Computing inner and outer approximations of forward reach sets of a nonlinear dynamical system

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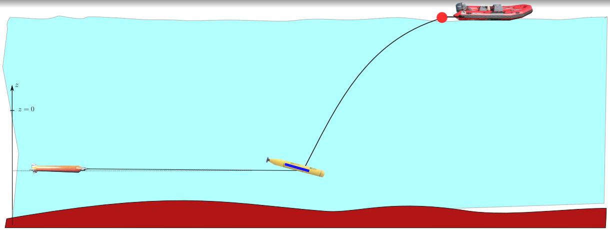
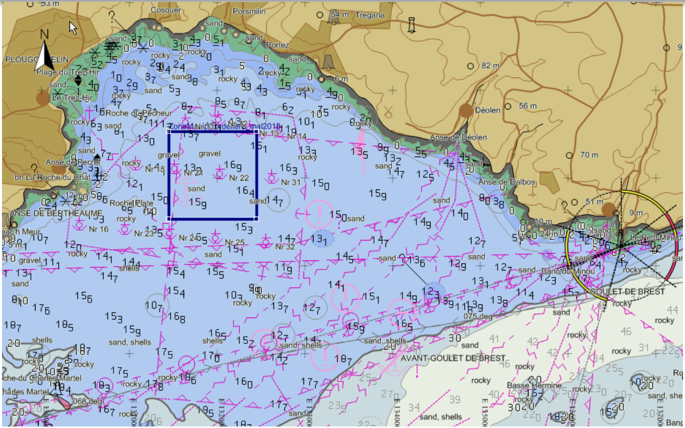
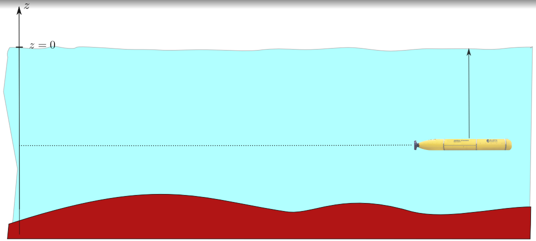
Lab-STICC, ENSTA-Bretagne
Grenoble, Verimag, 2018 May 15

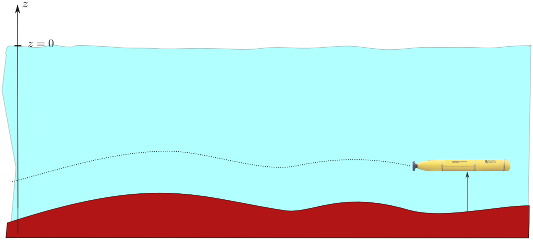
Navigating underwater

La Cordelière


z
$z = 0$


Cosquer
Porsmilin
Tregana
Portez
N
Plage du Trez-Hir
Le Trez-Hir
Roche du Pêcheur
Déolen
Anse de Déolen
Anse de Dalbos
Anse de Perzel
ANSE DE BERTHEAUME
Rochel Plate
Banc du Minou
GOULET DE BREST
AVANT-GOULET DE BREST
Basse Hermine
075 deg
068 deg


z
$z = 0$

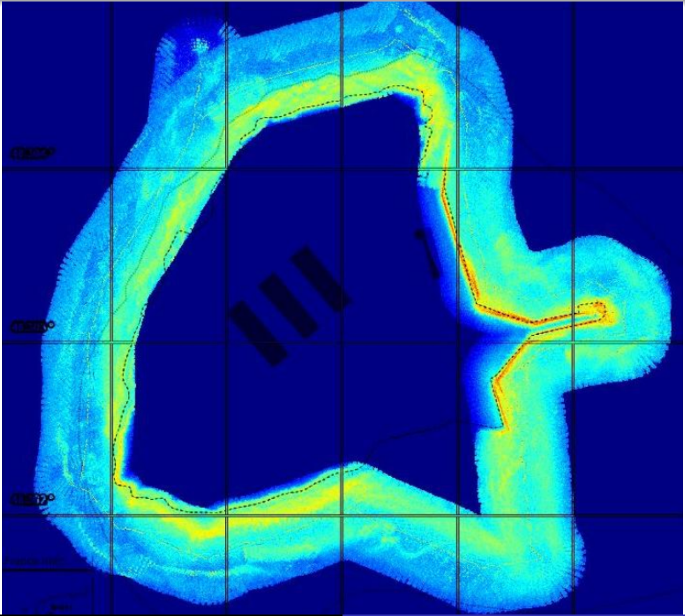

z
$z = 0$

Ile des morts experiment

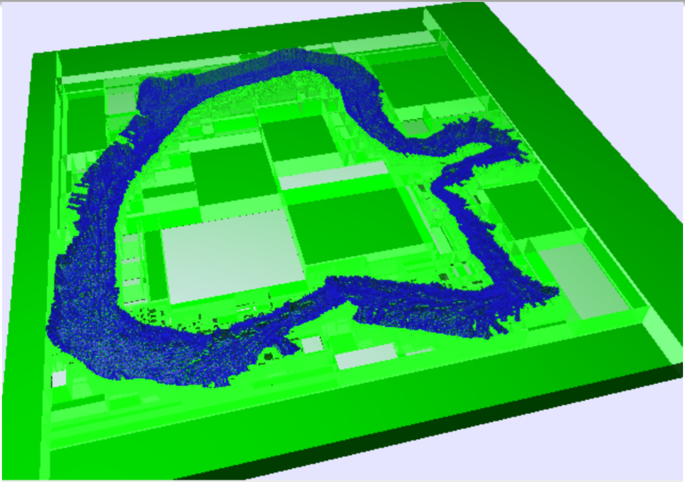

Saint-Renan
Guilers
Bohars
Ploumoguer
Plouzané
Locmaria-Plouzané
Plougonvelin
Lambézellec
Saint-Marc
Brest
Saint-Pierre
Le Relecq-Kerhuon
L'Elorn
La Forest-Landerneau
Dirinon
Saint-Urbain
Plougastel-Daoulas
Loperhet
N165
E60
E50
Irvillac
Daoulas
Hôpital-Camfrout
Logonna-Daoulas
Roscanvel
Lanvéoc
Landévennec
Camaret-sur-Mer
Crozon
Rue de Dinan
Argol
Telgruc-sur-Mer
Rosnoën
Saint-Nic
Ar Valare
Plomodiern


Île des Morts
ROSCANVEL
Route de Keruzou
Le Disloup
Route du Milieu
Route du Ménez
Route de Quélern
Camping Municipal
Route de la Fraternité
D355
Celtic Sea
Persuel
Rostellec
Taladec'h
Rue Auguste


8
9
7
10
11
6
5
12
4
2
3
23
13
22
21
20
14
15
17
16
19
18
50m



24 juillet 2013



Test-case

Consider an underwater robot:

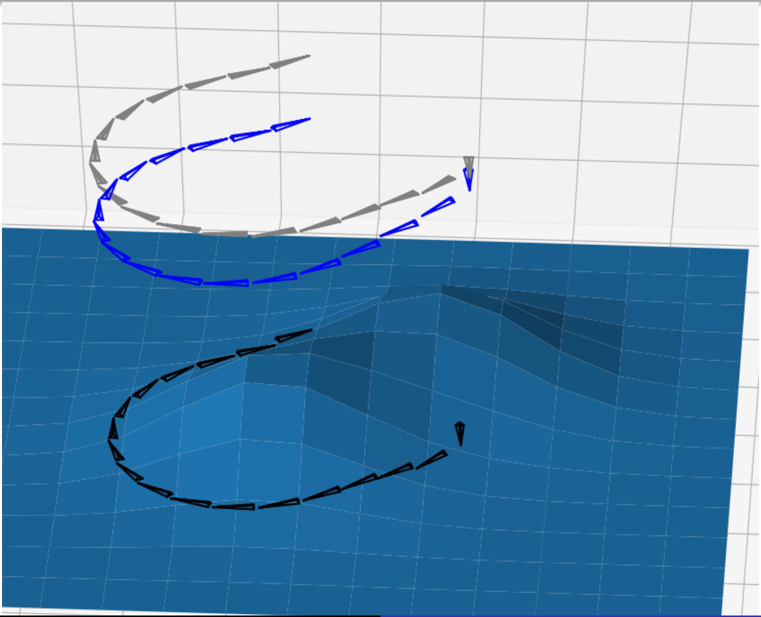
$$\begin{cases} \dot{x} & = & \cos\psi \\ \dot{y} & = & \sin\psi \\ \dot{z} & = & u_1 \\ \dot{\psi} & = & u_2 \end{cases}$$

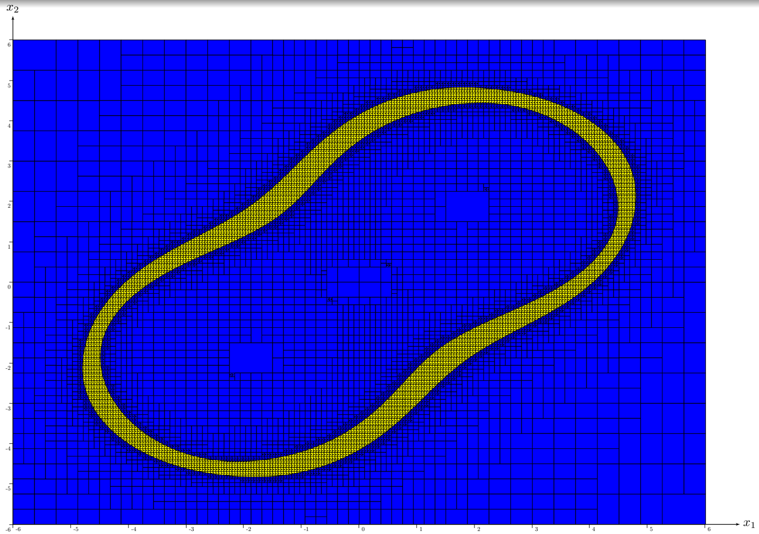
The robot is able to measure its altitude, the angle of the gradient of h and its depth

$$\begin{cases} y_1 &=& z - h(x,y) \\ y_2 &=& angle(\nabla h(x,y)) - \psi \\ y_3 &=& -z \end{cases}$$

We take the controller

$$\mathbf{u} = \begin{pmatrix} y_3 - \overline{y}_3 \\ \text{-tanh}(h_0 + y_3 + y_1) + \text{sawtooth}(y_2 + \frac{\pi}{2}) \end{pmatrix}.$$




x_2
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
x_1

Computing invariant sets

Guaranteed integration[6][5]

Consider the system

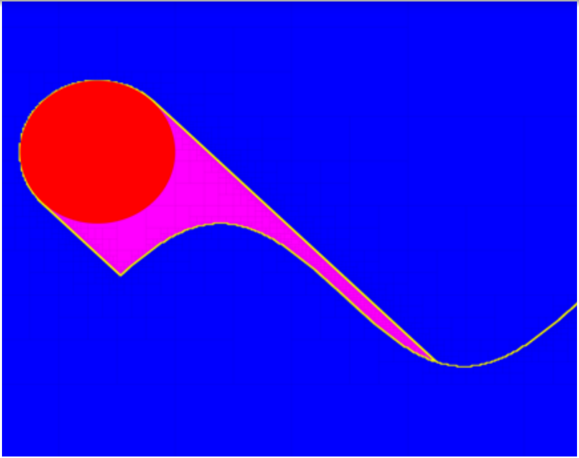
$$\mathscr{S} : \dot{\mathsf{x}}(t) = \gamma(\mathsf{x}(t))$$

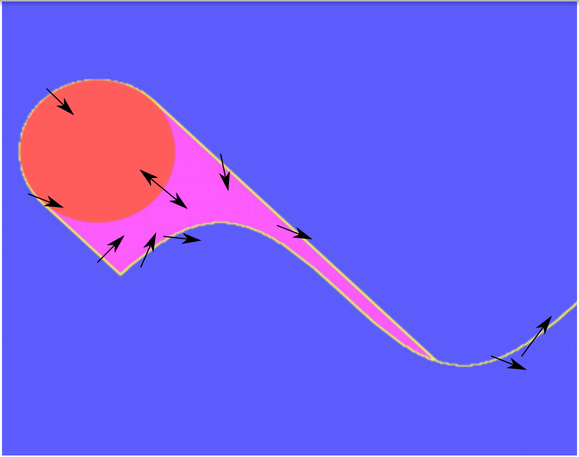
Denote by $\varphi_\gamma(t, \mathsf{x})$ the flow map.

The *forward reach set* of $\mathbb{X} \subset \mathbb{R}^n$ is:

$$\mathrm{Forw}\,(\mathbb{X}) = \left\{ \mathbf{x} \mid \exists \mathbf{x}_0 \in \mathbb{X}, \exists t \geq 0, \mathbf{x} = \varphi_\gamma(t, \mathbf{x}_0) \right\}.$$

$$\begin{aligned}\dot{x}_1 &= 1\\ \dot{x}_2 &= \mathrm{sign}(\sin(x_1) - x_2)\end{aligned}$$

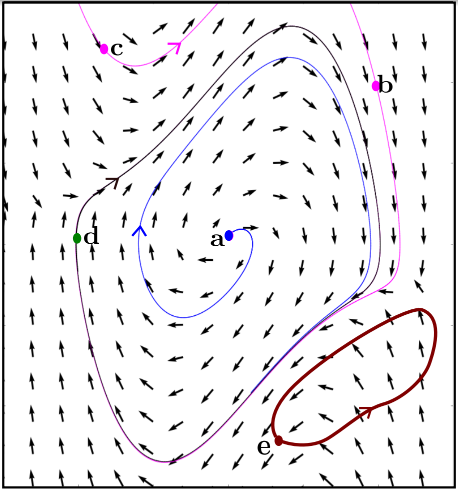




Largest positive invariant sets

Example: Consider

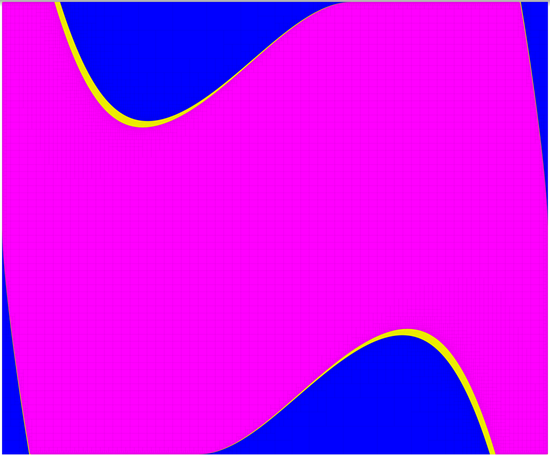
$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$



Positive invariant sets: $\mathrm{Inv}^{+}(\mathbb{X})$ with $\mathbb{X} = [-4,4] \times [-4,4]$.

The *largest positive invariant set* in $\mathbb{X} \subset \mathbb{R}^n$ is:

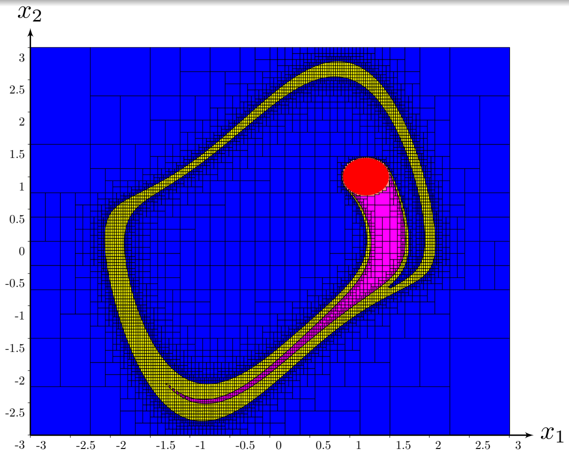
$$\mathrm{Inv}^+(\mathbb{X}) = \{\mathbf{x}_0 \mid \forall t \geq 0, \varphi(t, \mathbf{x}_0) \in \mathbb{X}\}.$$



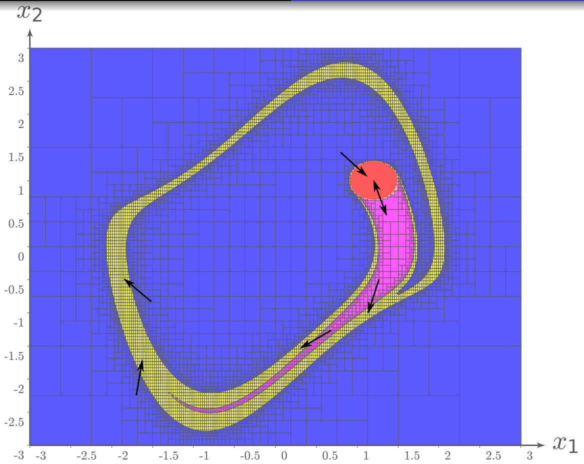
$\mathrm{Inv}^+(\mathbb{X})$ with $\mathbb{X} = [-4,4] \times [-4,4]$.

We have

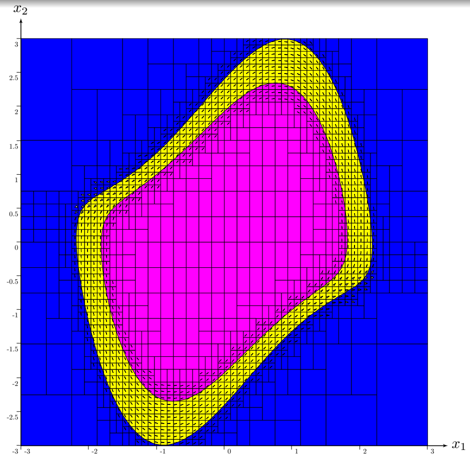
$$\begin{aligned}
\mathrm{Forw}(\mathbb{A}) &= \overline{\left\{\mathbf{x} \mid \exists t \geq 0, \varphi_\gamma(-t,\mathbf{x}) \in \mathbb{A}\right\}} \\
&= \overline{\left\{\mathbf{x} \mid \forall t \geq 0, \varphi_\gamma(-t,\mathbf{x}) \in \overline{\mathbb{A}}\right\}} \\
&= \overline{\mathrm{Inv}^{-}\left(\overline{\mathbb{A}}\right)}
\end{aligned}$$



$\mathsf{Forw}(\mathbb{A})$



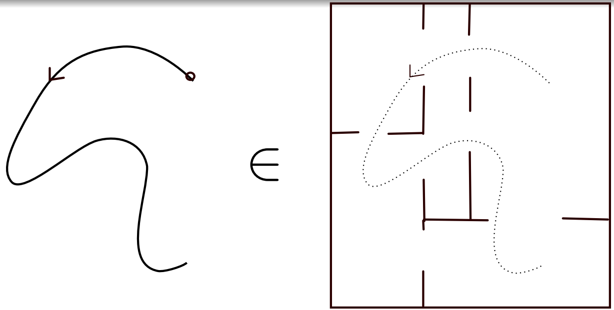
Forw($\mathbb{A}$)



$$\mathrm{Forw}(\mathbb{X}), \mathbb{X} = [-1,1] \times [-1,1]$$

Mazes

A maze [6][2] is a set of trajectories.



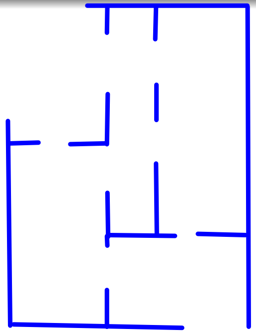
The trajectory $\mathbf{x}(\cdot)$ belongs to the maze $[\mathbf{x}](\cdot)$

Here, a **maze** $\mathscr{L}$ is composed of

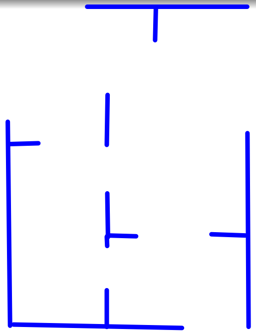
- A paving $\mathscr{P}$
- Doors between adjacent boxes

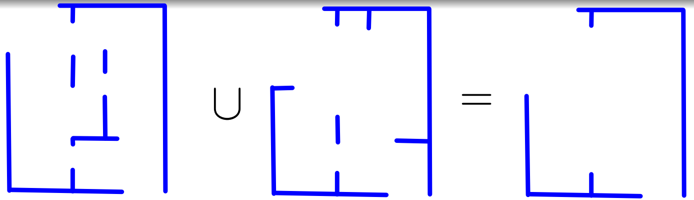
The set of mazes forms a lattice with respect to $\subset$.
$\mathscr{L}_a \subset \mathscr{L}_b$ means :

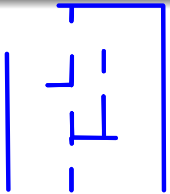
- the boxes of $\mathscr{L}_a$ are subboxes of the boxes of $\mathscr{L}_b$.
- The doors of $\mathscr{L}_a$ are thinner than those of $\mathscr{L}_b$.



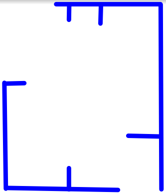
$\subset$



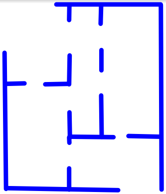




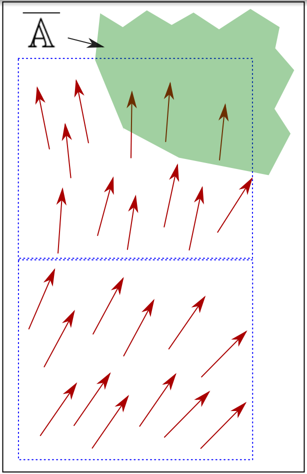
$\cap$

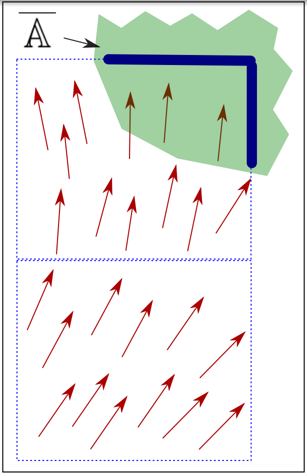


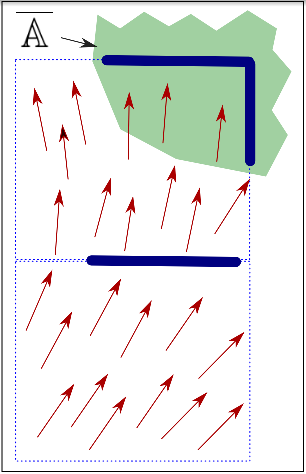
$=$

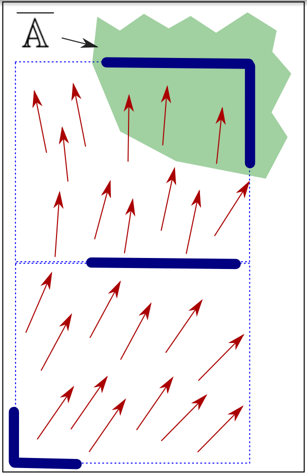


Contract trajectories that never go to $\overline{\mathbb{A}}$

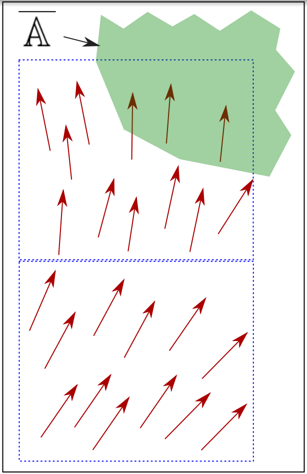

$\overline{A}$

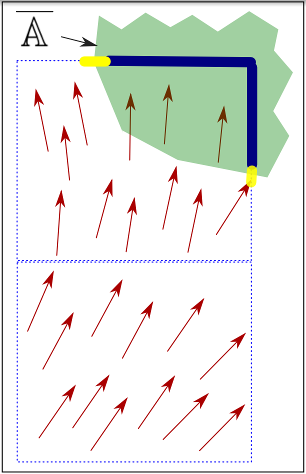

$\overline{A}$

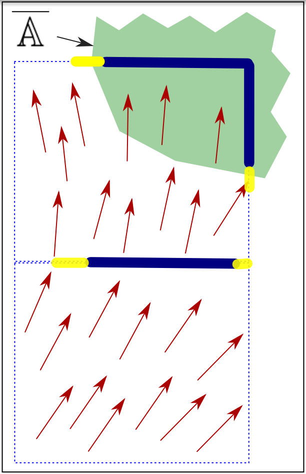

$\overline{A}$

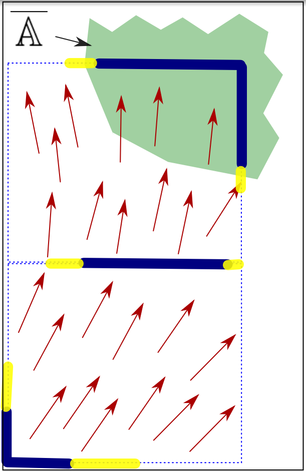

$\overline{A}$

Contract trajectories that possibly go to $\overline{\mathbb{A}}$

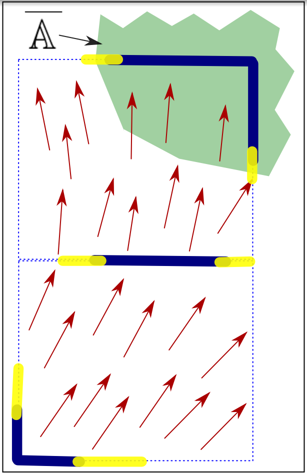

$\overline{\mathbb{A}}$

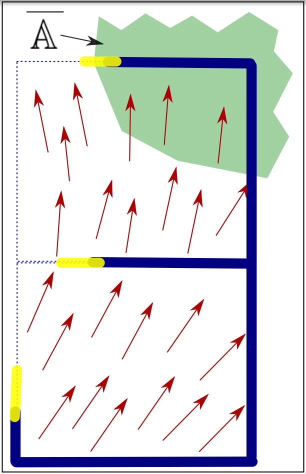

$\overline{A}$

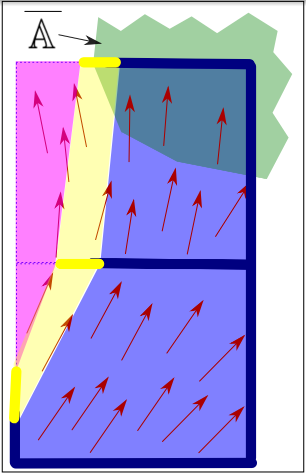

A


$\overline{A}$

Getting the largest positive invariant set

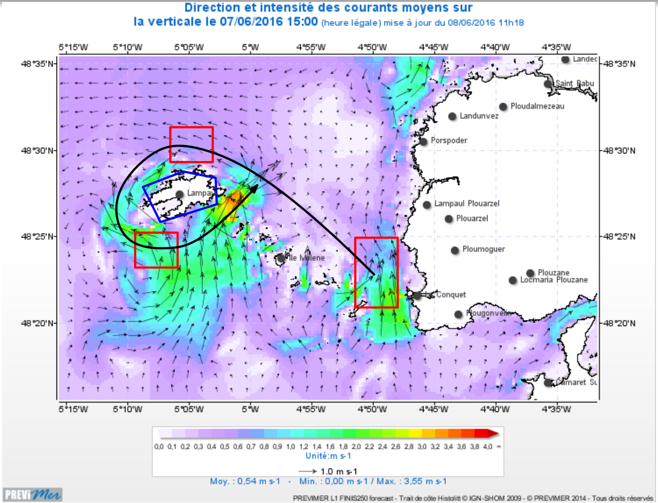

A


A

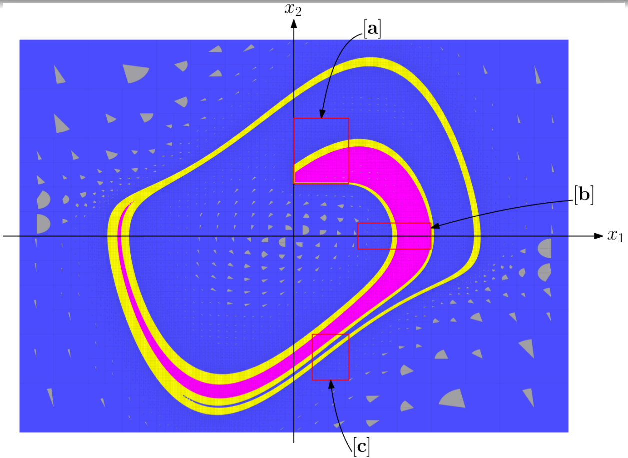

A

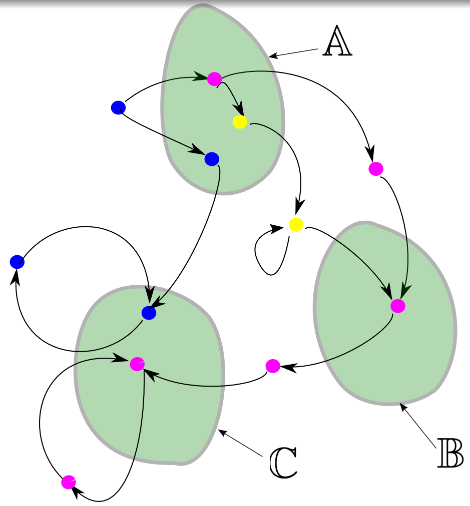
Kleene approach

Motivation



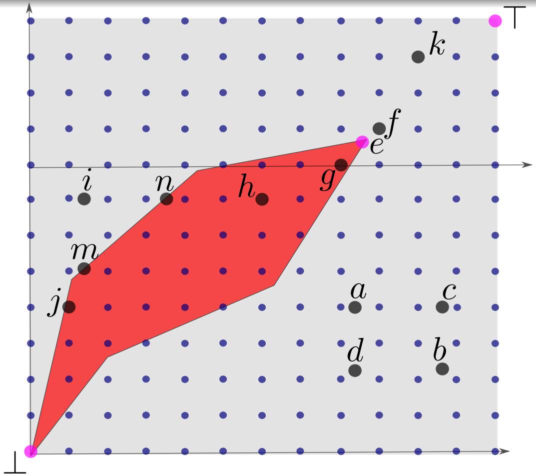
Visiting the three red boxes using a buoy that follows the currents is an Eulerian state estimation problem


x_2
[a]
[b]
x_1
[c]


$\mathbb{A}$
$\mathbb{B}$
$\mathbb{C}$

Lattice

A *lattice* $(\mathscr{L}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds [1].
A *machine lattice* $(\mathscr{L}_M, \leq)$ of $\mathscr{L}$ is complete sublattice of $(\mathscr{L}, \leq)$ which is finite.


$\top$
k
f
e
g
i
n
h
m
j
a
c
d
b
$\perp$

Kleene algebra

Kleene algebra	$(\mathscr{K},+,\cdot,*)$
Addition	$a+b$
Product	$a\cdot b$
Associativity	$a+(b+c)=(a+b)+c$
	$a\cdot(b\cdot c)=(a\cdot b)\cdot c$
Commutativity	$a+b=b+a$
Distributivity	$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$
	$(b+c)\cdot a=(b\cdot a)+(c\cdot a)$
zero	$a+\perp=a$
One	$a\cdot\top=\top\cdot a=a$
Annihilation	$a\cdot\perp=\perp\cdot a=\perp$
Idempotence	$a+a=a$
Partial order	$a\leq b\Leftrightarrow a+b=b$
Kleene star	$a^*=\top+a+a\cdot a+a\cdot a\cdot a+\dots$

A Kleene algebra $\mathscr{K}(\leq, +, \cdot, *, \perp, \top)$ is a lattice.
We can also define the machine Kleene algebra $(\mathscr{K}_M, \leq)$ of $\mathscr{K}$.

Automorphism

Given a lattice $(\mathscr{L}, \wedge, \vee, \bot, \top)$, an *automorphism* of $\mathscr{L}$ is a function $f{:}\mathscr{L} \to \mathscr{L}$ such that

$$\begin{array}{lccc} \text{(i)} & f(\top) & = & \top \\ \text{(ii)} & f(a \wedge b) & = & f(a) \wedge f(b) \end{array}$$

We denote by $\mathscr{A}(\mathscr{L})$ the set of automorphisms of $\mathscr{L}$.

Example. $f(\mathbb{A}) = \varphi(1, \mathbb{A})$ is an automorphism.

- $f(\mathbb{A} \cap \mathbb{B}) = f(\mathbb{A}) \cap f(\mathbb{B})$
- $f(\mathbb{R}^n) = \mathbb{R}^n$

- $\mathbb{A} \cap \varphi(1, \mathbb{A})$ is an automorphism
- $\mathbb{A} \cap \varphi(1, \mathbb{A}) \cap \varphi^2(1, \mathbb{A})$ is an automorphism
- $(\varphi(1, \mathbb{A}))^*$ is an automorphism

Factorization

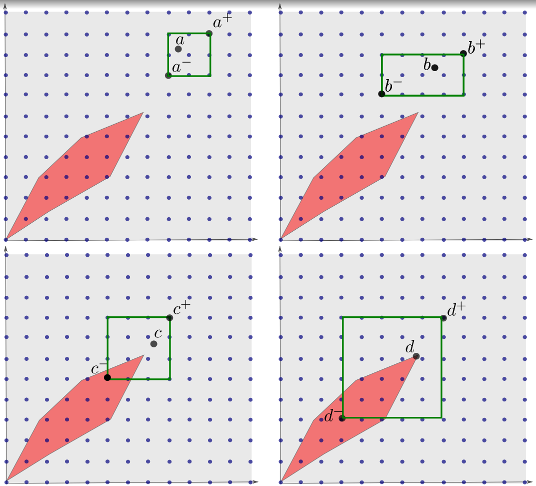
We want to compute expressions, such as

$$f^* (a) \wedge (g^* (b) \vee h^* (a))^* .$$

We have

$$f^* \wedge f^* = f^*$$
$$(f^*)^* = f^*$$
$$(f^* \wedge g^*)^* = (f \wedge g)^*$$
$$f^* \circ (f \circ g^*)^* = (f \wedge g)^*$$

Algorithm

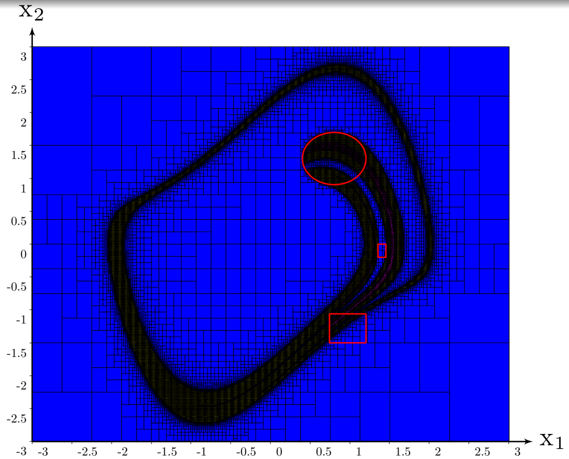

a^+
a
a^-
b^+
b
b^-
c^+
c
c^-
d^+
d
d^-

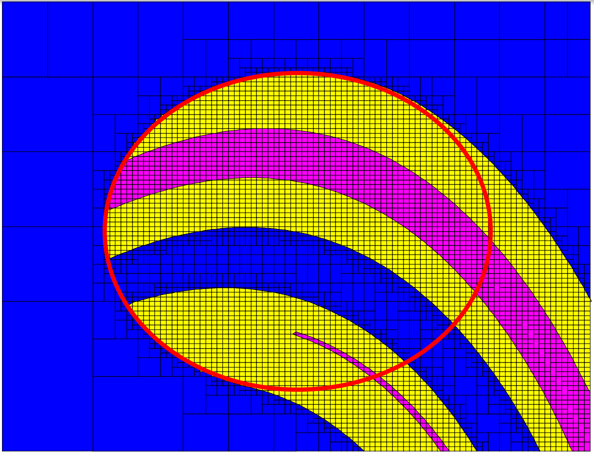
Dynamical systems

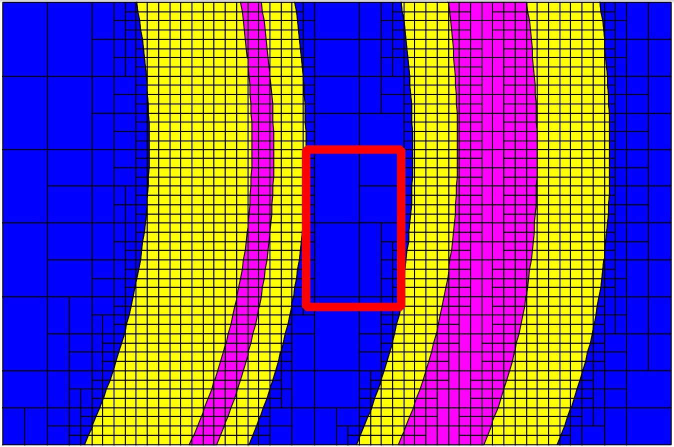
Path planning reach set

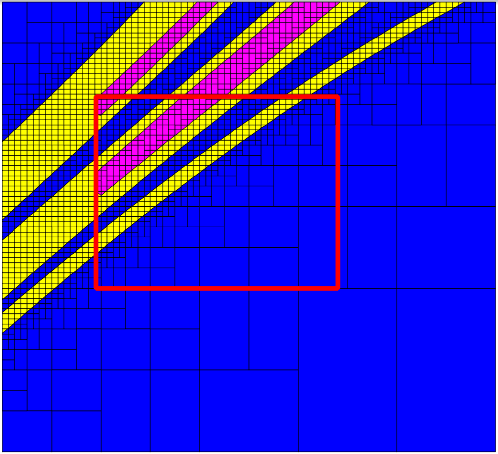
We want the set $\mathbb{X}$ of all paths that start in $\mathbb{A}$, avoid $\mathbb{B}$ and reach $\mathbb{C}$. We have

$$\mathbb{X} = \left(\overline{\overleftarrow{f_{\gamma|\overline{\mathbb{B}}}}^{*}\left(\overline{\mathbb{A}}\right)} \cap \overline{\overrightarrow{f_{\gamma|\overline{\mathbb{B}}}}^{*}\left(\overline{\mathbb{C}}\right)}\right)$$


x2
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
x1







Control reach set

Consider the system:

$$\mathscr{S} : \dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t), u),\ u \in \{0, 1\}$$

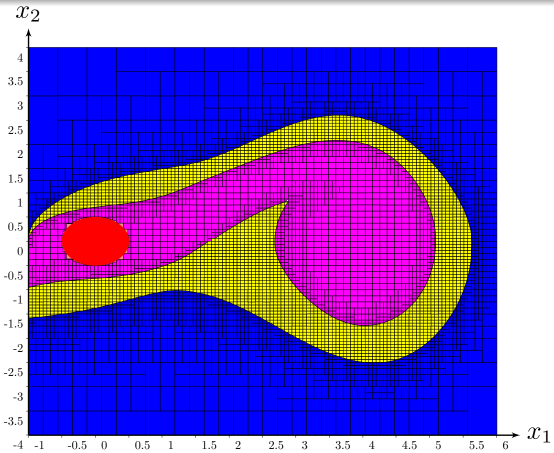
We want to compute the largest set $\mathbb{X}$ that can be reached from the set $\mathbb{A}$.

We have

$$\mathbb{X} = \overline{\left(\overleftarrow{f_1} \circ \overleftarrow{f_0}\right)^* (\overline{\mathbb{A}})}$$

Car on the hill system [3] :

$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -9.81\sin(0.55\sin(1.2x_1) - 0.6\sin(1.1x_1)) - 0.7x_2 + u \end{cases}$$

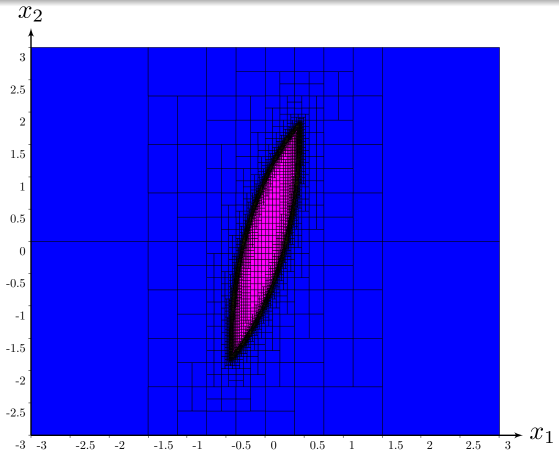

x_2
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
x_1

Minimal robust positive invariant set

The example is a continuous-time version of an example in [4]:

$$\begin{cases} \dot{x}_1 = & 0.2x_1 + 0.2x_2 - x_1 \\ \dot{x}_2 = & -0.2x_1 + 0.5x_2 + \omega - x_1 \end{cases}$$

where $\omega \in [-1,1]$.
We want the *minimal robust positively invariant set* containing $\mathbf{0}$


x_2
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
x_1

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