An interval approach for stability analysis of nonlinear systems

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1 Vaimos

Vaimos (IFREMER and ENSTA)

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right).$$

With the controller $\mathbf{u} = \mathbf{g}\left(\mathbf{x}\right)$, the robot satisfies an equation of the form

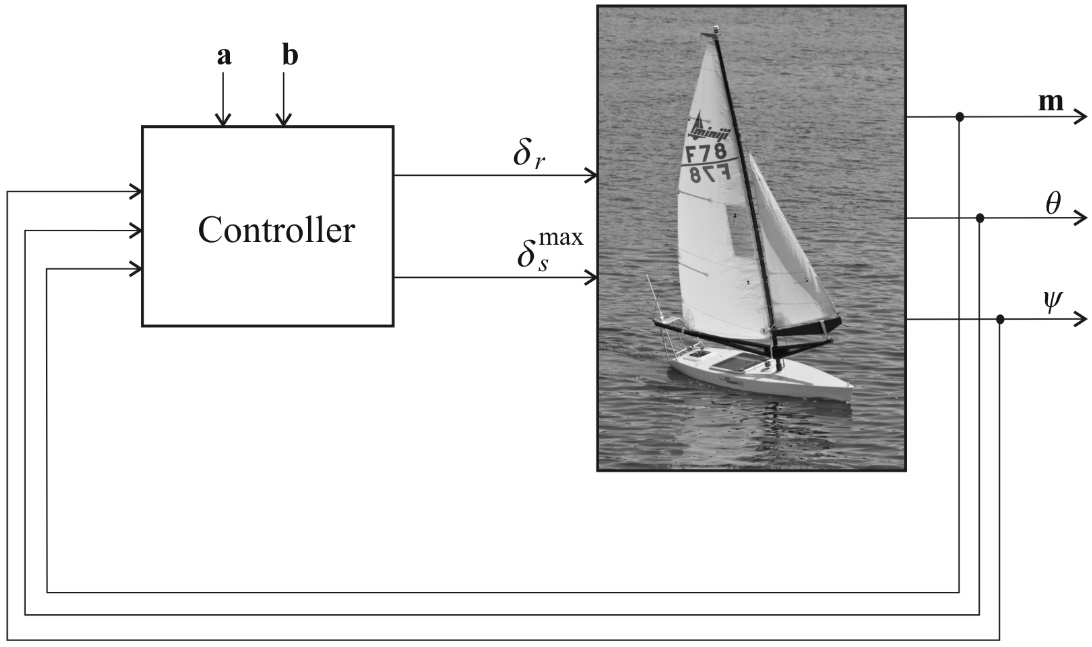
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right).$$

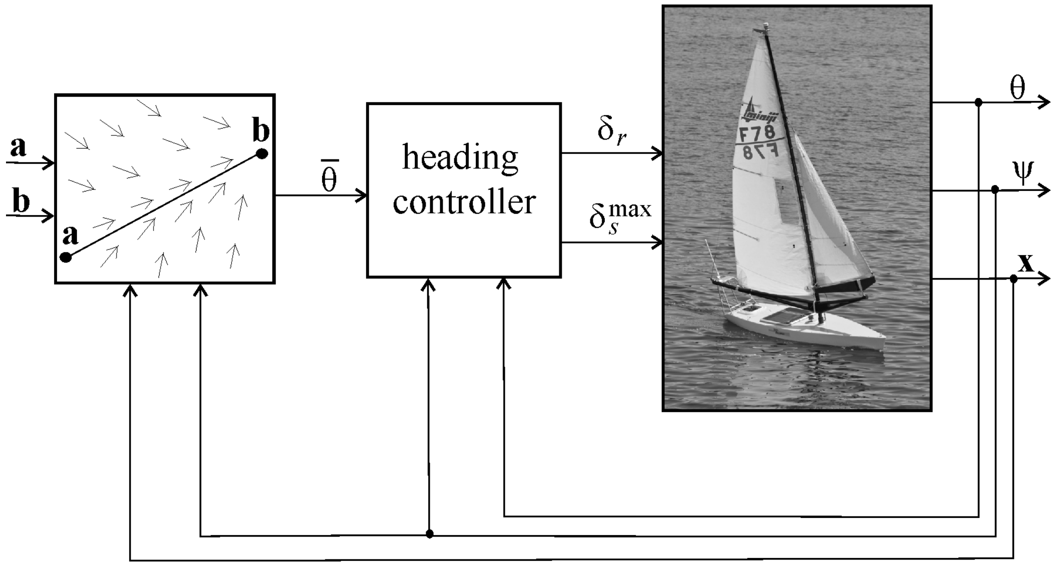
With all uncertainties, the robot satisfies.

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

2 Line following


a
b
Controller
δ_r
$\delta_s^{\max}$
F78
m
θ
ψ



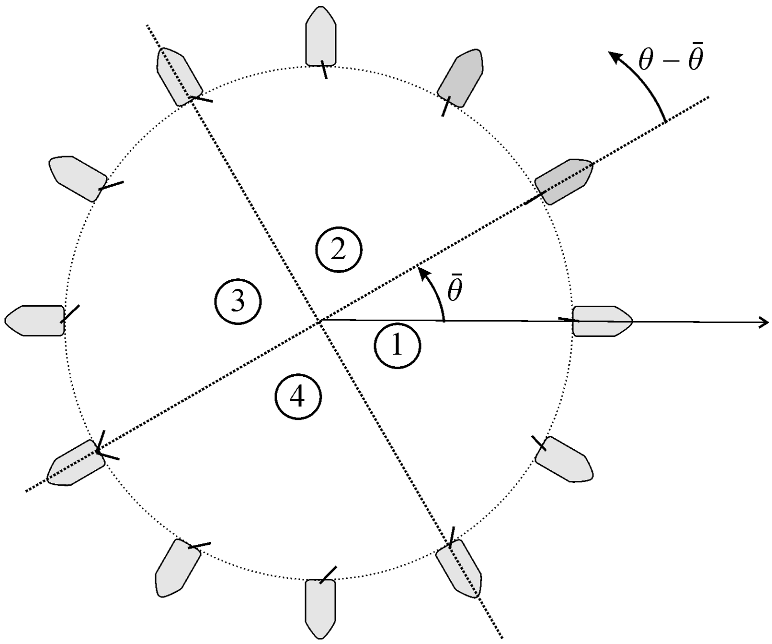
a
b
a
b
$\bar{\theta}$
heading
controller
δ_r
δ_s^{max}
F78
θ
ψ
x

2.1 Heading controller

$$\begin{cases} \delta_r & = \begin{cases} \delta_r^{\mathsf{max}}.\,\mathsf{sin}\left(\theta - \bar{\theta}\right) & \text{if } \mathsf{cos}\left(\theta - \bar{\theta}\right) \geq 0 \\ \delta_r^{\mathsf{max}}.\mathsf{sign}\left(\mathsf{sin}\left(\theta - \bar{\theta}\right)\right) & \text{otherwise} \end{cases} \\ \delta_s^{\mathsf{max}} & = \frac{\pi}{2}.\left(\frac{\mathsf{cos}(\psi - \bar{\theta}) + 1}{2}\right)^q. \end{cases}$$

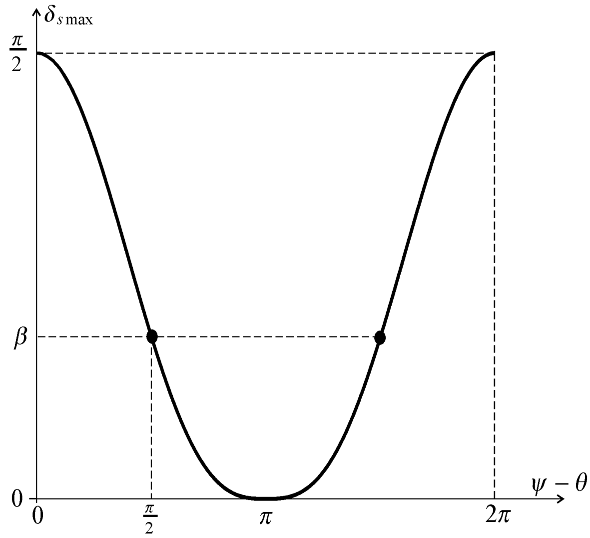
Rudder

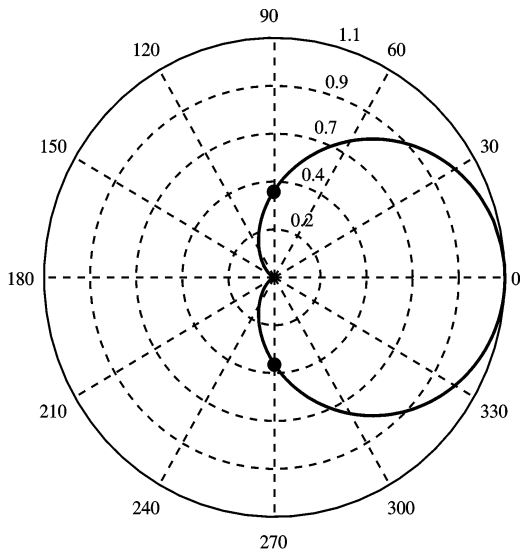
$$\delta_r = \begin{cases} \delta_r^{\max}.\sin\left(\theta-\bar{\theta}\right) & \text{if } \cos\left(\theta-\bar{\theta}\right)\geq 0 \\ \delta_r^{\max}.\text{sign}\left(\sin\left(\theta-\bar{\theta}\right)\right) & \text{otherwise} \end{cases}$$



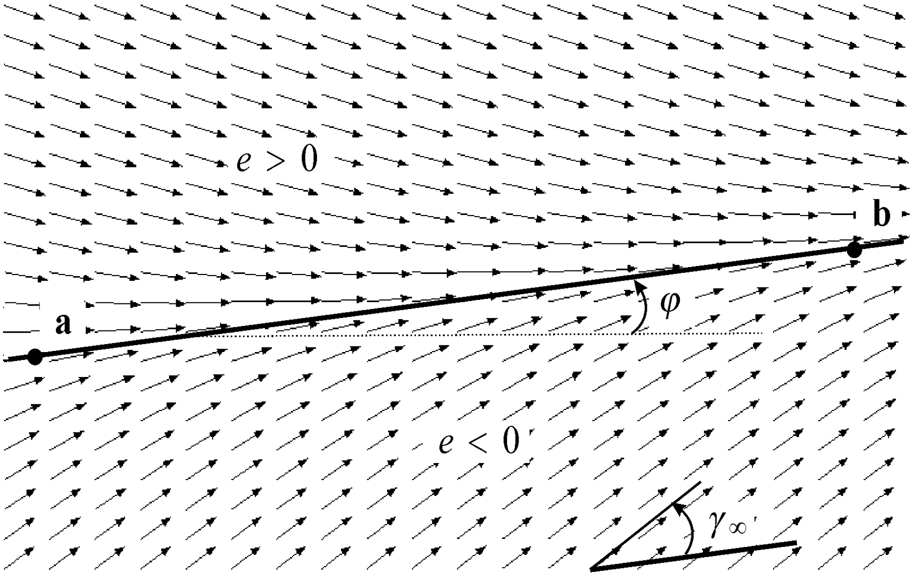
Sail

$$\delta_s^{\max} = \frac{\pi}{2} \cdot \left(\frac{\cos\left(\psi - \bar{\theta}\right) + 1}{2} \right)^q \text{ with } q = \frac{\log\left(\frac{\pi}{2\beta}\right)}{\log(2)}$$



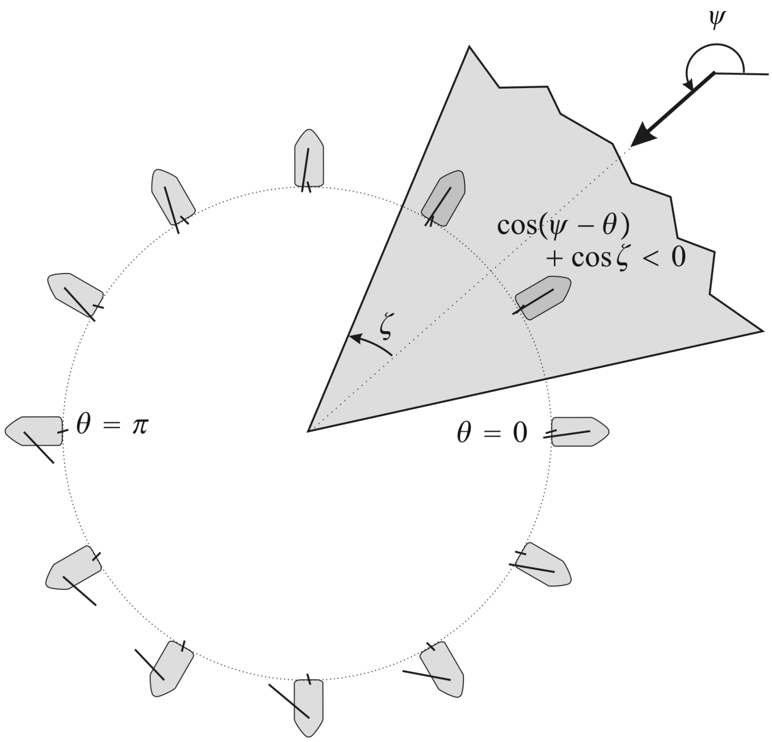


2.2 Vector field

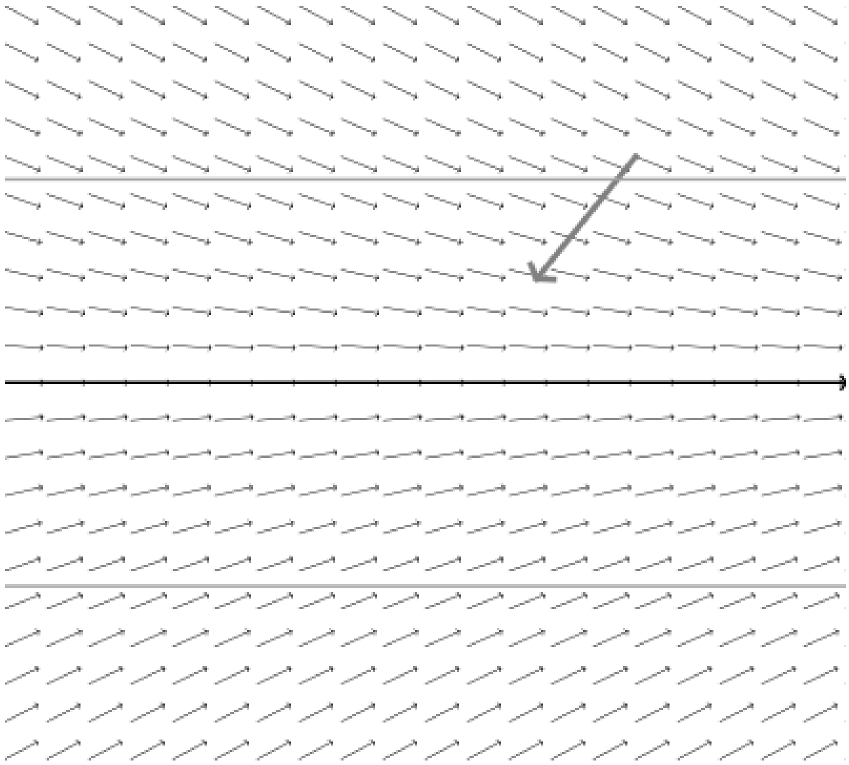


Nominal vector field: $\theta^* = \varphi - \frac{2.\gamma_\infty}{\pi}.\mathrm{atan}\left(\frac{e}{r}\right).$

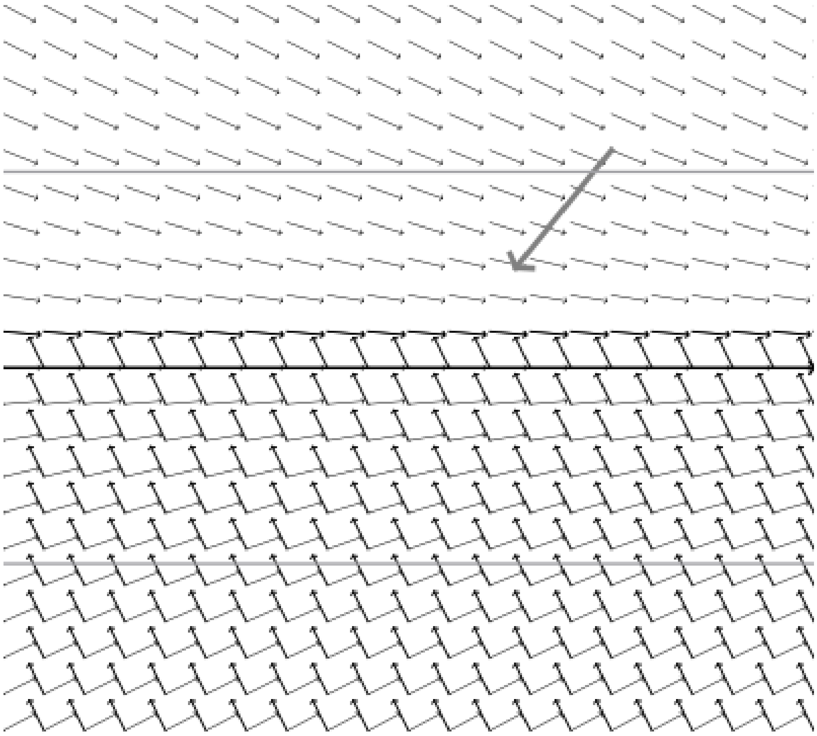
A course θ^* may be unfeasible



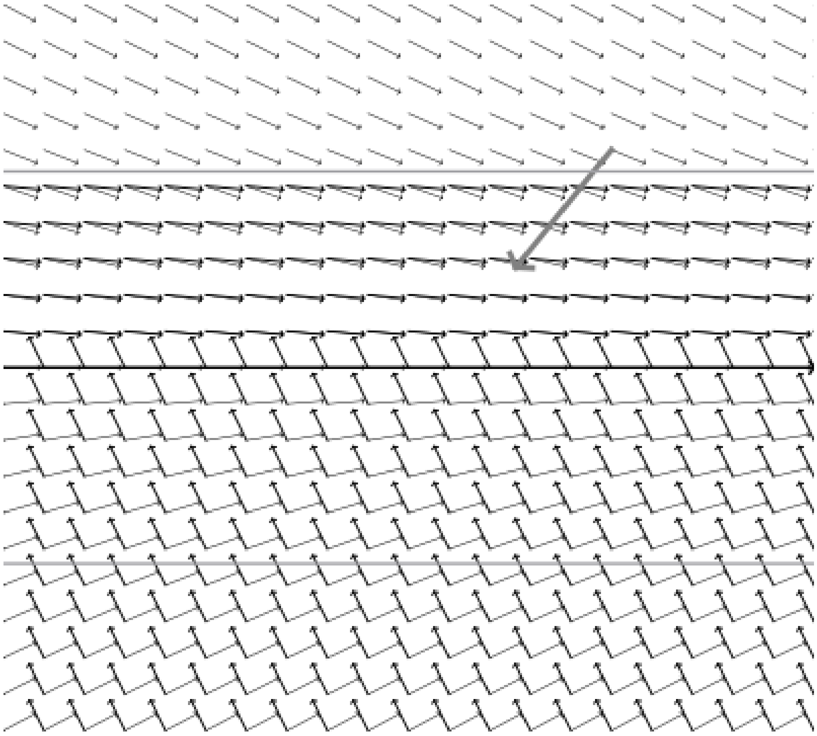
$$\cos\left(\psi - \bar{\theta}\right) + \cos\zeta \quad < \quad 0 \Rightarrow \bar{\theta} \text{ is unfeasible}$$
$$\text{In this case, take } \bar{\theta} \quad = \quad \pi + \psi - \zeta.sign\,(e)$$



$$\theta^* = -\frac{2.\gamma_\infty}{\pi}.\mathsf{atan}\left(\frac{e}{r}\right)$$



Polar projection strategy



Keep-close-hauled strategy. strategy: even if the route $\bar{\theta}$ is feasible, we keep the close hauled mode http://youtu.be/pHteidmZpnY

2.3 Controller

Controller $\bar{\theta}\left(\mathbf{m},\mathbf{a},\mathbf{b},\psi,\gamma_{\infty},r,\zeta\right)$

$e = \det\left(\frac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}, \mathbf{m}-\mathbf{a}\right)$
$\varphi =$atan2$(\mathbf{b}-\mathbf{a})$
$\theta^* = \varphi - \frac{2.\gamma_{\infty}}{\pi}.\text{atan}\left(\frac{e}{r}\right)$
if $\cos\left(\psi-\theta^*\right)+\cos\zeta<0$
    or $(|e|<r$ and $(\cos(\psi-\varphi)+\cos(\zeta)<0))$
      then $\bar{\theta} = \pi+\psi-\zeta.\text{sign}(e)$;
      else $\bar{\theta} = \theta^*$;
end

Without hysteresis

Controller in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$; out: $\delta_r, \delta_s^{\mathsf{max}}$; inout: q

$e = \mathrm{det}\left(\frac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}, \mathbf{m}-\mathbf{a}\right)$

if $|e| > \frac{r}{2}$ then $q = \mathrm{sign}(e)$

$\varphi = \mathrm{atan2}(\mathbf{b}-\mathbf{a})$

$\theta^* = \varphi - \frac{2.\gamma_\infty}{\pi}.\mathrm{atan}\left(\frac{e}{r}\right)$

if $\cos(\psi - \theta^*) + \cos\zeta < 0$

  or $(|e| < r$ and $(\cos(\psi - \varphi) + \cos\zeta < 0))$

    then $\bar{\theta} = \pi + \psi - q.\zeta.$

    else $\bar{\theta} = \theta^*$

end

if $\cos\left(\theta - \bar{\theta}\right) \geq 0$ then $\delta_r = \delta_r^{\mathsf{max}}.\sin\left(\theta - \bar{\theta}\right)$

else $\delta_r = \delta_r^{\mathsf{max}}.\mathrm{sign}\left(\sin\left(\theta - \bar{\theta}\right)\right)$

$\delta_s^{\mathsf{max}} = \frac{\pi}{2}.\left(\frac{\cos(\psi - \bar{\theta}) + 1}{2}\right)^q.$

With hysteresis

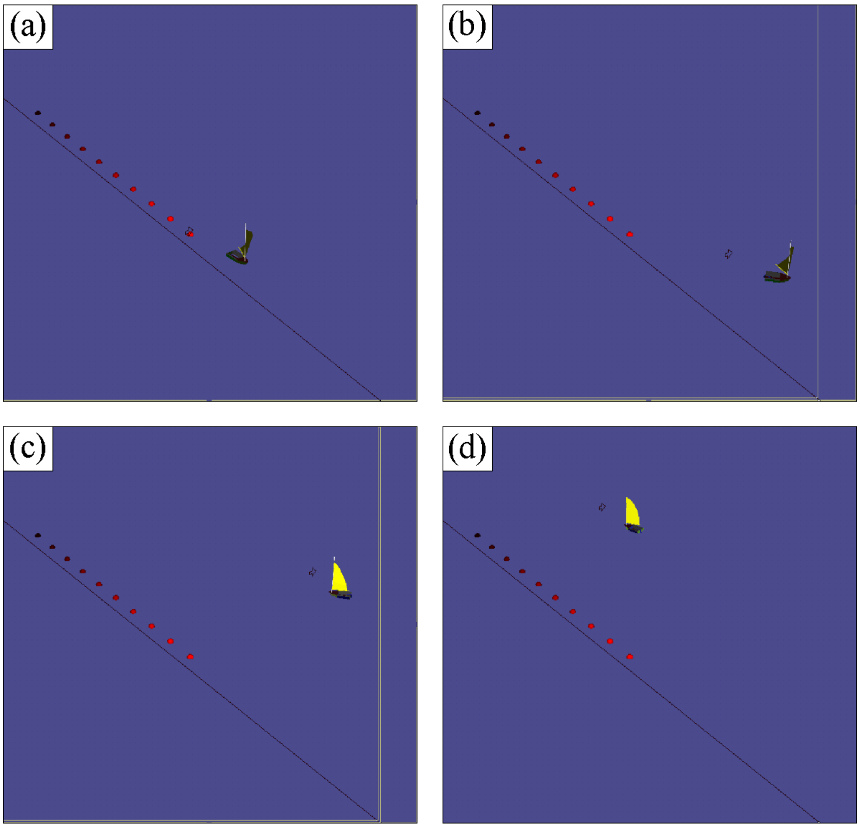
The motion of the sailboat robot satisfies

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is a differential inclusion.

The hysteresis variable q, the wind ψ, all uncertainties are enclosed inside $\mathbf{F}(\mathbf{x})$.

3 Validation by simulation


(a)
(b)
(c)
(d)

4 Theoretical validation

When the wind is known, the sailboat with the heading controller is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

The system

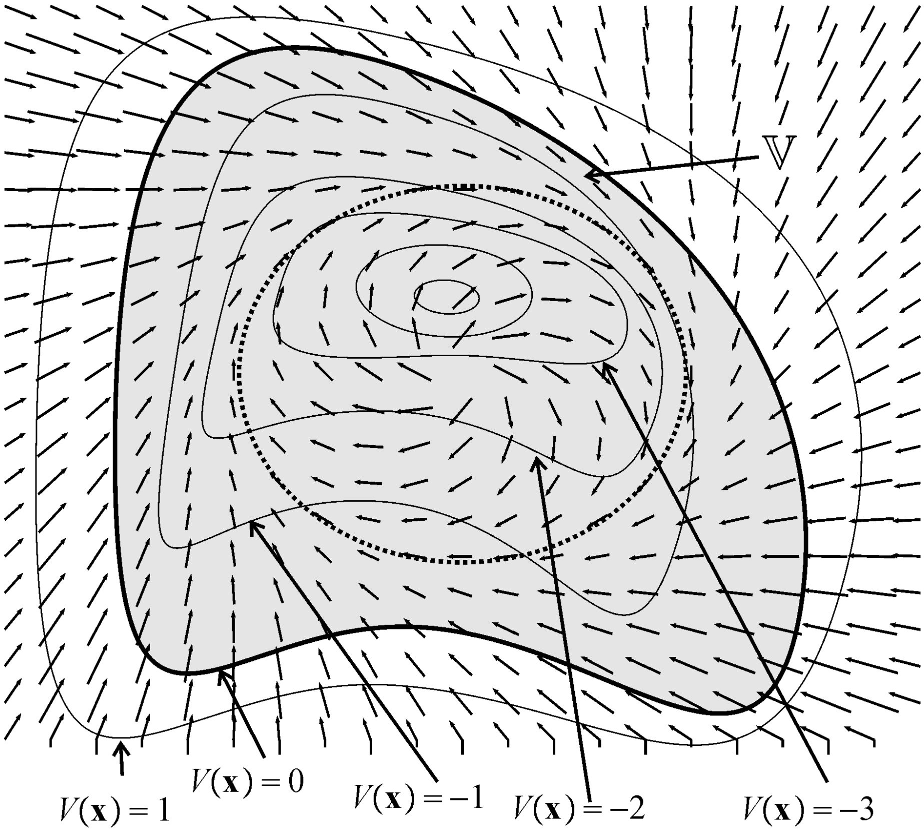
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) is there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned} \dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}. \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0} \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system is V-stable if

$$\left(V\left(\mathbf{x}\right) \geq 0 \;\Rightarrow\; \dot{V}\left(\mathbf{x}\right) < 0\right).$$



𝕍
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0$, $V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$\begin{aligned}
& \left(V(\mathbf{x}) \geq 0 \ \Rightarrow \ \dot{V}(\mathbf{x}) < 0\right) \\
\Leftrightarrow \ & \left(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0\right) \\
\Leftrightarrow \ & \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow \ & \neg \left(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) \geq 0 \text{ and } V(\mathbf{x}) \geq 0\right)
\end{aligned}$$

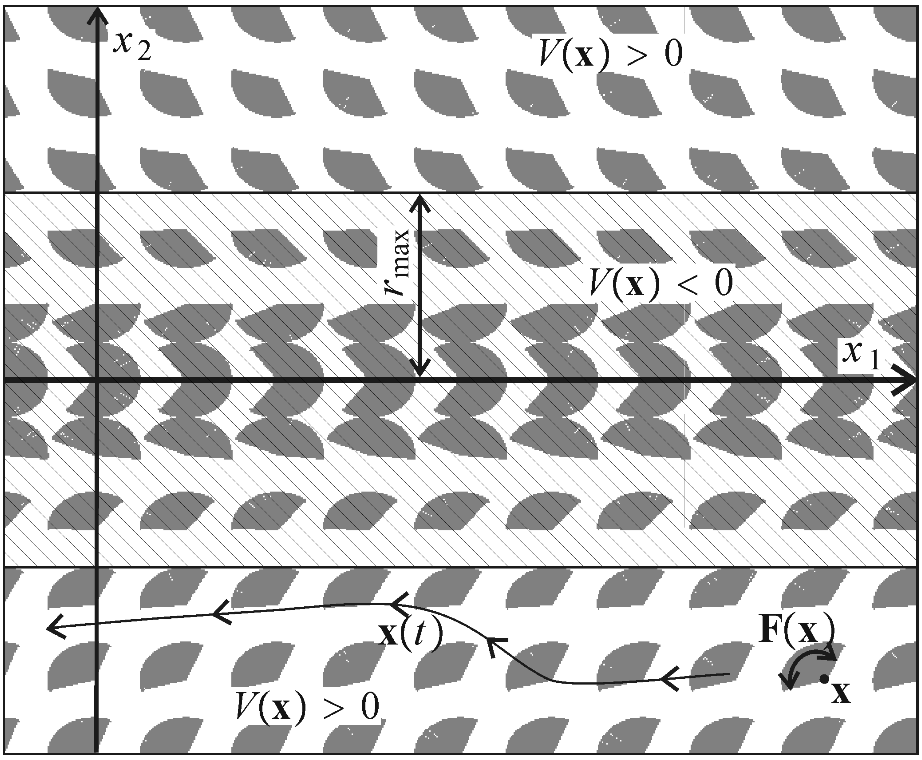
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{f}(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V-stability.

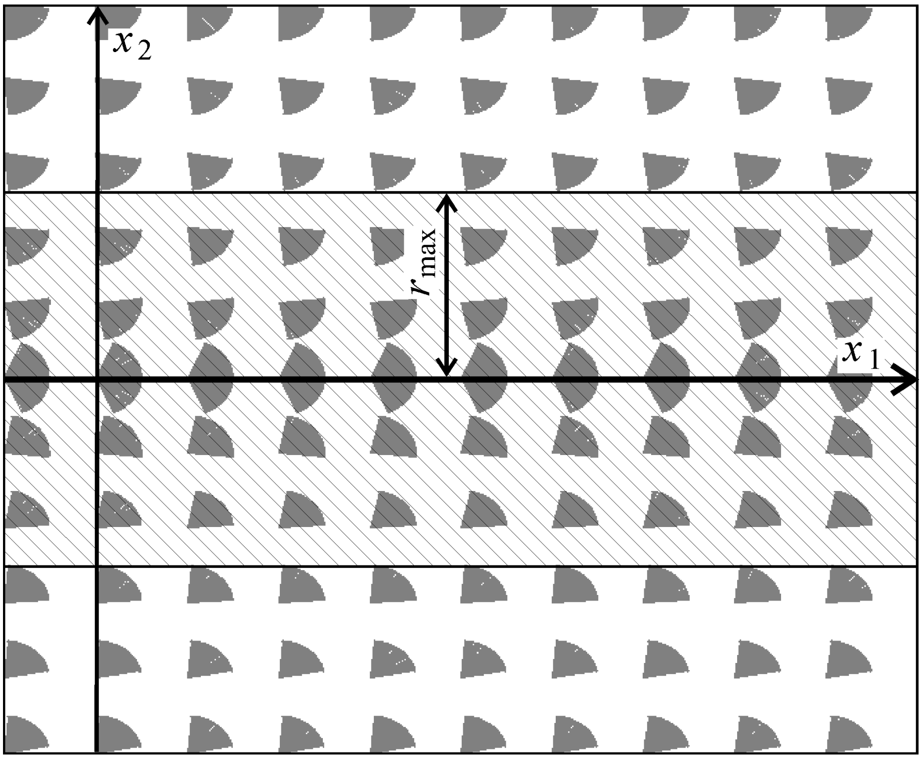
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{a} \geq 0 \\ \mathbf{a} \in \mathbf{F}(\mathbf{x}) \\ V(\mathbf{x}) \geq 0 \end{array} \right. \quad \text{inconsistent} \quad \Leftrightarrow \quad \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}) \text{ is } V\text{-stable}$$



Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\max}^2.$$



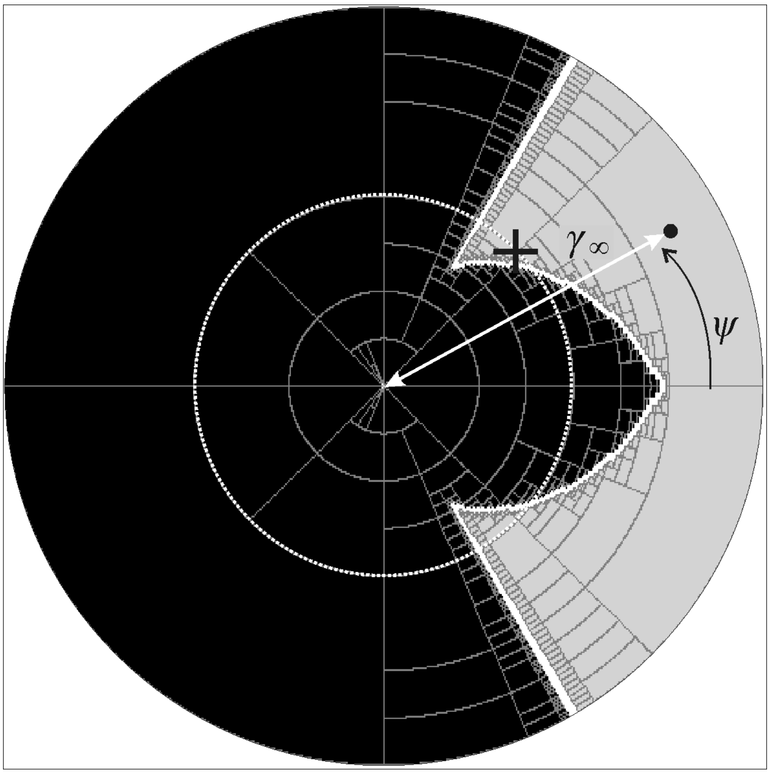
x_2
r_{max}
x_1

4.1 Parametric case

Consider the differential inclusion

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, \mathbf{p}).$$

We characterize the set $\mathbb{P}$ of all $\mathbf{p}$ such that the system is V-stable.



+
γ_∞
ψ

For Vaimos, we have found parameters for the controller such that

Property 1. If $|e(\mathbf{x})| < r_{\max}$ then, it will be the case for ever.

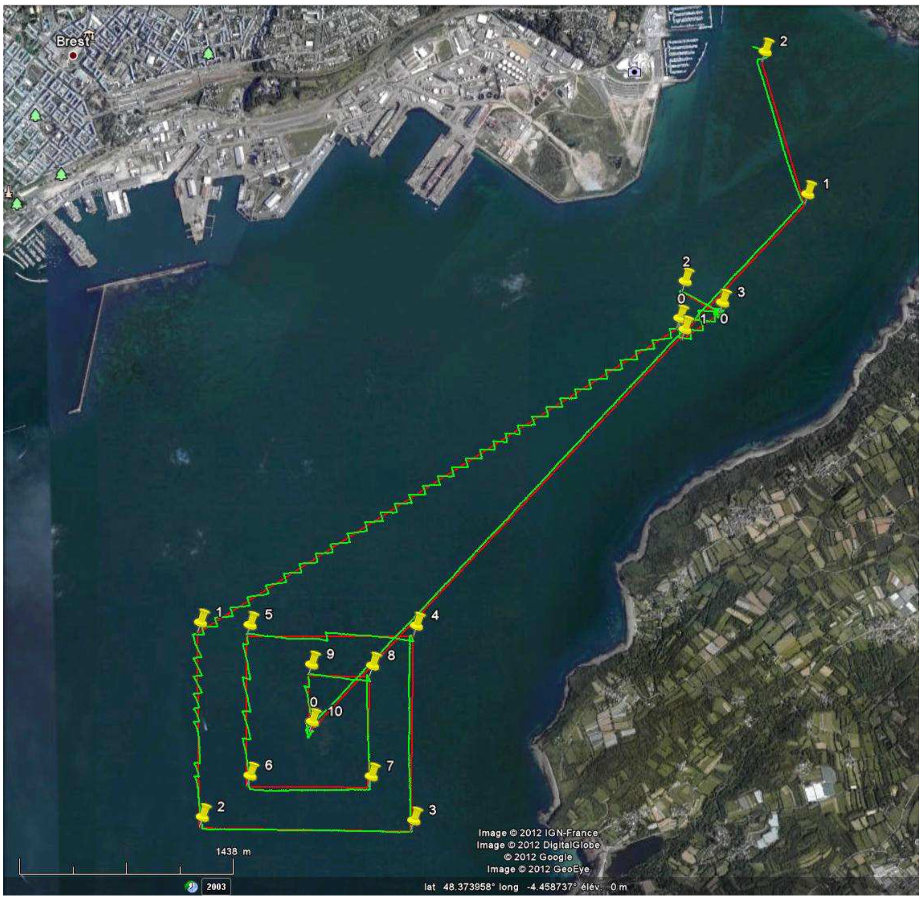
Property 2. If $|e(\mathbf{x})| > r_{\max}$ then $|e(\mathbf{x})|$ will decrease until $|e(\mathbf{x})| < r_{\max}$.

Property 3. The course is be feasible, i.e.,

$$\cos\left(\psi - \bar{\theta}\right) + \cos\zeta \geq 0.$$

4.2 Experimental validation

Brest



Brest
Image © 2012 IGN-France
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1438 m
2003
lat 48.373958° long -4.458737° élév. 0 m

Show the dashboard

Brest-Douarnenez. January 17, 2012, 8am


F7
8


Brest
PRO
BR D98 505
YAMAHA


miniji
F78


Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Le Conquet
Plougastel-Daoulas
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google

Middle of Atlantic ocean

350 km made by Vaimos in 53h, September 6-9, 2012.

Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.

Main reference: L. Jaulin, F. Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE Transaction on Robotics, Volume 27, Issue 5,