rightarrow & \mathbb{IR}^n & \rightarrow & \{0,1\} \\ \underbrace{\begin{pmatrix} x_1^- \\ x_1^+ \\ \vdots \\ x_n^- \\ x_n^+ \end{pmatrix}}_{\underline{\overline{\mathbf{x}}}} & \overset{i}{\rightarrow} & \underbrace{\begin{pmatrix} \left[x_1^-, x_1^+\right] \\ \vdots \\ \left[x_n^-, x_n^+\right] \end{pmatrix}}_{[\mathbf{x}]} & \overset{C}{\rightarrow} & C([\mathbf{x}]) \end{array} \right.$$

From an expression of $C([\mathbf{x}])$ we can get an expression for $\underline{\overline{C}}(\underline{\overline{\mathbf{x}}})$.

The procedure to get such an expression is called *boundarification*.

For instance the boundarification of

$$C([\mathbf{x}]) \overset{\text{def}}{=} ([x_1] \subset [x_2] \text{ and } [\mathbf{x}] \neq \emptyset)$$

is

$$\underline{\overline{C}} \begin{pmatrix} x_1^- \\ x_1^+ \\ x_2^- \\ x_2^+ \end{pmatrix} : \begin{cases} x_1^- \geq x_2^- \text{ and} \\ x_1^+ \leq x_2^+ \text{ and} \\ x_1^- \leq x_1^+ \text{ and} \\ x_2^- \leq x_2^+ \end{cases} .$$

The boundarification can be made easier using symbolic interval arithmetic.

3 Symbolic-intervals

A *term* is a word (a finite sequence of elements of the alphabet $\{a, b, \ldots, Y, Z, +, -, /, *,), (, \ldots\}$) which can be obtained by the following rules

$$
\begin{gathered}
'a', \ldots' z',' A', \ldots' Z' \in \mathcal{S} \\
\mathcal{A} \in \mathcal{S}, \mathcal{B} \in \mathcal{S} \Rightarrow \mathcal{A}\mathcal{B} \in \mathcal{S} \\
\mathcal{A} \in \mathcal{S}, \mathcal{B} \in \mathcal{S} \Rightarrow \mathcal{A} + \mathcal{B} \in \mathcal{S} \\
\mathcal{A} \in \mathcal{S}, \mathcal{B} \in \mathcal{S} \Rightarrow \mathcal{A} * \mathcal{B} \in \mathcal{S} \\
\mathcal{A} \in \mathcal{S} \Rightarrow \mathsf{sin}\,(\mathcal{A}) \in \mathcal{S} \\
\ldots
\end{gathered}
$$

For instance

$$\text{sin("aaa")}+\text{cos("bbb")}$$

is a term.

A *symbolic interval* is a couple $[\mathcal{A},\mathcal{B}]$ of terms. We define the following operations or functions for symbolic intervals.

$$
\begin{array}{lcl}
[\mathcal{A},\mathcal{B}]+[\mathcal{C},\mathcal{D}] & = & [\mathcal{A}+\mathcal{C},\mathcal{B}+\mathcal{D}] \\
[\mathcal{A},\mathcal{B}]-[\mathcal{C},\mathcal{D}] & = & [\mathcal{A}-\mathcal{D},\mathcal{B}-\mathcal{C}] \\
[\mathcal{A},\mathcal{B}]*[\mathcal{C},\mathcal{D}] & = & \begin{array}{l}[\min\left(\mathcal{A}*\mathcal{C},\mathcal{A}*\mathcal{D},\mathcal{B}*\mathcal{C},\mathcal{B}*\mathcal{D}\right)\\ \quad ,\max\left(\mathcal{A}*\mathcal{C},\ldots\right)]\end{array} \\
[\mathcal{A},\mathcal{B}]^2 & = & \begin{array}{l}[\max(0,\mathrm{sign}\left(\mathcal{A}*\mathcal{B}\right)\min\left(\mathcal{A}^2,(\mathcal{B})\right)\\ \quad ,\max\left(\mathcal{A}^2,\mathcal{B}^2\right)]\end{array} \\
\exp\left([\mathcal{A},\mathcal{B}]\right) & = & [\exp\left(\mathcal{A}\right),\exp\left(\mathcal{B}\right)]\,. \\
1/\left[\mathcal{A},\mathcal{B}\right] & = & \begin{array}{l}[\min\left(1/\mathcal{B},\infty*\mathcal{A}*\mathcal{B}\right)\\ \quad ,\max\left(1/\mathcal{A},-\infty*\mathcal{A}*\mathcal{B}\right)]\end{array} \\
[\mathcal{A},\mathcal{B}]\cap[\mathcal{C},\mathcal{D}] & = & [\max\left(\mathcal{A},\mathcal{C}\right),\min\left((\mathcal{B},\mathcal{D})\right)] \\
[\mathcal{A},\mathcal{B}]\sqcup[\mathcal{C},\mathcal{D}] & & [\min\left(\mathcal{A},\mathcal{C}\right),\max\left((\mathcal{B},\mathcal{D})\right)] \\
w\left([\mathcal{A},\mathcal{B}]\right) & = & \mathcal{B}-\mathcal{A}
\end{array}
$$

For instance ,

$$\begin{aligned}
\mathbf{exp}\,([\text{aaa,bbb}] - [\text{ccc,aaa}]) &= \mathbf{exp}\,([\text{aaa} - \text{aaa}\ ,\ \text{bbb} - \text{ccc}]) \\
&= [\exp(\text{aaa} - \text{aaa}),\ \exp(\text{bbb} - \text{ccc})
\end{aligned}$$

Define the following relations on symbolic intervals

$$\begin{aligned}([\mathcal{A},\mathcal{B}] = [\mathcal{C},\mathcal{D}]) &= (\mathcal{A}-\mathcal{C}=0 \text{ and } \mathcal{B}-\mathcal{D}=0)\\ ([\mathcal{A},\mathcal{B}] \subset [\mathcal{C},\mathcal{D}]) &= (\mathcal{A}-\mathcal{C}\geq 0 \text{ and } \mathcal{D}-\mathcal{B}\geq 0)\end{aligned}$$

For instance

$$([\text{aaa} , \text{bbb}] = [\text{ccc} , \text{ddd}]) = (\text{aaa} = \text{ccc and bbb=ddd})$$

Another example is the following

$$\left([a,b] \subset [a,b]^2\right) = \begin{cases} a - \max(0, \text{sign}(a.b) * \min(a^2, b^2) & \geq 0 \\ \text{and} & \\ \max(a^2, b^2) - b & \geq 0 \end{cases}.$$

4 Implementation

```
struct sint
{
AnsiString lb;
AnsiString ub;
};
```

```
void plus(sint& r,sint& a,sint& b)
{
r.lb=a.lb+"+"+b.lb;
r.ub=a.ub+"+"+b.ub;
}
```

```
void moins(sint& r,sint& a,sint& b)
{
r.lb=a.lb+"-("+b.ub+")";
r.ub=a.ub+"-("+b.lb+")";
}

void moins(sint& r,sint& a,AnsiString b)
{
r.lb=a.lb+"-"+b;
r.ub=a.ub+"-"+b;
}

void moins(sint& r,AnsiString a, sint& b)
{
r.lb=a+"-"+b.ub;
r.ub=a+"-"+b.lb;
}
```

```
void mult(sint& r,sint& a,sint& b)
{
AnsiString z11="("+a.lb+")*("+b.lb+")";
AnsiString z12="("+a.lb+")*("+b.ub+")";
AnsiString z21="("+a.ub+")*("+b.lb+")";
AnsiString z22="("+a.ub+")*("+b.ub+")";
AnsiString z =z11+","+z12+","+z21+","+z22;
r.lb="min("+z+")";
r.ub="max("+z+")";
}
```

```
void exp(sint& r,sint& a)
{
r.lb="exp("+a.lb+")";
r.ub="exp("+a.ub+")";
}
```

```
void sqr(sint& r,sint& a)
{ AnsiString z1="sqr("+a.lb+")";
AnsiString z2="sqr("+a.ub+")";
AnsiString z3="sign("+a.lb+"*"+a.ub+")*min("+z1+","+z
r.lb="max(0,"+z3+")";
r.ub="max("+z1+","+z2+")";
}

void sqrt(sint& r,sint& a)
{ r.lb="sqrt("+a.lb+")";
r.ub="sqrt("+a.ub+")";
}
```

```
void inv(sint& r,sint& a)
{
AnsiString z1="1/("+a.ub+")";
AnsiString z2="1/("+a.lb+")";
AnsiString z3="+oo*("+a.lb+"*"+a.ub+")";
AnsiString z4="-"+z3+"";
r.lb="min("+z1+","+z3+")";
r.ub="max("+z2+","+z4+")";
}
```

```
void div(sint& R,AnsiString a, sint& B)
{
sint Z1;
inv(Z1,B);
mult(R,a,Z1);
}
```

```
void inter(sint& r,sint& a,sint& b)
{
r.lb="max("+a.lb+","+b.lb+")";
r.ub="min("+a.ub+","+b.ub+")";
}

AnsiString subset(sint& a,sint& b)
{
return a.lb+"-("+b.lb+") in [0,+oo] \n"
+ b.ub+"-("+a.ub+") in [0,+oo]";
}
```

5 Experimental design

Example

Tomorrow, we will make an experiment with a moving object.
Its speed will be measured using a speed sensor with an accuracy less that $\pm 1 ms^{-1}$.
Its weight will be measured with an accuracy less than $0.1 kg$.
We are interested by its kinenic energy $E = \frac{1}{2} m v^2$.
We will use the interval formula $[E] = \frac{1}{2} [m] . [v]^2$.

Question : With which accuracy will we be able to measure E ?

Formalism

Quantities x_i will be measured with an accuracy can be bounded a priori.
The quantity y of interest satisfies $y = f(x_1, \ldots, x_n)$.
An interval for $[y]$ will be obtained using a known interval function $[f]$.

Question : With which accuracy will we be able to measure y ?

Interval analysis makes it possible to build an interval function

$$[f] : \left\{ \begin{array}{lcl} \mathbb{IR}^n & \rightarrow & \mathbb{IR} \\ [\mathbf{x}] & \rightarrow & [y] = [f]\left([\mathbf{x}]\right) \end{array} \right.$$

that computes an enclosure for y.

Assuming that $\mathbf{x}$ will be measured with an accuracy less that $\bar{w}$, the worst-case uncertainty for $[y]$ is

$$\max_{\substack{[\mathbf{x}]\in\mathbb{IR}^n \\ w([\mathbf{x}])\leq\bar{w}}} w\left([f]\left(\mathbf{x}\right)\right)$$

Example

A boundarification of the following interval optimization problem

$$\max_{\substack{[x]\in\mathbb{IR}\\ w([x])\leq 1}} w\left(\mathsf{exp}\left([x]-[x]^2\right)\right)$$

is

$$\max_{b-a\in[0,1]} e^{b-max(0,sign(ab).min(a^2,b^2))}-e^{a-(max(a^2,b^2)}$$

The maximum is inside [3.324807; 3.324808] and the global optimizer satisfies

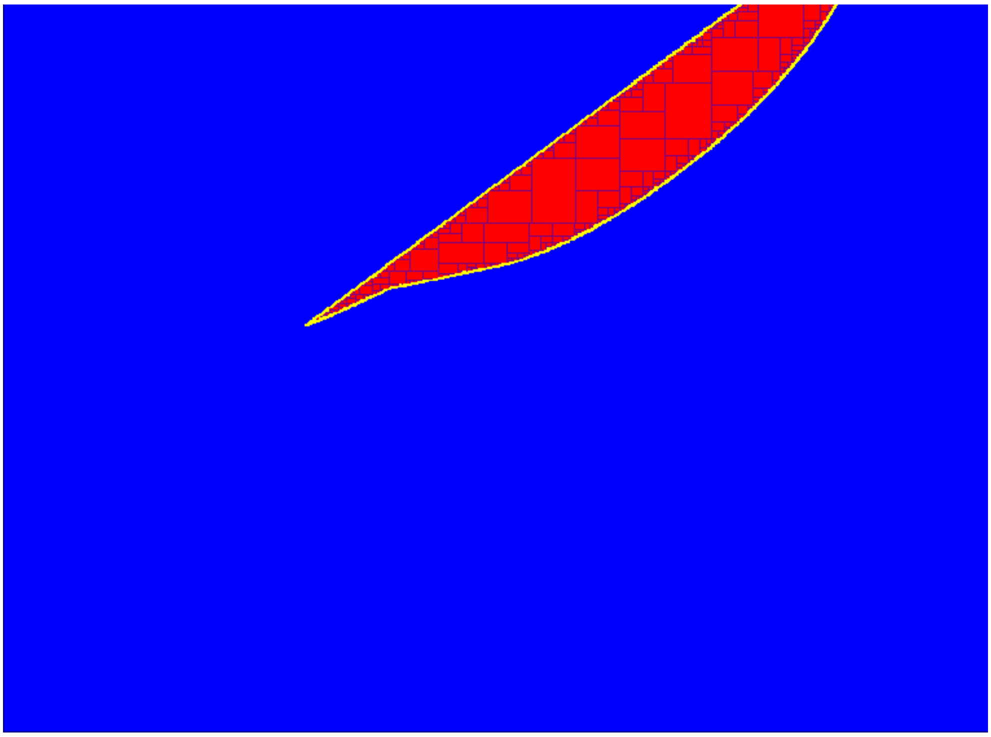
$$(a^*,b^*)\in[0.547,0.548]\times[1.547,1.548].$$

i.e., the interval optimizer is an interval $[a^*,b^*]$ which satisfies the previous relation.

The figure shows the set

$$\mathbb{S} = \left\{ [x] \in \mathbb{IR}, w([x] \leq 1 \text{ and } w\left(\exp\left([x] - [x]^2\right)\right) > 1 \right\}.$$

inside the box $[-2, 2] \times [-2, 2]$.



6 Comparing two inclusion functions

Consider the two following inclusion functions

$$\begin{array}{rcl} [f]([x]) & = & [x] * ([x] - 1) \\ [g]([x]) & = & [x]^2 - [x] \end{array}$$

We would like to known for which intervals $[x]$, $[f]$ is more accurate than $[g]$.

We have

$$\begin{aligned}
[f]([x]) &= [a, b] * ([a, b] - 1) \\
&= [a, b] * [a - 1, b - 1] \\
&= [\min(a(a-1), b(a-1), a(b-1), b(b-1)), \\
&\quad \max(a(a-1), b(a-1), a(b-1), b(b-1))]
\end{aligned}$$

Moreover

$$
\begin{aligned}
[g]([x]) &= [a,b]^2 - [a,b] \\
&= \left[\max(0, \operatorname{sign}(a.b)\min\left(a^2, b^2\right)), \max\left(a^2, b^2\right)\right] \\
&\quad -[a,b] \\
&= [\max(0, \operatorname{sign}(a.b)\min\left(a^2, b^2\right) - b \\
&\quad , \max\left(a^2, b^2\right) - a]
\end{aligned}
$$

Thus

$$
\begin{aligned}
& [f]\,([x]) \subset [g]\,([x]) \\
\Leftrightarrow\ & \left\{
\begin{array}{ll}
\mathrm{min}\,(a\,(a-1)\,, b\,(a-1)\,, a\,(b-1)\,, b\,(b-1)) & \\
-\max(0, \mathrm{sign}\,(a.b)\,\mathrm{min}\left(a^2, b^2\right)) + b & \geq 0 \\
\mathrm{max}\left(a^2, b^2\right) - a- & \\
\max(a\,(a-1)\,, b\,(a-1)\,, a\,(b-1)\,, b\,(b-1)) & \geq 0
\end{array}
\right.
\end{aligned}
$$

- If $[x] = [1,2]$

$$\begin{aligned} [f]([x]) &= [1,2] * ([1,2] - 1) = [0,2] \\ [g]([x]) &= [1,2]^2 - [1,2] = [-1,3] \end{aligned}$$

We have $[f]([x]) \subset [g]([x])$.

- If $[x] = [-2,-1]$,

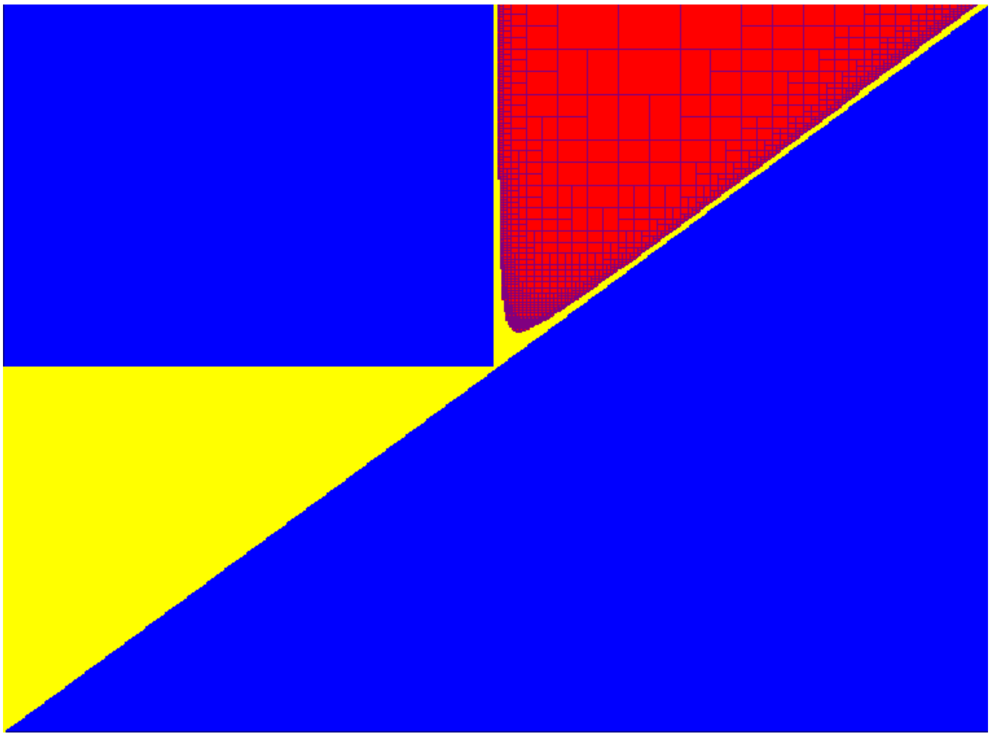
$$\begin{aligned} [f]([x]) &= [-2,-1] * ([-2,-1] - 1) = [2,6] \\ [g]([x]) &= [-2,-1]^2 - [-2,-1] = [2,6] \end{aligned}$$

  We have $[f]([x]) \subset [g]([x])$ but we are not able to prove it.

- If $[x] = [-1,1]$,

$$\begin{aligned} [f]([x]) &= [-1,1] * ([-1,1] - 1) = [-2,2] \\ [g]([x]) &= [-1,1]^2 - [-1,1] = [-1,2] \end{aligned}$$

  we have $[f]([x]) \not\subset [g]([x])$.



In red $[f]$ is more accurate that $[g]$

In blue $[f]$ is not more accurate that $[g]$

In yellow, we don't know.

The frame box is $[-2,2] \times [-2,2]$.

7 Analysis of the Newton operator

Consider the equation $f(x) = 0$ with $f(x) = e^x - 1$.

The interval Newton operator is defined by

$$\mathcal{N}([x]) = x_0 - \frac{f(x_0)}{[f']([x])},$$

where x_0 is any point in $[x]$. Here, we shall take $x_0 = x^-$ and thus

$$\mathcal{N}([x]) = x^- - \frac{f\left(x^-\right)}{[f']([x])}. = x^- - \frac{e^{x^-} - 1}{\exp([x^-, x^+])}$$

The Newton operator is contracting if

$$\mathcal{N}([x]) \subset [x].$$

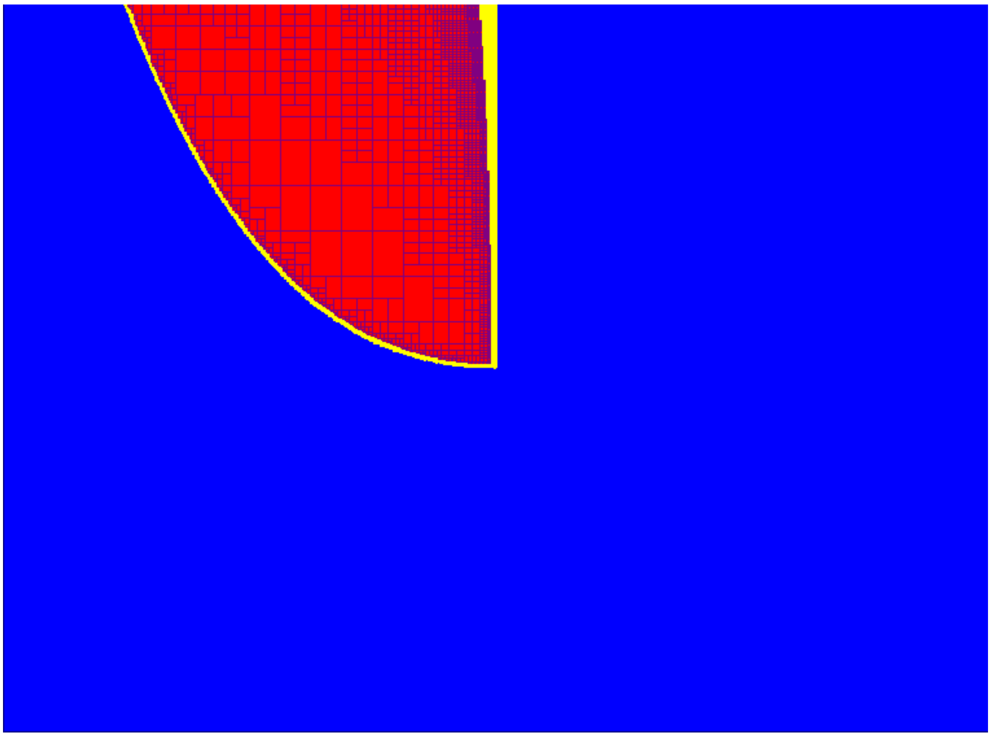
The interval Newton set is the set of all $[x]$ such that $\mathcal{N}$ is contracting.

If we set $[x] = [a, b]$, we get

$$\mathcal{N}([a, b]) = a - \frac{a - 1}{\exp([a, b])}$$

A boundarification of the relation $\mathcal{N}([a, b]) \subset [a, b]$ yields

$$\begin{cases} a - \max\left(\frac{e^a - 1}{e^b}, \frac{e^a - 1}{e^a}\right) - a & \geq 0 \\ b - a + \min\left(\frac{e^a - 1}{e^b}, \frac{e^a - 1}{e^a}\right) & \geq 0 \\ b - a & \geq 0 \end{cases}$$



Set of all intervals such that the interval Newton operator is contracting

The frame box is $[-2, 2] \times [-2, 2]$.

8 Proving global consistency

8.1 Motivation

Consider the *angle* constraint

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

The corresponding optimal contractor $\mathcal{C}^*$ is defined by

$$\begin{cases} \mathbb{IR}^5 & \rightarrow & \mathbb{IR}^5 \\ [\mathbf{x}] & \rightarrow & \left[\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \theta \end{pmatrix} \in [\mathbf{x}], \begin{cases} x_2 & = & x_1 \cos\theta - y_1 \sin\theta \\ y_2 & = & x_1 \sin\theta + y_1 \cos\theta \end{cases} \right] \end{cases}$$

Conjecture: Consider the set of constraints

$$\left\{ \begin{array}{lll} x_2 & = & x_1 \cos \theta - y_1 \sin \theta \\ y_2 & = & x_1 \sin \theta + y_1 \cos \theta \end{array} \right.$$

If we add the following redundant constraints

$$\left\{ \begin{array}{lll} x_1 & = & x_2 \cos \theta + y_2 \sin \theta \\ y_1 & = & -x_2 \sin \theta + y_2 \cos \theta \\ x_1^2 + y_1^2 & = & x_2^2 + y_2^2 \\ \tan \theta & = & \frac{x_1 y_2 - y_1 x_2}{x_1 x_2 + y_1 y_2} \end{array} \right.$$

A hull consistency algorithm with input $[\mathbf{x}]$ will to converge toward $\mathcal{C}^* ([\mathbf{x}])$.

With Xavier Baguenard, we tried to prove it by hand, but we failed.

Question : Can we automatically prove this conjecture with interval methods ?

8.2 Example

Consider a simpler constraint given by

$$x^2 - x = 0$$

A hull consistency contractor for this constraint amounts to iterate the two statements

$$\begin{array}{rcl} [x] & = & [x] \cap [x]^2 \\ [x] & = & [x] \cap \sqrt{[x]} \end{array}$$

from an initial interval $[x]$ until a steady interval is reached.

The resulting contractor is said optimal if it always converge to the smallest box which encloses all solutions that belongs to $[x]$.

Question: Is the hull contractor optimal ?

Step 1. Compute all solutions of the equation $x^2 - x = 0$. With an interval method (with bisections), we get that we have exactly two solutions

$$x_1 \simeq 0 \text{ and } x_2 \simeq 1$$

Thus any safe contractor has at least 3 steady boxes (those corresponding to $[0,0], [1,1], [0,1]$).

Step 2. Since the hull contractor will converge the biggest box inside $[x](0)$ which satisfies

$$\begin{array}{rcl} [x] & \subset & [x]^2 \\ [x] & \subset & \sqrt{[x]} \end{array}$$

The interval CSP translates into the following bound-CSP

$$\begin{array}{lcl} a - \max(0, \operatorname{sign}(a.b).\min(a^2, b^2)) & \geq & 0 \\ \max(a^2, b^2) - b & \geq & 0 \\ \min(a - \sqrt{a}, \sqrt{b} - b) & \geq & 0 \\ b - a & \geq & 0 \end{array}$$

This bound-CSP has three solutions enclosed by

$$
\begin{aligned}
&[0.999999999, 1] \times [0.999999999, 1] \\
&[0, 3.10^{-39}] \times [0, 3.10^{-39}] \\
&[0, 3.10^{-39}] \times [0.999999999, 1]
\end{aligned}
$$

A unicicity test concludes that each of the three boxes contains a unique solution.

Thus, we know that we have exactly three steady boxes.

Thus, we have proven that the hull contractor is optimal.