Resolution of nonlinear interval problems
using symbolic interval arithmetic

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1 Interval problem
Interval optimization

$$\min_{[x] \in \mathbb{IR}^n} f ([x])$$

where $f : \mathbb{IR}^n \rightarrow \mathbb{R}$ and $\mathbb{IR}^n$ is the set of boxes in $\mathbb{R}^n$. 
Interval inequality

Characterize the set

$$S = \{ [x] \in \mathbb{R}^n, f([x]) \leq 0 \},$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$. 
Interval inclusion

Characterize the set

\[ S = \{ [x] \in \mathbb{IR}^n, [x] \subset [f]([x]) \} \]

where \([f] : \mathbb{IR}^n \rightarrow \mathbb{IR}^n\).
Quantified interval inequalities

Characterize the set

\[ S = \{ [x] \in \mathbb{IR}^n, \exists [y] \in \mathbb{IR}^p, f([x],[y]) \leq 0 \} \]

where \( f: \mathbb{IR}^n \times \mathbb{IR}^p \rightarrow \mathbb{R}^m \).
An interval constraint is a function from $\mathbb{IR}^n$ to $\{0, 1\}$. An example of interval constraint is

$$C ([x]) : [x_1] \subset [x_2],$$

where $[x] = [x_1] \times [x_2]$. 

2 Boundarification
An interval constraint is monotonic if

\[ [x] \subset [y] \Rightarrow (C'(x) \Rightarrow C'(y)). \]

For instance \( C([x]) \overset{\text{def}}{=} (0 \in [x]) \) is monotonic.
Define the *intervalization* function $i$ as follows

$$
i : \begin{cases} 
\mathbb{R}^{2n} & \rightarrow \mathbb{I} \mathbb{R}^n \\
\begin{pmatrix} 
x_1^- \\
x_1^+ \\
\vdots \\
x_n^- \\
x_n^+ 
\end{pmatrix} & \rightarrow \ [x] = \begin{pmatrix} 
x_1^- , x_1^+ \\
\vdots \\
x_n^- , x_n^+ 
\end{pmatrix} \text{ if } \forall i, x_i^- \leq x_i^+ \\
[x] &= \emptyset \text{ otherwise}
\end{cases}
$$
An interval constraint $C ([x])$ from $\mathbb{I}R^n$ to $\{0, 1\}$ is equivalent to a constraint $\overline{C}$ on their bounds:

$$\overline{C} : \begin{cases} \mathbb{R}^{2n} & \rightarrow \mathbb{I}R^n & \rightarrow \{0, 1\} \\ \left( x_1^-, x_1^+ \right) & \vdots & \left( x_n^-, x_n^+ \right) \end{cases}$$

$$i \quad \rightarrow \quad \begin{pmatrix} [x_1^-, x_1^+] \\ \vdots \\ [x_n^-, x_n^+] \end{pmatrix} \quad \rightarrow \quad C ([x])$$
From an expression of $C ([x])$ we can get an expression for $\overline{C} (\overline{x})$.
The procedure to get such an expression is called \textit{boundarification}.
For instance the boundarification of

$$C ([x]) \overset{\text{def}}{=} ([x_1] \subset [x_2] \text{ and } [x] \neq \emptyset)$$

is

$$\overline{C} \left(\begin{array}{c}
  x_1^- \\
  x_1^+ \\
  x_2^- \\
  x_2^+
\end{array}\right) : \begin{cases}
  x_1^- \geq x_2^- \text{ and } \\
  x_1^+ \leq x_2^+ \text{ and } \\
  x_1^- \leq x_1^+ \text{ and } \\
  x_2^- \leq x_2^+
\end{cases}.$$
The boundarification can be made easier using symbolic interval arithmetic.
3 Symbolic-intervals
A term is a word (a finite sequence of elements of the alphabet \{a, b, \ldots, Y, Z, +, -, /, *, ), (, \ldots \}\}) which can be obtained by the following rules

\[
\begin{align*}
  'a', \ldots 'z', 'A', \ldots 'Z' \in S \\
  A \in S, B \in S \Rightarrow AB \in S \\
  A \in S, B \in S \Rightarrow A + B \in S \\
  A \in S, B \in S \Rightarrow A * B \in S \\
  A \in S \Rightarrow \sin (A) \in S \\
  \ldots
\end{align*}
\]
For instance

$$\sin("aaa") + \cos("bbb")$$

is a term.
A symbolic interval is a couple \([A, B]\) of terms. We define the following operations or functions for symbolic intervals.

\[
[A, B] + [C, D] = [A + C, B + D] \\
[A, B] - [C, D] = [A - D, B - C] \\
[A, B] \cdot [C, D] = \left[\min (A \cdot C, A \cdot D, B \cdot C, B \cdot D), \max (A \cdot C, \ldots)\right] \\
[A, B]^2 = \left[\max (0, \text{sign}(A \cdot B) \min (A^2, (B)), \max (A^2, B^2))\right] \\
\exp ([A, B]) = [\exp (A), \exp (B)] . \\
1/ [A, B] = \left[\min (1/B, \infty \cdot A \cdot B), \max (1/A, -\infty \cdot A \cdot B)\right] \\
[A, B] \cap [C, D] = [\max (A, C), \min ((B, D))] \\
[A, B] \cup [C, D] = [\min (A, C), \max ((B, D))] \\
\omega ([A, B]) = B - A
\]
For instance,

\[
\exp([\text{aaa,bbb}] - [\text{ccc,aaa}]) = \exp([\text{aaa - aaa, bbb - ccc}]) = [\exp(\text{aaa - aaa}), \exp(\text{bbb - ccc})]
\]
Define the following relations on symbolic intervals

\[
([A, B] = [C, D]) = (A - C = 0 \text{ and } B - D = 0)
\]
\[
([A, B] \subset [C, D]) = (A - C \geq 0 \text{ and } D - B \geq 0)
\]

For instance

\[
([aaa, bbb] = [ccc, ddd]) = (aaa = ccc \text{ and } bbb = ddd)
\]

Another example is the following

\[
([a, b] \subset [a, b]^2) = \begin{cases} 
  a - \max(0, \text{sign}(a.b) \times \min(a^2, b^2)) \geq 0 \\
  \text{and} \\
  \max(a^2, b^2) - b \geq 0
\end{cases}
\]
struct sint
{
  AnsiString lb;
  AnsiString ub;
};
void plus(sint& r, sint& a, sint& b)
{
    r.lb = a.lb + "+" + b.lb;
    r.ub = a.ub + "+" + b.ub;
}
void moins(sint& r, sint& a, sint& b)
{
    r.lb = a.lb + "-" + b.ub + "";
    r.ub = a.ub + "-" + b.lb + "";
}

void moins(sint& r, sint& a, AnsiString b)
{
    r.lb = a.lb + "-" + b;
    r.ub = a.ub + "-" + b;
}

void moins(sint& r, AnsiString a, sint& b)
{
    r.lb = a + "-" + b.ub;
    r.ub = a + "-" + b.lb;
}
void mult(sint& r, sint& a, sint& b) 
{
    AnsiString z11="("+a.lb+")*("+b.lb+")";
    AnsiString z12="("+a.lb+")*("+b.ub+")";
    AnsiString z21="("+a.ub+")*("+b.lb+")";
    AnsiString z22="("+a.ub+")*("+b.ub+")";
    AnsiString z =z11","+z12","+z21","+z22;
    r.lb="min("+z+")";
    r.ub="max("+z+")";
}
void exp(sint& r, sint& a) 
{
    r.lb="exp("+a.lb+")";
    r.ub="exp("+a.ub+")";
}
void sqr(sint& r, sint& a)
{
    AnsiString z1 = "sqr(" + a.lb + ")";
    AnsiString z2 = "sqr(" + a.ub + ")";
    AnsiString z3 = "sign(" + a.lb + "*" + a.ub + ")*min(" + z1 + "," + z2 + ")";
    r.lb = "max(0," + z3 + ")";
    r.ub = "max(" + z1 + "," + z2 + ")";
}

void sqrt(sint& r, sint& a)
{
    r.lb = "sqrt(" + a.lb + ")";
    r.ub = "sqrt(" + a.ub + ")";
}
void inv(sint& r, sint& a)  
{
    AnsiString z1="1/"("+a.ub+")";
    AnsiString z2="1/"("+a.lb+")";
    AnsiString z3="+oo*("+a.lb+"*"+a.ub+")";
    AnsiString z4="-"+z3+"";
    r.lb="min("+z1+","+z3+")";
    r.ub="max("+z2+","+z4+")";
}
void div(sint& R, AnsiString a, sint& B)
{
    sint Z1;
    inv(Z1, B);
    mult(R, a, Z1);
}
void inter(sint& r, sint& a, sint& b)
{
  r.lb="max("+a.lb+","+b.lb+")";
  r.ub="min("+a.ub+","+b.ub+")";
}

AnsiString subset(sint& a, sint& b)
{
  return a.lb+"-("+b.lb+") in [0,+oo] \n"
  + b.ub+"-("+a.ub+") in [0,+oo]";
}
5 Experimental design
Example

Tomorrow, we will make an experiment with a moving object. 
Its speed will be measured using a speed sensor with an accuracy less than \( \pm 1 \text{ms}^{-1} \). 
Its weight will be measured with an accuracy less than 0.1kg. 
We are interested by its kinetic energy \( E = \frac{1}{2}mv^2 \). 
We will use the interval formula \([E] = \frac{1}{2} [m] \cdot [v]^2\). 

Question : With which accuracy will we be able to measure \( E \)?
Formalism

Quantities $x_i$ will be measured with an accuracy can be bounded a priori. The quantity $y$ of interest satisfies $y = f(x_1, \ldots, x_n)$. An interval for $[y]$ will be obtained using a known interval function $[f]$.

**Question**: With which accuracy will we be able to measure $y$?
Interval analysis makes it possible to build an interval function

\[ [f] : \left\{ \begin{array}{c}
\mathbb{IR}^n \rightarrow \mathbb{IR} \\
[x] \rightarrow [y] = [f]([x])
\end{array} \right. \]

that computes an enclosure for \( y \).
Assuming that \( x \) will be measured with an accuracy less than \( \bar{w} \), the worst-case uncertainty for \([y]\) is

\[
\max_{[x] \in \mathbb{IR}^n} \quad w([f](x)) \quad \text{subject to} \quad w([x]) \leq \bar{w}
\]
Example

A boundarification of the following interval optimization problem

\[
\max_{x \in \mathbb{R}} \ w \left( \exp \left( [x] - [x]^2 \right) \right)
\]

is

\[
\max_{b-a \in [0,1]} e^{b - \max(0, \text{sign}(ab) \cdot \min(a^2, b^2))} - e^{a - (\max(a^2, b^2))}
\]

The maximum is inside \([3.324807; 3.324808]\) and the global optimizer satisfies

\[
(a^*, b^*) \in [0.547, 0.548] \times [1.547, 1.548].
\]

i.e., the interval optimizer is an interval \([a^*, b^*]\) which satisfies the previous relation.
The figure shows the set
\[ S = \{ x \in \mathbb{R} \mid \text{w}([x] \leq 1 \text{ and } \text{w}(\exp([x] - [x]^2)) > 1) \} \]
inside the box \([-2, 2] \times [-2, 2]\).
6 Comparing two inclusion functions

Consider the two following inclusion functions

\[ f([x]) = [x] \times ([x] - 1) \]
\[ g([x]) = [x]^2 - [x] \]

We would like to known for which intervals \([x]\), \([f]\) is more accurate than \([g]\).
We have

\[ [f] ([x]) = [a, b] * ([a, b] - 1) \]
\[ = [a, b] * [a - 1, b - 1] \]
\[ = [\min(a(a - 1), b(a - 1), a(b - 1), b(b - 1)), \max(a(a - 1), b(a - 1), a(b - 1), b(b - 1))] \]
Moreover

\[ [g](x) = [a, b] - [a, b] \]
\[ = \left[ \max(0, \text{sign}(a \cdot b) \min(a^2, b^2)), \max(a^2, b^2) \right] - [a, b] \]
\[ = \left[ \max(0, \text{sign}(a \cdot b) \min(a^2, b^2) - b, \max(a^2, b^2) - a \right] \]
Thus

\[ f([x]) \subset g([x]) \iff \begin{cases} \min(a(a - 1), b(a - 1), a(b - 1), b(b - 1)) \\ -\max(0, \text{sign}(a.b) \min(a^2, b^2)) + b \geq 0 \\ \max(a^2, b^2) - a- \\ \max(a(a - 1), b(a - 1), a(b - 1), b(b - 1)) \geq 0 \end{cases} \]
• If \([x] = [1, 2]\)

\[
[f]([x]) = [1, 2] \ast ([1, 2] - 1) = [0, 2]
\]
\[
[g]([x]) = [1, 2]^2 - [1, 2] = [-1, 3]
\]

We have \([f]([x]) \subset [g]([x])\).

• If \([x] = [-2, -1]\),

\[
[f]([x]) = [-2, -1] \ast ([{-2, -1]} - 1) = [2, 6]
\]
\[
[g]([x]) = [-2, -1]^2 - [-2, -1] = [2, 6]
\]

We have \([f]([x]) \subset [g]([x])\) but we are not able to prove it.

• If \([x] = [-1, 1]\),

\[
[f]([x]) = [-1, 1] \ast ([{-1, 1]} - 1) = [-2, 2]
\]
\[
[g]([x]) = [-1, 1]^2 - [-1, 1] = [-1, 2]
\]

we have \([f]([x]) \not\subset [g]([x])\).
In red \([f]\) is more accurate that \([g]\)
In blue \([f]\) is not more accurate that \([g]\)
In yellow, we don’t know.
The frame box is \([-2, 2] \times [-2, 2]\).
Consider the equation $f(x) = 0$ with $f(x) = e^x - 1$.

The interval Newton operator is defined by

$$\mathcal{N} ([x]) = x_0 - \frac{f (x_0)}{[f']([x])},$$

where $x_0$ is any point in $[x]$. Here, we shall take $x_0 = x^-$ and thus

$$\mathcal{N} ([x]) = x^- - \frac{f (x^-)}{[f']([x])}. = x^- - \frac{e^{x^-} - 1}{\exp ([x^-, x^+]})$$
The Newton operator is contracting if

\[ N([x]) \subset [x]. \]

The interval Newton set is the set of all \([x]\) such that \(N\) is contracting.
If we set \([x] = [a, b]\), we get

\[
\mathcal{N}([a, b]) = a - \frac{a - 1}{\exp([a, b])}
\]

A boundarification of the relation \(\mathcal{N}([a, b]) \subset [a, b]\) yields

\[
\begin{cases}
  a - \max \left( \frac{e^{a-1}}{e^b}, \frac{e^{a-1}}{e^a} \right) - a \geq 0 \\
  b - a + \min \left( \frac{e^{a-1}}{e^b}, \frac{e^{a-1}}{e^a} \right) \geq 0 \\
  b - a \geq 0
\end{cases}
\]
Set of all intervals such that the interval Newton operator is contracting
The frame box is $[-2, 2] \times [-2, 2]$. 
8 Proving global consistency

8.1 Motivation

Consider the angle constraint

\[
\begin{pmatrix}
  x_2 \\
  y_2
\end{pmatrix} = \begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  y_1
\end{pmatrix}.
\]

The corresponding optimal contractor \( C^* \) is defined by

\[
\begin{align*}
\mathbb{IR}^5 \rightarrow \mathbb{IR}^5, \\
[x] & \rightarrow \begin{pmatrix}
  x_1 \\
  y_1 \\
  x_2 \\
  y_2 \\
  \theta
\end{pmatrix} \in [x], \\
\end{align*}
\]

\[
\begin{align*}
  x_2 &= x_1 \cos \theta - y_1 \sin \theta, \\
  y_2 &= x_1 \sin \theta + y_1 \cos \theta.
\end{align*}
\]
**Conjecture:** Consider the set of constraints

\[
\begin{align*}
x_2 &= x_1 \cos \theta - y_1 \sin \theta \\
y_2 &= x_1 \sin \theta + y_1 \cos \theta
\end{align*}
\]

If we add the following redundant constraints

\[
\begin{align*}
x_1 &= x_2 \cos \theta + y_2 \sin \theta \\
y_1 &= -x_2 \sin \theta + y_2 \cos \theta \\
x_1^2 + y_1^2 &= x_2^2 + y_2^2 \\
\tan \theta &= \frac{x_1y_2 - y_1x_2}{x_1x_2 + y_1y_2}
\end{align*}
\]

A hull consistency algorithm with input \([x]\) will to converge toward \(C^*([x])\).
With Xavier Baguenard, we tried to prove it by hand, but we failed.

**Question** : Can we automatically prove this conjecture with interval methods?
Consider a simpler constraint given by

\[ x^2 - x = 0 \]
A hull consistency contractor for this constraint amounts to iterate the two statements

\[
[x] = [x] \cap [x]^2
\]

\[
[x] = [x] \cap \sqrt{[x]}
\]

from an initial interval \([x]\) until a steady interval is reached.
The resulting contractor is said optimal if it always converge to the smallest box which encloses all solutions that belongs to \([x]\).

**Question:** Is the hull contractor optimal?
Step 1. Compute all solutions of the equation \( x^2 - x = 0 \). With an interval method (with bisections), we get that we have exactly two solutions

\[ x_1 \simeq 0 \text{ and } x_2 \simeq 1 \]

Thus any safe contractor has at least 3 steady boxes (those corresponding to \([0, 0], [1, 1], [0, 1]\)).
Step 2. Since the hull contractor will converge the biggest box inside $[x](0)$ which satisfies

$$[x] \subset [x]^2$$

$$[x] \subset \sqrt{|x|}$$

The interval CSP translates into the following bound-CSP

$$a - \max(0, \text{sign}(a.b).\min(a^2, b^2)) \geq 0$$

$$\max(a^2, b^2) - b \geq 0$$

$$\min(a - \sqrt{a}, \sqrt{b} - b) \geq 0$$

$$b - a \geq 0$$
This bound-CSP has three solutions enclosed by

\[
[0.999999999, 1] \times [0.999999999, 1] \\
[0, 3.10^{-39}] \times [0, 3.10^{-39}] \\
[0, 3.10^{-39}] \times [0.999999999, 1]
\]

A unicity test concludes that each of the three boxes contains a unique solution.

Thus, we know that we have exactly three steady boxes.
Thus, we have proven that the hull contractor is optimal.