Winding contractor

Brest (virtual)
2021, September 02

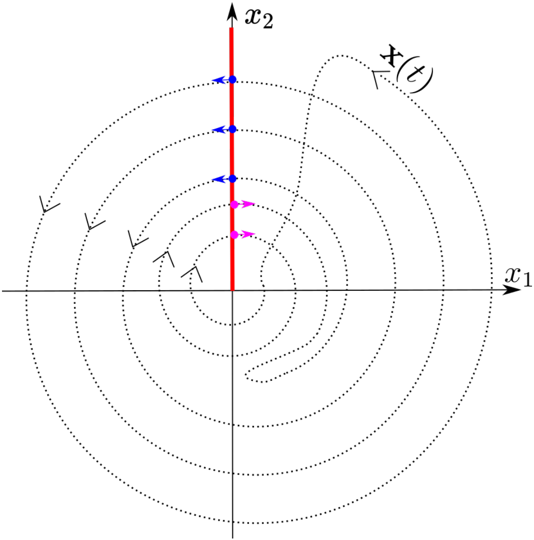
Problem

We consider a trajectory $\mathbf{x}(t)$ of $\mathbb{R}^2$, $t \in [0, t_{\max}]$. Here, $t_{\max} = 1$. Define the half-line $\mathscr{D} = \{0\} \times [0, \infty]$. Define

$$\begin{array}{rcl} \mathbb{T}^+ & = & \{t, 0 \leq t < t_{\max} \,|\, \mathbf{x}(t) \in \mathscr{D}, x_1(t+dt) < 0\} \\ \mathbb{T}^- & = & \{t, 0 \leq t < t_{\max} \,|\, \mathbf{x}(t) \in \mathscr{D}, x_1(t+dt) > 0\} \end{array}$$

The crossing number is $\eta(\mathbf{x}(\cdot)) = \text{card}(\mathbb{T}^+) - \text{card}(\mathbb{T}^-)$.

If $\mathbf{x}(0) = \mathbf{x}(t_{\max})$, $\eta(\mathbf{x}(\cdot))$ is the winding number


x_2
$\mathbf{x}(t)$
x_1

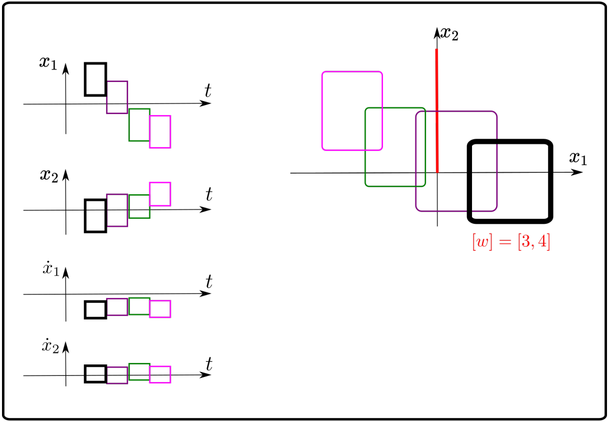
Winding contractor

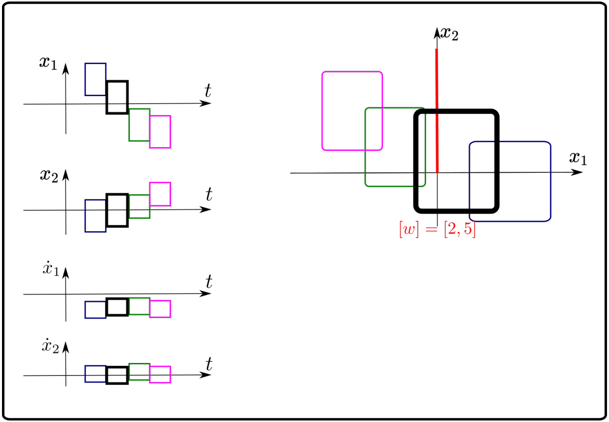
$$\mathbf{x}(\cdot) \in [\mathbf{x}](.)$$
$$\dot{\mathbf{x}}(\cdot) \in [\dot{\mathbf{x}}](.)$$
$$\eta(\mathbf{x}(\cdot)) \in [\eta]$$

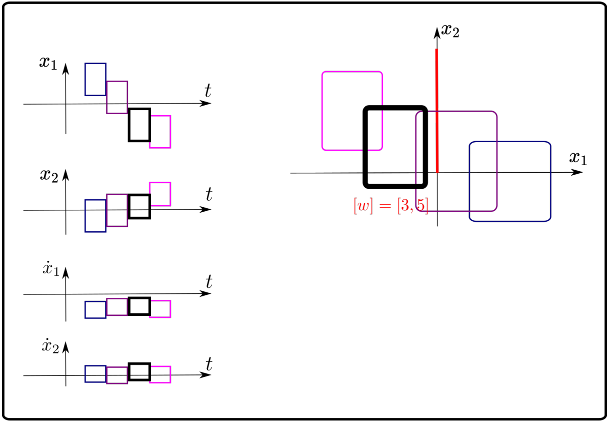
Contract as much as possible the tubes $[\mathbf{x}](.), [\dot{\mathbf{x}}](.)$ and the interval $[\eta]$, without removing solutions.

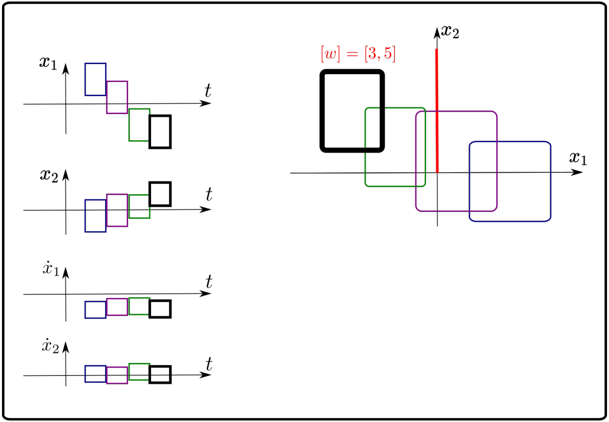
Approach

Denote by $[\mathbf{x}](k), [\dot{\mathbf{x}}](k)$ the kth slides of the tubes $[\mathbf{x}](\cdot), [\dot{\mathbf{x}}](\cdot)$.
We assume that for $k = 0$, $[\mathbf{x}](k) \cap \mathscr{D} = \emptyset$.
Denote by $w(t)$ the crossing number if we take $t_{\max} = t$.
We want to compute the tube $[w](k)$ recursively in a forward manner.


x_1
t
x_2
t
$\dot{x}_1$
t
$\dot{x}_2$
t
x_2
x_1
$[w] = [3, 4]$


x_1
t
x_2
t
$\dot{x}_1$
t
$\dot{x}_2$
t
x_2
x_1
$[w] = [2, 5]$

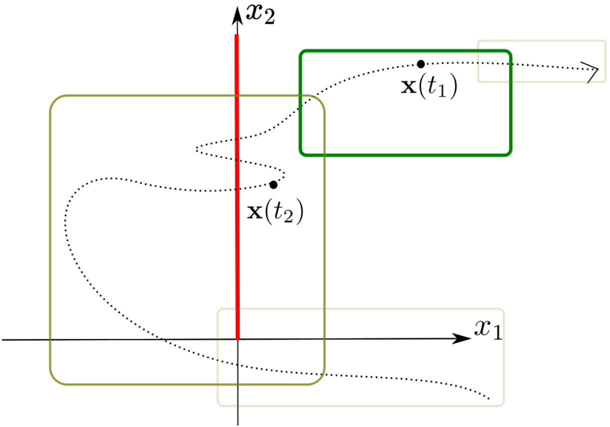

x_1
t
x_2
t
$\dot{x}_1$
t
$\dot{x}_2$
t
x_2
x_1
$[w] = [3, 5]$


x_1
t
x_2
t
$\dot{x}_1$
t
$\dot{x}_2$
t
$[w] = [3, 5]$
x_2
x_1

Contractor rules

Proposition 1.

$$\left\{\begin{array}{c} [\mathbf{x}](k)\cap\mathscr{D}=\emptyset \\ |\ell-k|=1 \\ t_1\in[t](k) \end{array}\right\}\Rightarrow \exists t_2\in[t](\ell)\,|\,w(t_1)=w(t_2)$$

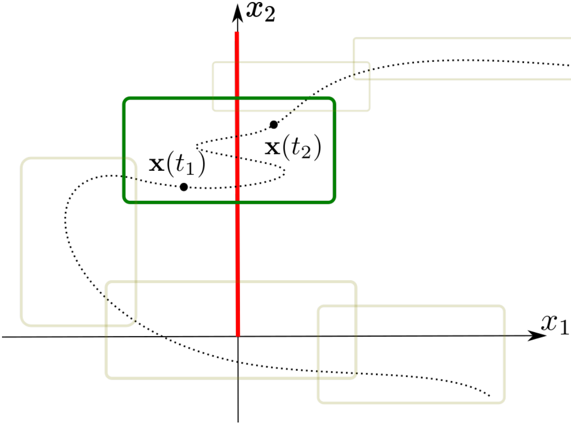

x_2
$\mathbf{x}(t_1)$
$\mathbf{x}(t_2)$
x_1

Rule 1

$$[\mathbf{x}](k) \cap \mathscr{D} = \emptyset \quad \Rightarrow \; [w](k) := [w](k) \cap [w](k-1) \cap [w](k+1)$$

Proposition 2.

$$\left\{\begin{array}{c} [\mathbf{x}](k) \cap \mathscr{D} \neq \emptyset \\ \mathbf{0} \notin [\mathbf{x}](k) \\ t_1, t_2 \in [t](k) \\ x_1(t_1) < 0 < x_1(t_2) \end{array}\right\} \Rightarrow w(t_1) = w(t_2) + 1$$

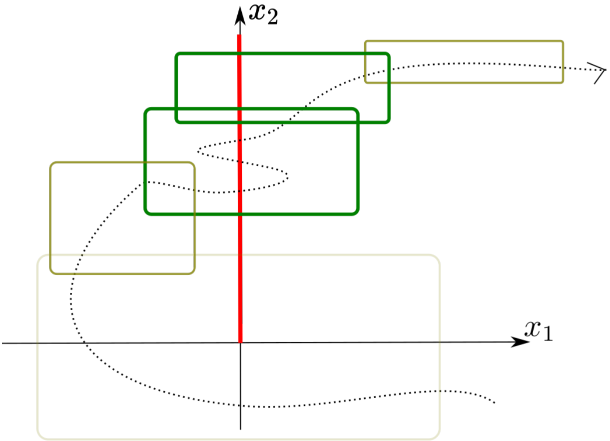

x_2
x_1
$\mathbf{x}(t_1)$
$\mathbf{x}(t_2)$

Rule 2

$$\left\{\begin{array}{c}[\mathbf{x}](k)\cap\mathscr{D}\neq\emptyset\\ |\ell-k|=1\\ \mathbf{0}\notin[\mathbf{x}](k)\end{array}\right.\quad\Rightarrow\quad [w](k):=[w](k)\cap([w](\ell)+[\varepsilon])$$

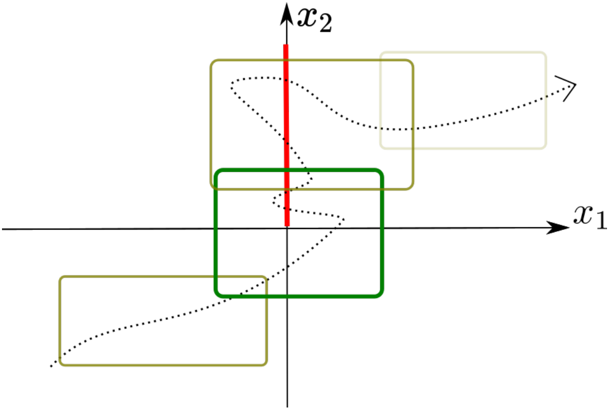
with

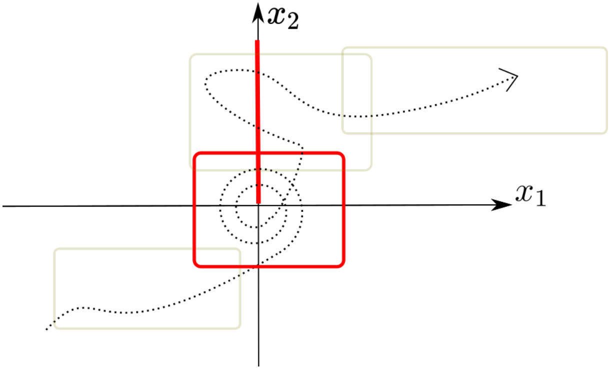
$$[\varepsilon]=\left\{\begin{array}{ccc}[0,1] & \text{if} & [x_1](\ell)\subset\mathbb{R}^+\\ [-1,0] & \text{if} & [x_1](\ell)\subset\mathbb{R}^-\\ [0,0] & \text{if} & 0\in[x_1](\ell)\end{array}\right.$$


x_2
x_1

Rule 3

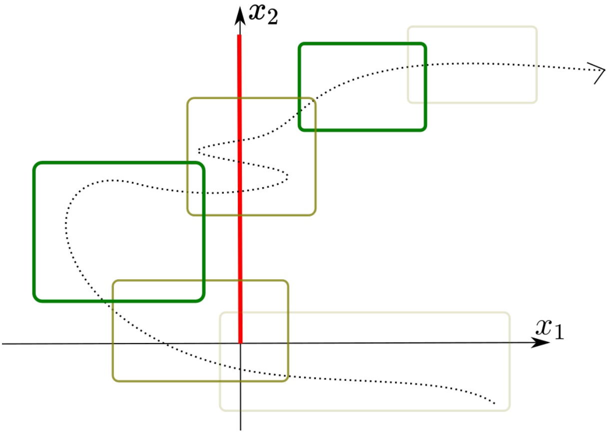
$$\left\{ \begin{array}{l} \mathbf{0} \in [\mathbf{x}](k) \\ \mathbf{0} \notin [\dot{\mathbf{x}}](k) \\ |\ell - k| = 1 \end{array} \right. \Rightarrow [w](k) := [w](k) \cap ([w](\ell) + [-1,1])$$


x_2
x_1


x_2
x_1

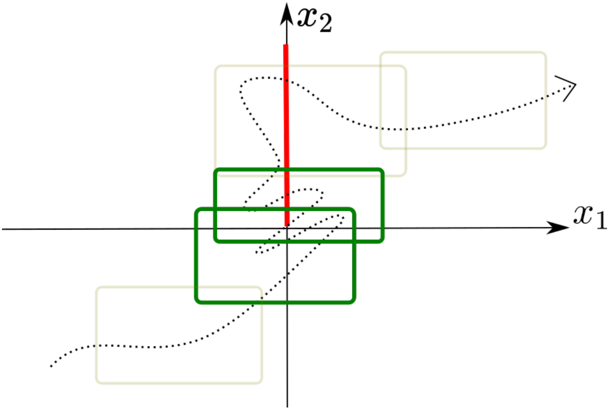
Rule 4

$$\left\{\begin{array}{c} [\mathbf{x}](k)\cap\mathscr{D}=\emptyset \\ |\ell-k|=1 \\ [\mathbf{x}](\ell)\cap\mathscr{D}\neq\emptyset \end{array}\right. \Rightarrow \left\{\begin{array}{c} [w](k):=[w](k)\cap[w^-(\ell),w^+(\ell)-1] \\ \text{if } [x_1](k)\subset\mathbb{R}^+ \\ [w](k):=[w](k)\cap[w^-(\ell)+1,w^+(\ell)] \\ \text{if } [x_1](k)\subset\mathbb{R}^- \end{array}\right.$$


x_2
x_1

Rule 5

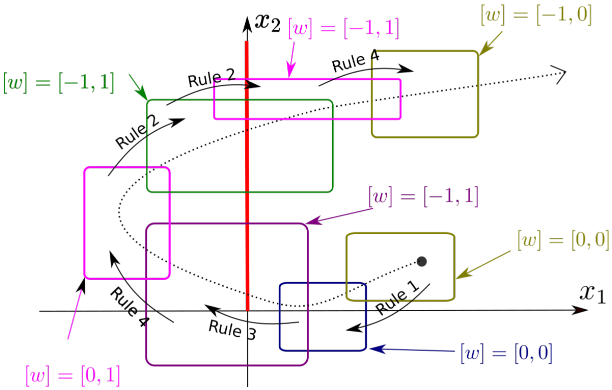
$$\left\{\begin{array}{c} \mathbf{0} \in [\mathbf{x}](k) \\ \mathbf{0} \in [\mathbf{x}](\ell) \\ |\ell - k| = 1 \\ \mathbf{0} \notin [\dot{\mathbf{x}}](k) \\ \mathrm{sign}([\dot{\mathbf{x}}](\ell)) = \mathrm{sign}([\dot{\mathbf{x}}](k)) \end{array}\right. \quad \Rightarrow [w](k) := [w](k) \cap [w](\ell)$$


x_2
x_1

Computing the winding interval

Initialization

$$\begin{cases} [w](0) = 0 \\ [w](k) = [-\infty, \infty], k \geq 1 \end{cases}$$


x_2
$[w] = [-1, 1]$
$[w] = [-1, 0]$
$[w] = [-1, 1]$
Rule 2
Rule 4
Rule 2
$[w] = [-1, 1]$
$[w] = [0, 0]$
x_1
Rule 1
Rule 4
Rule 3
$[w] = [0, 0]$
$[w] = [0, 1]$