Cooperative Robotics and Intervals

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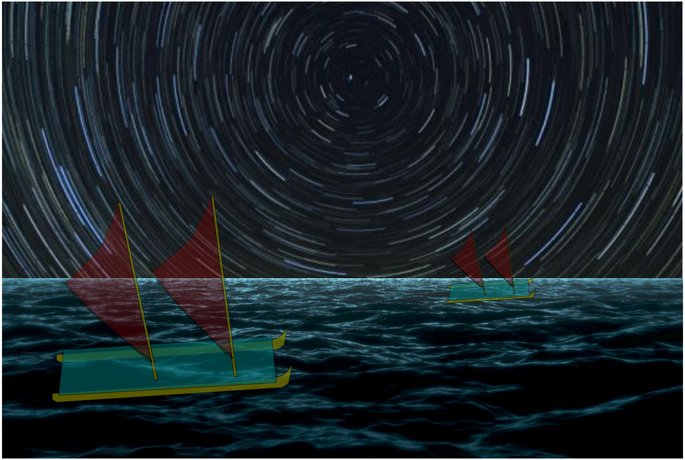
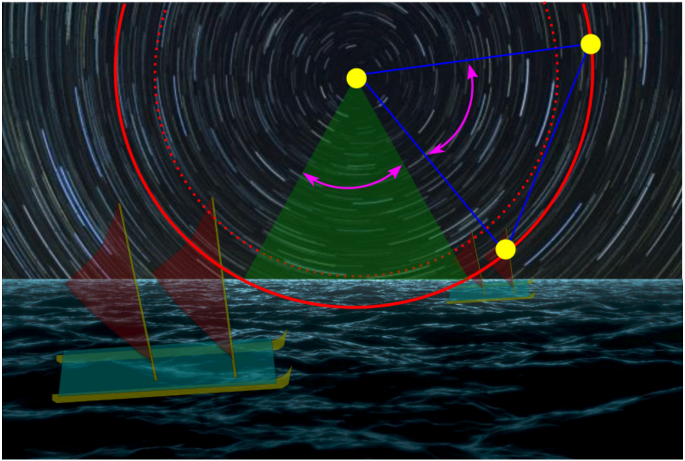
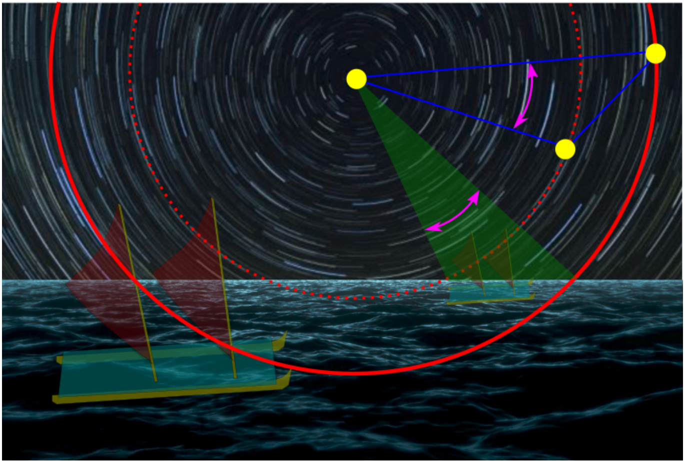
Polynesian navigation



Find the route without GPS, compass and clocks with *wa'a kaulua*[5]

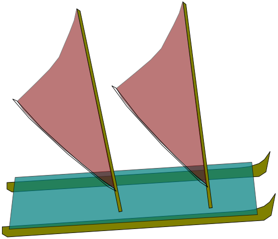


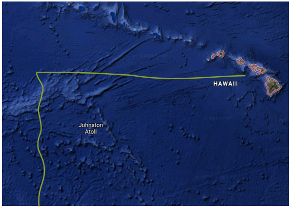
Pair of stars technique

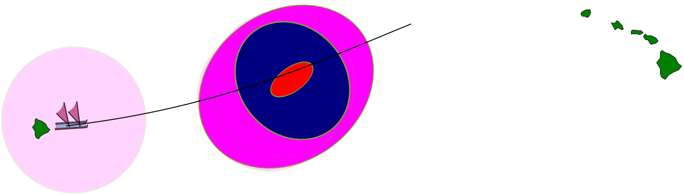


Pair of stars technique

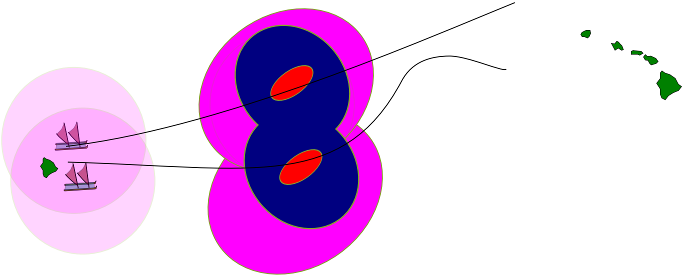

Arctic Ocean
Russia
Canada
Bering Sea
Sea of Okhotsk
Mongolia
Sea of Japan
China
South Korea
Japan
East China Sea
North Pacific Ocean
Myanmar (Burma)
Philippine Sea
Thailand
Vietnam
Malaysia
Indonesia
Banda Sea
Papua New Guinea
Arafura Sea
Coral Sea
Australia
South Pacific Ocean
Great Australian Bight
Tasman Sea
New Zealand
Google




HAWAII
Johnston Atoll

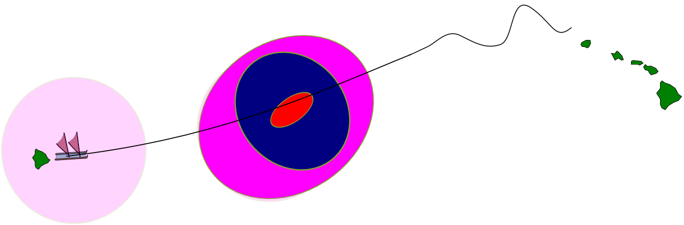


Prove that islands will be reached by one boat

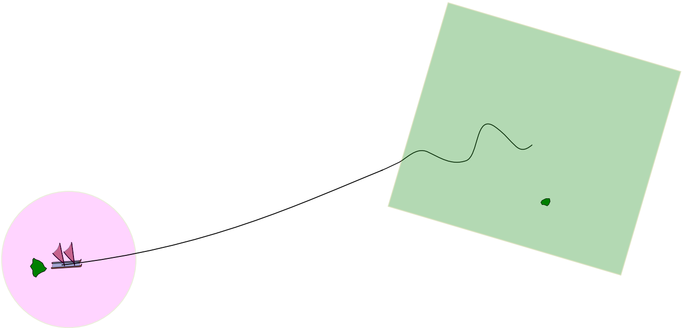


Prove that islands will be reached by the n boats

Alignment to keep the heading in case of clouds



Find a control to reach the geo-localized islands



Explore a given area entirely to find new islands

Secure a zone

Secure a zone

INFO OBS. Un sous-marin nucléaire russe repéré dans le Golfe de Gascogne

Le navire a été repéré en janvier. Ce serait la première fois depuis la fin de la Guerre Froide qu'un tel sous-marin, doté de missiles nucléaires, se serait aventuré dans cette zone au large des côtes françaises.

Bay of Biscay 220 000 km^2

An intruder

- Several robots $\mathscr{R}_1, \dots, \mathscr{R}_n$ at positions $\mathbf{a}_1, \dots, \mathbf{a}_n$ are moving in the ocean.
- If the intruder is in the visibility zone of one robot, it is detected.[8]

Complementary approach

- We assume that a virtual intruder exists inside $\mathbb{G}$.
- We localize it with a set-membership observer inside $\mathbb{X}(t)$.
- The secure zone corresponds to the complementary of $\mathbb{X}(t)$.

Assumptions

- The intruder satisfies

$$\dot{\mathbf{x}} \in \mathbb{F}(\mathbf{x}(t)).$$

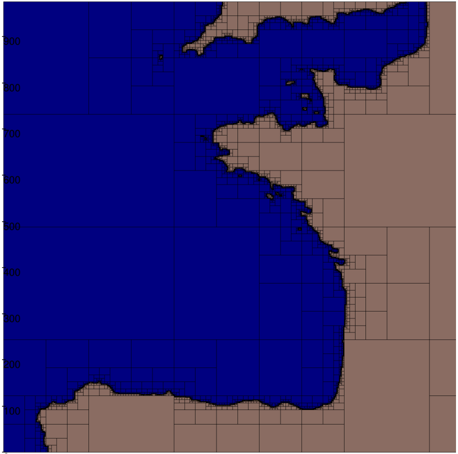
- Each robot $\mathscr{R}_i$ has the visibility zone $g_{\mathbf{a}_i}^{-1}([0, d_i])$ where d_i is the scope.

Theorem. An (undetected) intruder has a state vector $\mathbf{x}(t)$ inside the set

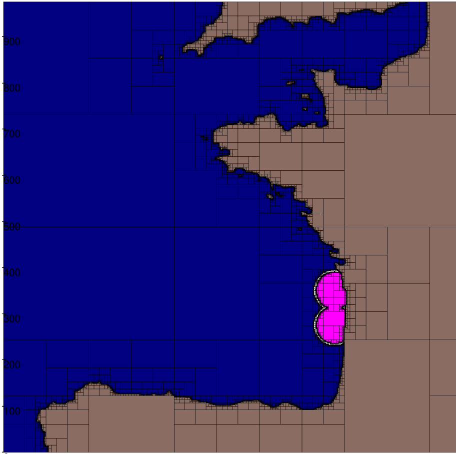
$$\mathbb{X}(t) = \mathbb{G} \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt)) \cap \bigcap_i g_{\mathbf{a}_i(t)}^{-1}([d_i(t), \infty]),$$

where $\mathbb{X}(0) = \mathbb{G}$. The secure zone is

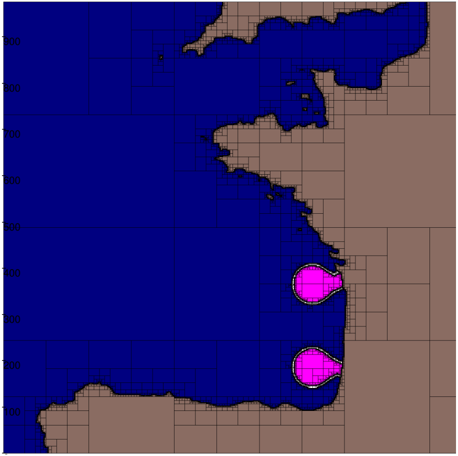
$$\mathbb{S}(t) = \overline{\mathbb{X}(t)}.$$



Set $\mathbb{G}$ in blue

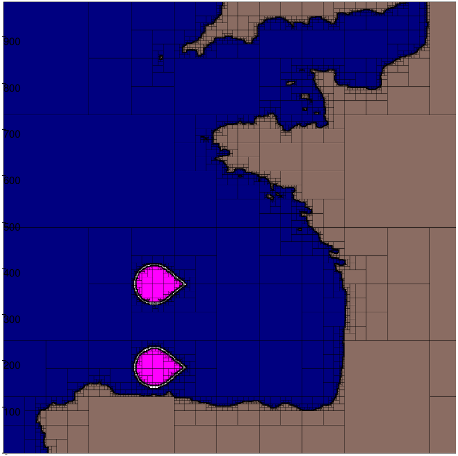


Magenta: $\mathbb{G} \cap \bigcup_i g_{\mathbf{a}_i(t)}^{-1}([0, d_i(t)])$ Blue:$\mathbb{G} \cap \bigcap_i g_{\mathbf{a}_i(t)}^{-1}([d_i(t), \infty])$



Blue:

$$\mathbb{X}(t) = \mathbb{G} \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt))) \cap \bigcap_i g^{-1}_{\mathbf{a}_i(t)}([d_i(t), \infty]).$$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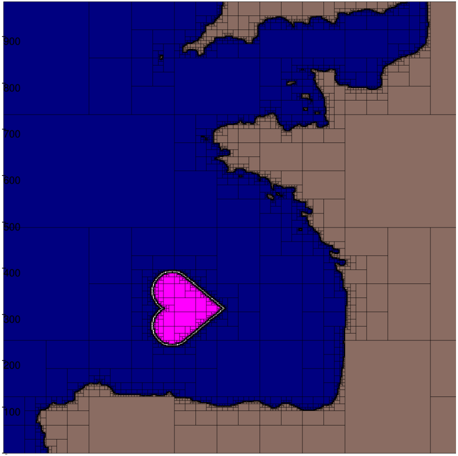
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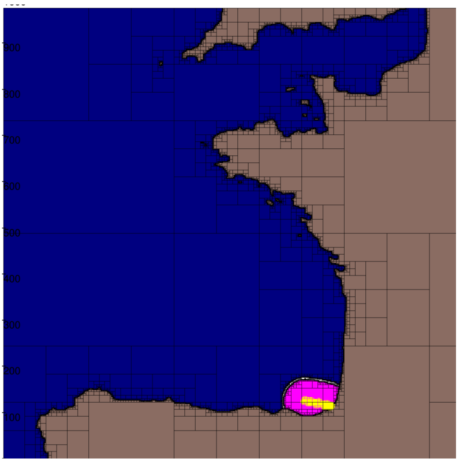
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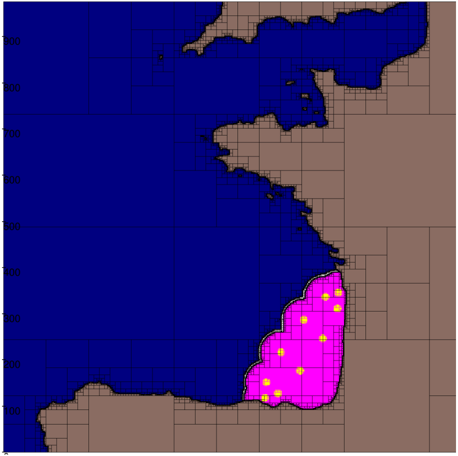
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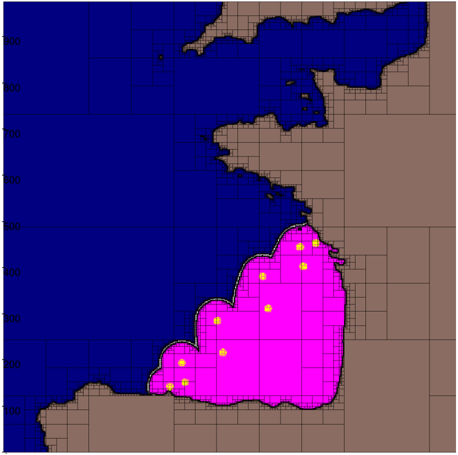
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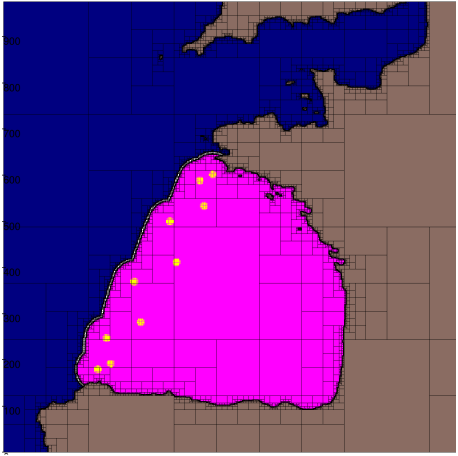


Video : https://youtu.be/rNcDW6npLfE

Smoother

Idea: Take into account the future.
The feasible set can be obtained by the following contractions

$$\begin{array}{rcl} \overrightarrow{\mathbb{X}}(t) & = & \overrightarrow{\mathbb{X}}(t) \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt))) \\ \overleftarrow{\mathbb{X}}(t) & = & \overleftarrow{\mathbb{X}}(t) \cap (\mathbb{X}(t+dt) - dt \cdot \mathbb{F}(\mathbb{X}(t+dt))) \\ \mathbb{X}(t) & = & \overrightarrow{\mathbb{X}}(t) \cap \overleftarrow{\mathbb{X}}(t) \end{array}$$

with the initialization

$$\mathbb{X}(t) = \overrightarrow{\mathbb{X}}(t) = \overleftarrow{\mathbb{X}}(t) = \mathbb{G}.$$

Intervals

Intervals

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$\begin{aligned} [-1,3]+[2,5] &= ?, \\ [-1,3]\cdot[2,5] &= ?, \\ \mathrm{abs}([-7,1]) &= ? \end{aligned}$$

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\operatorname{abs}([-7,1]) &= [0,7]
\end{aligned}
$$

The interval extension of

$$f\,(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{array}{rcl} [f]\,([x_1], [x_2]) & = & [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ & & + \sin [x_1] \cdot \sin [x_2] + 2. \end{array}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^+ \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

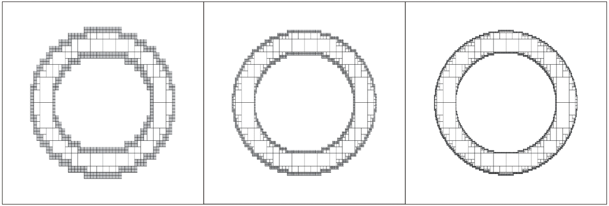
Set Inversion

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.
Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \in [1,2]\}.$$



Let $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbb{Y}$ be a subset of $\mathbb{R}^m$. Set inversion is the characterization of

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

We shall use the following tests.

$$\begin{array}{lll}\text{(i)} & [\mathbf{f}]([\mathbf{x}])\subset\mathbb{Y} & \Rightarrow & [\mathbf{x}]\subset\mathbb{X}\\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}])\cap\mathbb{Y}=\emptyset & \Rightarrow & [\mathbf{x}]\cap\mathbb{X}=\emptyset.\end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

Contractors

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* [4] for the equation $f(\mathbf{x}) = 0$, if

$$\begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathscr{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

Building contractors

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$\begin{array}{rl} x_3 = x_1 + x_2 \Rightarrow & x_3 \in & [x_3] \cap ([x_1] + [x_2]) \\ x_1 = x_3 - x_2 \Rightarrow & x_1 \in & [x_1] \cap ([x_3] - [x_2]) \\ x_2 = x_3 - x_1 \Rightarrow & x_2 \in & [x_2] \cap ([x_3] - [x_1]) \end{array}$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathscr{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1]\cap([x_3]-[x_2]) \\ [x_2]\cap([x_3]-[x_1]) \\ [x_3]\cap([x_1]+[x_2]) \end{pmatrix}$$

Tubes

A trajectory is a function $\mathbf{f}:\mathbb{R}\to\mathbb{R}^n$. [7, 6]. For instance

$$\mathbf{f}(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$$

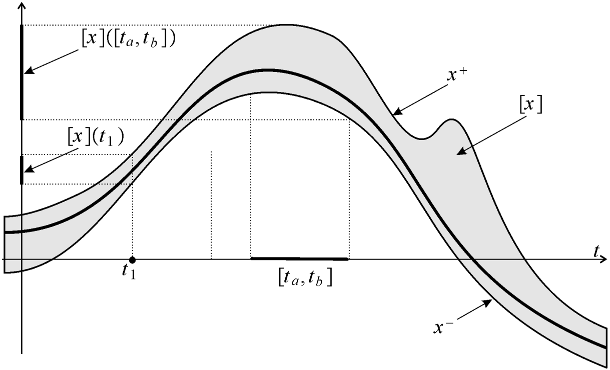
is a trajectory.

Order relation

$$\mathbf{f} \leq \mathbf{g} \Leftrightarrow \forall t, \forall i, f_i(t) \leq g_i(t).$$

We have

$$\begin{aligned}\mathbf{h} &= \mathbf{f}\wedge\mathbf{g}\Leftrightarrow\forall t,\forall i, h_i(t)=\min(f_i(t),g_i(t)),\\ \mathbf{h} &= \mathbf{f}\vee\mathbf{g}\Leftrightarrow\forall t,\forall i, h_i(t)=\max(f_i(t),g_i(t)).\end{aligned}$$



The set of trajectories is a lattice. Interval of trajectories (tubes) can be defined.

Example.

$$[\mathbf{f}](t)=\left(\begin{array}{c}\cos t+\left[0,t^2\right]\\ \sin t+[-1,1]\end{array}\right)$$

is an interval trajectory (or tube).

Tube arithmetics

If $[x]$ and $[y]$ are two scalar tubes [1], we have

$$\begin{array}{ll}
[z]=[x]+[y]\Rightarrow[z](t)=[x](t)+[y](t) & \text{(sum)}\\
[z]=\mathrm{shift}_a([x])\Rightarrow[z](t)=[x](t+a) & \text{(shift)}\\
[z]=[x]\circ[y]\Rightarrow[z](t)=[x]([y](t)) & \text{(composition)}\\
[z]=\int[x]\Rightarrow[z](t)=\left[\int_0^t x^-(\tau)\,d\tau,\int_0^t x^+(\tau)\,d\tau\right] & \text{(integral)}
\end{array}$$

Tube Contractors

Tube arithmetic allows us to build contractors [3].

Consider for instance the differential constraint

$$\begin{array}{rcl} \dot{x}(t) & = & x(t+\tau)\cdot u(t), \\ x(t) & \in & [x](t), \dot{x}(t) \in [\dot{x}](t), u(t) \in [u](t), \tau \in [\tau] \end{array}$$

We decompose as follows

$$\left\{ \begin{array}{rcl} x(t) & = & x(0) + \int_0^t y(\tau)\, d\tau \\ y(t) & = & a(t)\cdot u(t). \\ a(t) & = & x(t+\tau) \end{array} \right.$$

Possible contractors are

$$\left\{ \begin{array}{lll} [x](t) & = & [x](t) \cap \left([x](0) + \int_0^t [y](\tau)\, d\tau\right) \\ [y](t) & = & [y](t) \cap [a](t) \cdot [u](t) \\ [u](t) & = & [u](t) \cap \frac{[y](t)}{[a](t)} \\ [a](t) & = & [a](t) \cap \frac{[y](t)}{[u](t)} \\ [a](t) & = & [a](t) \cap [x](t + [\tau]) \\ [x](t) & = & [x](t) \cap [a](t - [\tau]) \\ [\tau] & = & [\tau](t) \cap \dots \end{array} \right.$$

Example. Consider $x(t) \in [x](t)$ with the constraint

$$\forall t,\ x(t) = x(t+1)$$

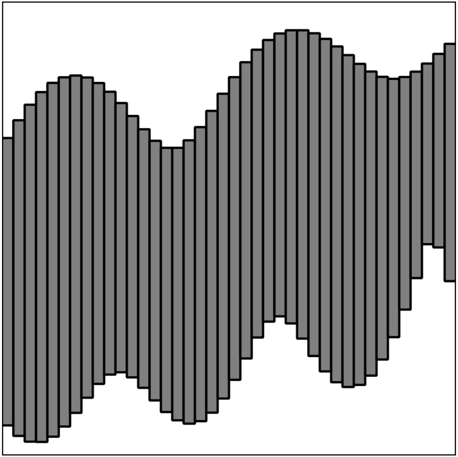
Contract the tube $[x](t)$.

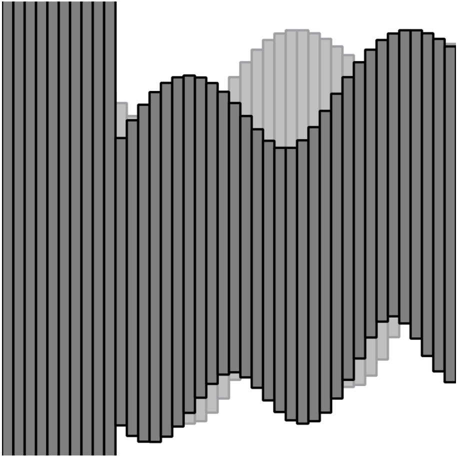
We first decompose into primitive trajectory constraints

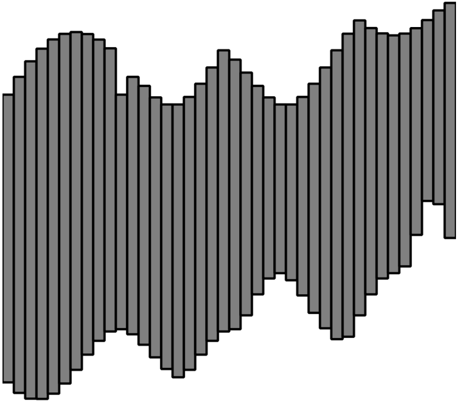
$$\begin{array}{rcl} x(t) & = & a(t+1) \\ x(t) & = & a(t). \end{array}$$

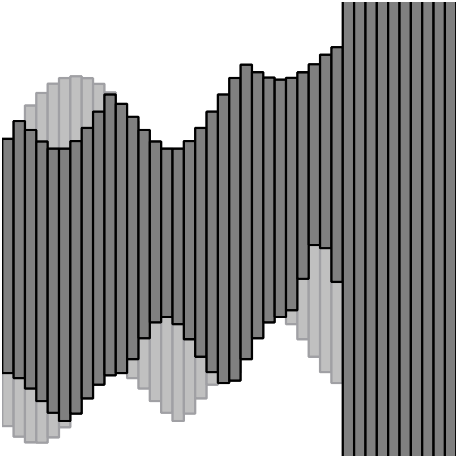
Contractors

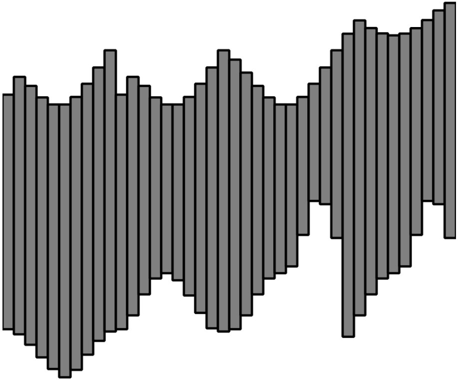
$$
\begin{aligned}
[x](t) &: = [x](t) \cap [a](t+1) \\
[a](t) &: = [a](t) \cap [x](t-1) \\
[x](t) &: = [x](t) \cap [a](t) \\
[a](t) &: = [a](t) \cap [x](t)
\end{aligned}
$$

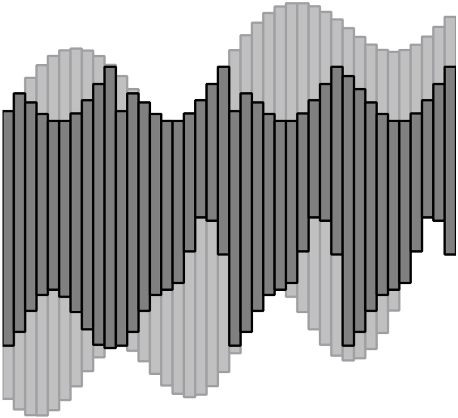


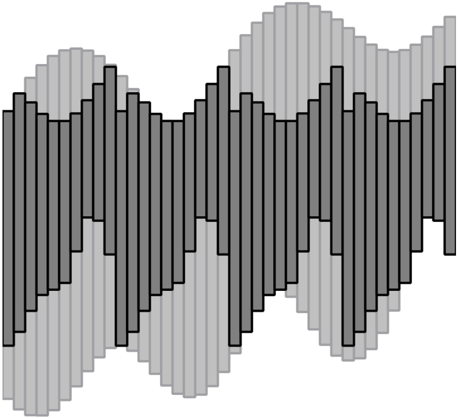












tubex-lib
1.0
Search docs

Definition

A tube $[\mathbf{x}](\cdot)$ is defined as an envelope enclosing an uncertain trajectory $\mathbf{x}(\cdot):\mathbb{R}\to\mathbb{R}^n$. It is built as an interval of two functions $[\mathbf{x}^-(\cdot),\mathbf{x}^+(\cdot)]$ such that $\forall t,\ \mathbf{x}^-(t)\leqslant\mathbf{x}^+(t)$. A trajectory $\mathbf{x}(\cdot)$ belongs to the tube $[\mathbf{x}](\cdot)$ if $\forall t,\ \mathbf{x}(t)\in[\mathbf{x}](t)$. Fig. 1 illustrates a tube implemented with a set of boxes. This sliced implementation is detailed hereinafter.

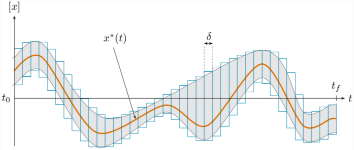


Fig. 1 A tube $[x](\cdot)$ represented by a set of slices. This representation can be used to enclose signals such as $x^*(\cdot)$.

Code example:

```
float timestep = 0.1;
Interval domain(0,10);
Tube x(domain, timestep, Function("t", "(t-5)^2 + [-0.5,0.5]"));
```

http://www.simon-rohou.fr/research/tubex-lib/ [7]

Cooperative localization

Time-space estimation

Classical state estimation

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t),t) & t \in \mathbb{T} \subset \mathbb{R}. \end{cases}$$

Space constraint $\mathbf{g}(\mathbf{x}(t),t) = 0$.

Example.

$$\begin{cases} \dot{x}_1 = x_3 \cos x_4 \\ \dot{x}_2 = x_3 \cos x_4 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = u_2 \\ (x_1(5) - 1)^2 + (x_2(5) - 2)^2 - 4 = 0 \\ (x_1(7) - 1)^2 + (x_2(7) - 2)^2 - 9 = 0 \end{cases}$$

With time-space constraints

$$\left\{ \begin{array}{lll} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t),\mathbf{x}(t'),t,t') & (t,t') \in \mathbb{T} \subset \mathbb{R} \times \mathbb{R}. \end{array} \right.$$

Example. An ultrasonic underwater robot with state

$$\mathbf{x} = (x_1, x_2, \dots) = (x, y, \theta, v, \dots)$$

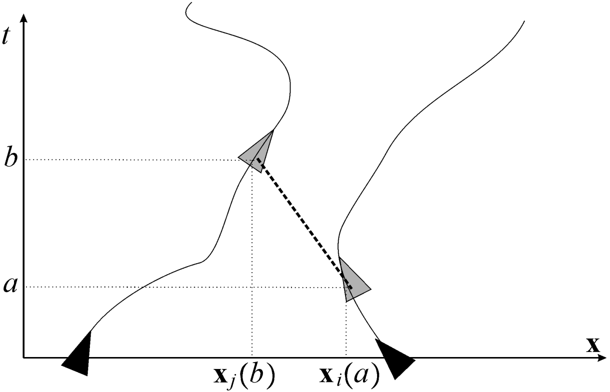
At time t the robot emits an omni-directional sound. At time t' it receives it

$$\left(x_1 - x_1^{'}\right)^2 + \left(x_2 - x_2^{'}\right)^2 - c\left(t - t'\right)^2 = 0.$$

Consider n robots $\mathscr{R}_1, \dots, \mathscr{R}_n$ described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i].$$

Omnidirectional sounds are emitted and received.
A *ping* is a 4-uple (a, b, i, j) where a is the emission time, b is the reception time, i is the emitting robot and j the receiver.


t
b
a
x
xj(b)
xi(a)

With the time space constraint

$$\begin{array}{l}\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\ g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0\end{array}$$

where

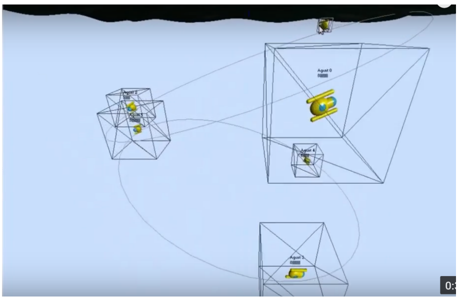
$$g(\mathbf{x}_i, \mathbf{x}_j, a, b) = \|x_1 - x_2\| - c(b - a).$$

Clocks are uncertain. We only have measurements $\tilde{a}(k), \tilde{b}(k)$ of $a(k), b(k)$ thanks to clocks h_i. Thus

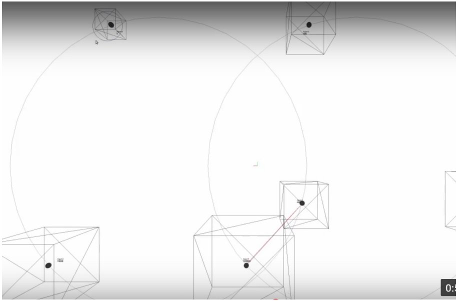
$$
\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}(a(k)) \\
\tilde{b}(k) = h_{j(k)}(b(k))
\end{array}
$$

The drift of the clocks is bounded

$$
\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}\left(\mathbf{x}_i, \mathbf{u}_i\right), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}\left(a(k)\right), \mathbf{x}_{j(k)}\left(b(k)\right), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}\left(a(k)\right) \\
\tilde{b}(k) = h_{j(k)}\left(b(k)\right) \\
\dot{h}_i = 1 + n_h,\ n_h \in [n_h]
\end{array}
$$



https://youtu.be/j-ERcoXF1Ks [2]



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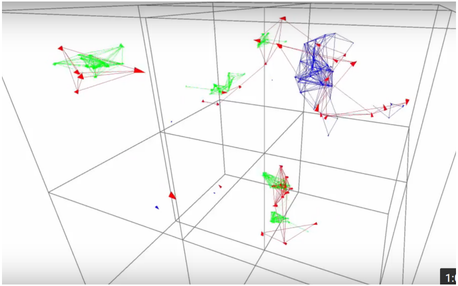

https://youtu.be/GycJxGFvYE8

https://youtu.be/GVGTwnJ_dpQ

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