Interval integration of an ODE

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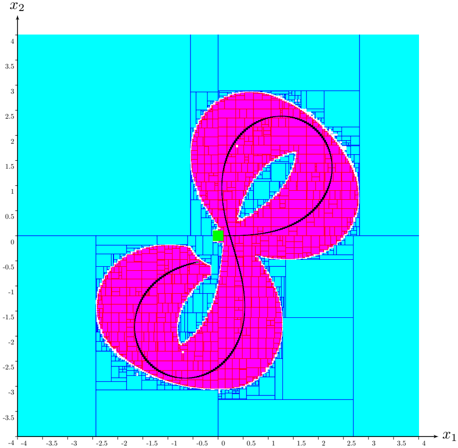
Consider the system

$$\begin{cases} \dot{x}_1 & = & u_1 \cdot \cos x_3 \\ \dot{x}_2 & = & u_1 \cdot \sin x_3 \\ \dot{x}_3 & = & u_2 \end{cases}$$

where u_1, u_2 is time dependent.



$$\operatorname{Proj}_{(x_1,x_2)} \bigcup_{t\in[0,14]} \mathbb{X}_t$$

Interval analysis

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$\begin{aligned}
[-1,3]+[2,5] &= ?, \\
[-1,3]\cdot[2,5] &= ?, \\
\text{abs}([-7,1]) &= ?
\end{aligned}$$

Interval arithmetic

$$\begin{aligned} [-1,3]+[2,5] &= [1,8], \\ [-1,3]\cdot[2,5] &= [-5,15], \\ \mathrm{abs}([-7,1]) &= [0,7] \end{aligned}$$

The interval extension of

$$f\left(x_1,x_2\right)=x_1\cdot x_2-\left(x_1+x_2\right)\cdot\cos x_2+\sin x_1\cdot\sin x_2+2$$

is

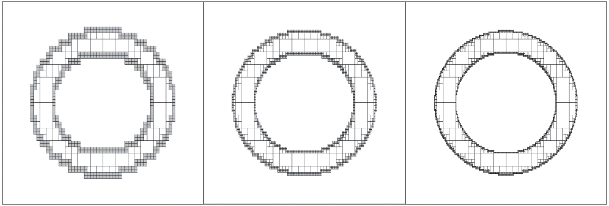
$$\begin{aligned}[f]\left([x_1],[x_2]\right) &= [x_1]\cdot[x_2]-\left([x_1]+[x_2]\right)\cdot\cos[x_2]\\ &\quad +\sin[x_1]\cdot\sin[x_2]+2.\end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Set inversion.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \in [1,4]\}.$$



Consider $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{Y} \subset \mathbb{R}^m$. Set inversion is the characterization of

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

We use the following tests.

$$\begin{array}{lllll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Interval integration (classic)

With Simon Rohou

Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is Lipschitz continuous.
The initial condition $\mathbf{x}_0^*$ is known.
We want an interval enclosure for the solution $\mathbf{x}^*(\cdot)$.

The following is an extract from the paper Guaranteed computation of robot trajectories.

Definition

A tube $[\mathbf{x}](\cdot)$ is defined as an envelope enclosing an uncertain trajectory $\mathbf{x}(\cdot):\mathbb{R}\to\mathbb{R}^n$. It is built as an interval of two functions $[\mathbf{x}^-(\cdot),\mathbf{x}^+(\cdot)]$ such that $\forall t,\ \mathbf{x}^-(t)\leqslant\mathbf{x}^+(t)$. A trajectory $\mathbf{x}(\cdot)$ belongs to the tube $[\mathbf{x}](\cdot)$ if $\forall t,\ \mathbf{x}(t)\in[\mathbf{x}](t)$. Fig. 1 illustrates a tube implemented with a set of boxes. This sliced implementation is detailed hereinafter.

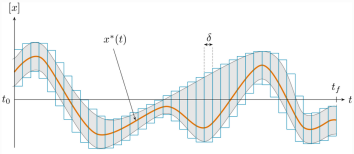


Fig. 1 A tube $[x](\cdot)$ represented by a set of slices. This representation can be used to enclose signals such as $x^*(\cdot)$.

Code example:

```
float timestep = 0.1;
Interval domain(0,10);
Tube x(domain, timestep, Function("t", "(t-5)^2 + [-0.5,0.5]"));
```

http://www.simon-rohou.fr/research/tubex-lib/ [7]

Brouwer theorem. Given a continuous function f and a compact convex set $\mathbb{X}$. We have

$$f(\mathbb{X}) \subset \mathbb{X} \Rightarrow \exists x \in \mathbb{X}, f(x) = x.$$

Example. Take $f(x) = \sin(x) \cdot \cos(x)$ and $\mathbb{X} = [-2, 2]$. Since

$f([-2,2]) \subset \sin([-2,2]) \cdot \cos([-2,2]) = [-1,1] \cdot [-1,1] = [-1,1] \subset \mathbb{X}.$

From the Brouwer theorem, we have

$$\exists x \in [-2,2] \mid \sin(x) \cdot \cos(x) = x.$$

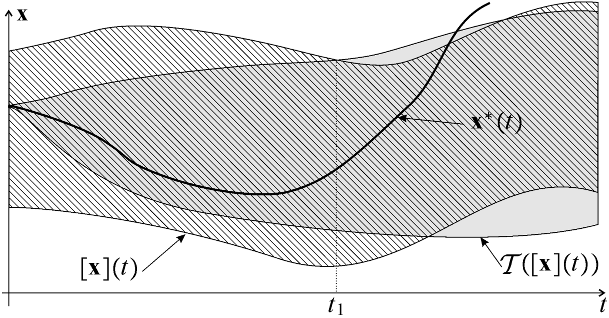
Define the Picard-Lindelöf operator associated with $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is

$$\mathscr{T} : \mathbf{x}(\cdot) \to \left(t \mapsto \mathbf{x}_0^* + \int_0^t \mathbf{f}(\mathbf{x}(\tau))\, d\tau \right).$$

$\mathscr{T}$ has a unique fixed point $\mathbf{x}^*(\cdot)$.

Take a tube $[\mathbf{x}](\cdot)$.
From the Brouwer theorem

$$\mathscr{T}([\mathbf{x}](\cdot)) \subset [\mathbf{x}](\cdot) \Rightarrow \mathbf{x}^*(\cdot) \in [\mathbf{x}](\cdot).$$


x
$\mathbf{x}^*(t)$
$[\mathbf{x}](t)$
$\mathcal{T}([\mathbf{x}](t))$
t_1
t

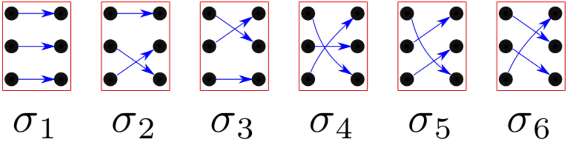
Group action

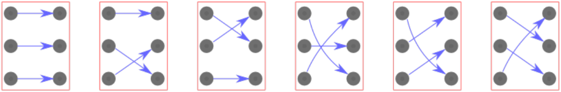
With Julien Damers

Define by $\mathbb{A} = \{a, b, c\}$.
The *symmetric group* is the set of all permutations in $\mathbb{X}$ is

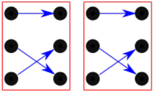
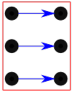
$$\begin{aligned} S_3 &= \{\sigma_1, \ldots, \sigma_6\} \\ &= \{abc \to abc, abc \to acb, abc \to bac, \\ &\quad abc \to cba, abc \to bca, abc \to cab\} \end{aligned}$$

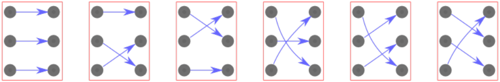
It is a group with respect to $\circ$.


σ1
σ2
σ3
σ4
σ5
σ6



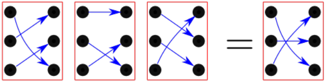
σ_1 σ_2 σ_3 σ_4 σ_5 σ_6

$\sigma_2 \circ \sigma_2 =$  $=$  $= \sigma_1$



σ_1 σ_2 σ_3 σ_4 σ_5 σ_6

$\sigma_6 \circ \sigma_2 \circ \sigma_6^{-1} =$



$= \sigma_4$

σ_6^{-1} σ_2 σ_6

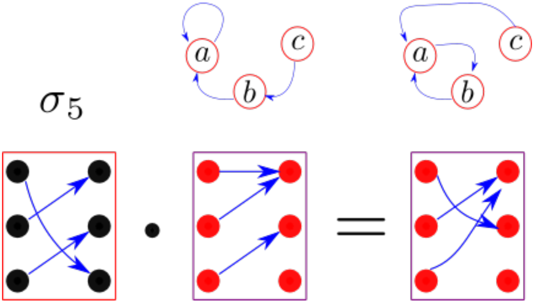
Action. Define by $\mathbb{F}$ the set of applications from $\mathbb{A}$ to $\mathbb{A}$. For instance

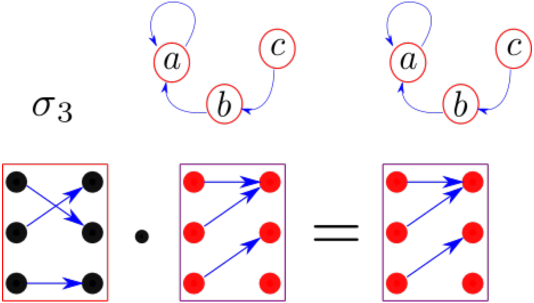
$$f_{aab} = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} a \\ a \\ b \end{pmatrix} \right) \in \mathbb{F}$$

Given $\sigma \in S_3$, we define the *action* of σ on f as

$$\sigma \bullet f = f \circ \sigma.$$

σ_5




σ3
a
b
c
a
b
c

$(S_3, \circ, \bullet)$ is a *group action* since

- $(S_3, \circ)$ is a group
- $\forall f \in \mathbb{F}, \sigma_1 \bullet f = f$ (identity)
- $(\sigma_i \circ \sigma_j) \bullet f = \sigma_i \circ (\sigma_j \bullet f)$ (compatibility)

Stabilizer. For f in $\mathbb{F}$, the stabilizer group (or symmetry group) of G with respect to f is

$$G_f = Sym(f) = \{\sigma \in S_3 | \sigma \bullet f = f\}.$$

In our example we can check that

$$G_{f_{aab}} = \{\sigma_1, \sigma_3, \sigma_5\}$$

Differential group action

With Julien Damers

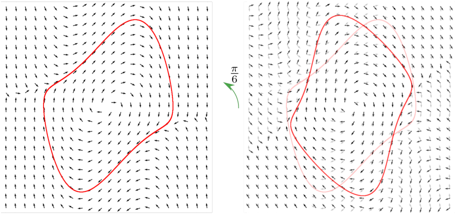
Action.
Define by $\mathbb{F}$ the set of all state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$
If $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, we define

$$\mathbf{g} \bullet \mathbf{f} = \left(\frac{d\mathbf{g}}{d\mathbf{x}} \circ \mathbf{g}^{-1} \right) \cdot \left(\mathbf{f} \circ \mathbf{g}^{-1} \right).$$

Proposition. If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y} = \mathbf{g}(\mathbf{x})$, we have $\dot{\mathbf{y}} = \mathbf{g} \bullet \mathbf{f}(\mathbf{y})$.

Example. If $\mathbf{g}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, we get

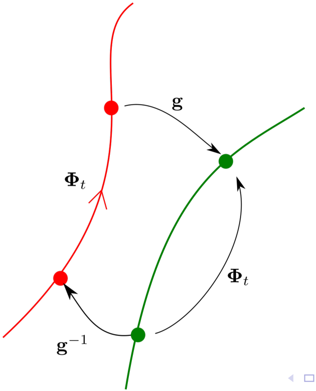
$$\begin{aligned}(\mathbf{g} \bullet \mathbf{f})(\mathbf{x}) &= \underbrace{\left(\frac{d\mathbf{g}}{d\mathbf{x}}(\mathbf{g}^{-1}(\mathbf{x}))\right)}_{\mathbf{A}} \cdot (\mathbf{f}(\mathbf{g}^{-1}(\mathbf{x}))) \\ &= \mathbf{A} \cdot \mathbf{f}(\mathbf{A}^{-1} \cdot \mathbf{x}).\end{aligned}$$

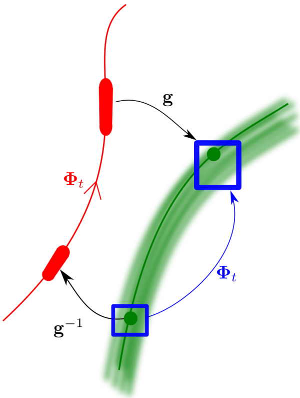


A transformation $\mathbf{g}$ is a *stabilizer* of $\mathbf{f}$ if $\mathbf{g} \bullet \mathbf{f} = \mathbf{f}$.
Equivalently, $\mathbf{g} \in Sym(\mathbf{f})$.

Proposition. Define $\mathbf{\Phi}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ as the flow associated to $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$. We have:

$$\mathbf{g}\bullet\mathbf{f}=\mathbf{f}\Leftrightarrow\mathbf{\Phi}_t=\mathbf{g}\circ\mathbf{\Phi}_t\circ\mathbf{g}^{-1}.$$



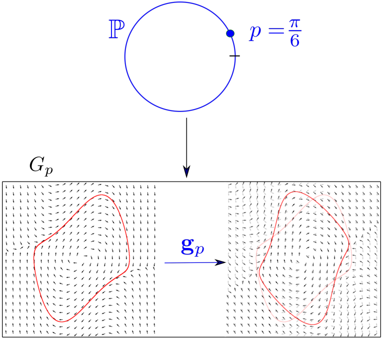

g
Φ_t
Φ_t
g^{-1}

Lie group of symmetries

With Julien Damers

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and a manifold $\mathbb{P}$.
A Lie group $G_{\mathbf{p}}$ of symmetries is a family of transformations $\mathbf{g}_{\mathbf{p}}$, $\mathbf{p} \in \mathbb{P}$ such that

- $(G_{\mathbf{p}}, \circ)$ is a Lie group
- $\forall \mathbf{p} \in \mathbb{P}, \mathbf{g}_{\mathbf{p}} \bullet \mathbf{f} = \mathbf{f}$.



Here, $(G_\mathbf{p}, \circ)$ is a Lie group but $\mathbf{g_p} \bullet \mathbf{f} \neq \mathbf{f}$

Transport function [1]

Given a Lie group of symmetries $G_{\mathbf{p}}$.

A *transport function* $\mathbf{h}(\mathbf{x},\mathbf{a})$ returns $\mathbf{p}$ so that $\mathbf{g}_{\mathbf{p}}(\mathbf{a}) = \mathbf{x}$, *i.e.*,

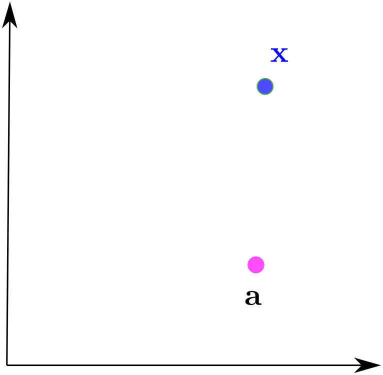
$$\mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a})}(\mathbf{a}) = \mathbf{x}$$

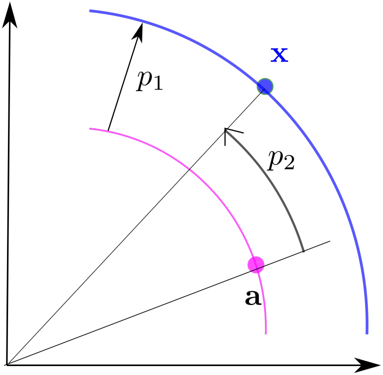
Example. Consider the Lie group:

$$G_{\mathbf{p}} = \left\{ \mathbf{g}_{\mathbf{p}} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to p_1 \cdot \begin{pmatrix} \cos p_2 & -\sin p_2 \\ \sin p_2 & \cos p_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

To get the transport function $\mathbf{h}(\mathbf{x}, \mathbf{a})$, we use the equivalence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{p_1 \cdot \begin{pmatrix} \cos p_2 & -\sin p_2 \\ \sin p_2 & \cos p_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\mathbf{g}_{\mathbf{p}}(\mathbf{a})} \Leftrightarrow \mathbf{p} = \mathbf{h}(\mathbf{x}, \mathbf{a})$$


x
a



$\mathbf{p} = \mathbf{h}(\mathbf{x}, \mathbf{a})$ and $\mathbf{g}_\mathbf{p}(\mathbf{a}) = \mathbf{x}$

Interval integration with symmetries

With Julien Damers

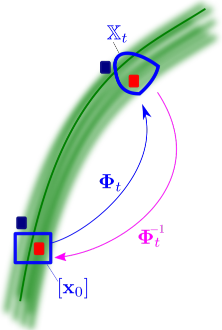
Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x}_0 \in [\mathbf{x}_0]$.
$\mathbf{\Phi}_t$ is the flow associated to $\mathbf{f}$

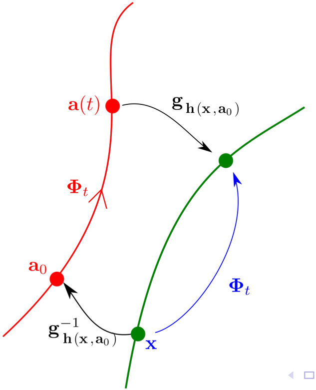
The set of all states $\mathbf{x}$ at time t consistent with the initial box $[\mathbf{x}_0]$ is $\mathbb{X}_t = \mathbf{\Phi}_t([\mathbf{x}_0])$,i.e. [4],

$$\mathbb{X}_t = \mathbf{\Phi}_{-t}^{-1}([\mathbf{x}_0])$$

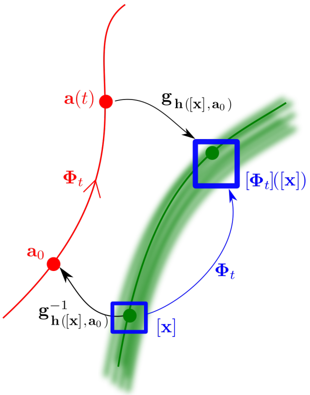


Theorem. We have one reference $\mathbf{a}(t) = \mathbf{\Phi}_t(\mathbf{a}_0)$.
If $\mathbf{h}(\mathbf{x}, \mathbf{a})$ is a transport function for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then

$$\mathbf{\Phi}_t(\mathbf{x}) = \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t).$$



An inclusion function for $\mathbf{\Phi}_t(\mathbf{x})$ is thus $\mathbf{\Phi}_t([\mathbf{x}]) = \mathbf{g}_{\mathbf{h}([\mathbf{x}],\mathbf{a}_0)} \circ \mathbf{a}(t)$.



Method [1][7]

- Enclose a reference $\mathbf{a}(t)$ in a thin tube $[\mathbf{a}(t)]$.
- Find a Lie group of symmetries $G_{\mathbf{p}}$.
- Give an expression for the transport function $\mathbf{h}(\mathbf{x},\mathbf{a})$.
- Solve the set inversion problem.

Using separators [5], we can compute

$$\bigcup_{t\in\mathbb{T}} \mathbb{X}_t = \bigcup_{t\in\mathbb{T}} \mathbf{\Phi}_{-t}^{-1}([\mathbf{x}_0])$$

with $\mathbf{\Phi}_t(\mathbf{x}) = \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t)$ and $\mathbb{T}$ is either

- a discrete set $\mathbb{T} = \{1, \dots, m\}$;
- an interval $\mathbb{T} = [0, t_{\max}]$.

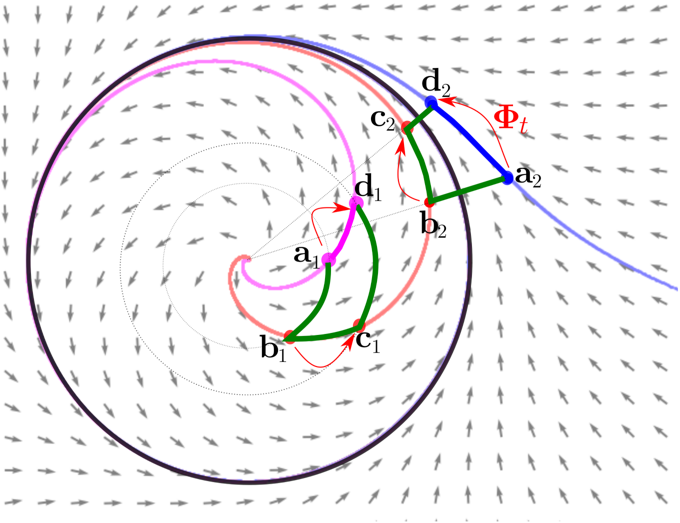
Hydon system

Example. Consider the system [2]

$$\begin{cases} \dot{x}_1 & = & -x_1^3 - x_1 x_2^2 + x_1 - x_2 \\ \dot{x}_2 & = & -x_2^3 - x_1^2 x_2 + x_1 + x_2 \end{cases}$$

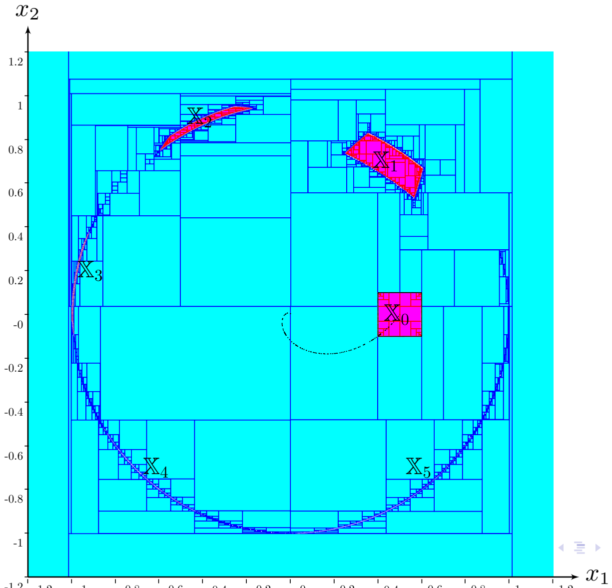
It has the following symmetry

$$\mathbf{g}_{\mathbf{p}}(\mathbf{x}) = \frac{1}{\sqrt{p_2 + \left(x_1^2 + x_2^2\right)\left(1 - p_2\right)}} \cdot \begin{pmatrix} \cos p_1 & -\sin p_1 \\ \sin p_1 & \cos p_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$


$\mathbf{d}_2$
$\mathbf{c}_2$
$\boldsymbol{\Phi}_t$
$\mathbf{a}_2$
$\mathbf{d}_1$
$\mathbf{b}_2$
$\mathbf{a}_1$
$\mathbf{b}_1$
$\mathbf{c}_1$

The reference $\mathbf{a}(t)$ is obtained with $\mathbf{a}_0 = (\frac{1}{2}, 0)^\mathsf{T}$.
We have

$$\begin{aligned}\mathbf{\Phi}_t(\mathbf{x}) &= \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t) \\ &= \frac{\sqrt{3}\begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \cdot \mathbf{a}(t)}{\sqrt{1-\|\mathbf{a}(t)\|^2+(4\|\mathbf{a}(t)\|^2-1)\|\mathbf{x}\|^2}}\end{aligned}$$


x_2
1.2
1
0.8
0.6
0.4
0.2
-0
-0.2
-0.4
-0.6
-0.8
-1
$\mathbb{X}_0$
$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
x_1

Dead reckoning

Consider the system [3]

$$\begin{cases} \dot{x}_1 = u_1 \cdot \cos x_3 \\ \dot{x}_2 = u_1 \cdot \sin x_3 \\ \dot{x}_3 = u_2 \end{cases}$$

To avoid the time dependence in $\mathbf{u}$, we rewrite the system into

$$\begin{cases} \dot{x}_1 = u_1(x_4) \cdot \cos x_3 \\ \dot{x}_2 = u_1(x_4) \cdot \sin x_3 \\ \dot{x}_3 = u_2(x_4) \\ \dot{x}_4 = 1 \end{cases}$$

where x_4 is the clock variable.

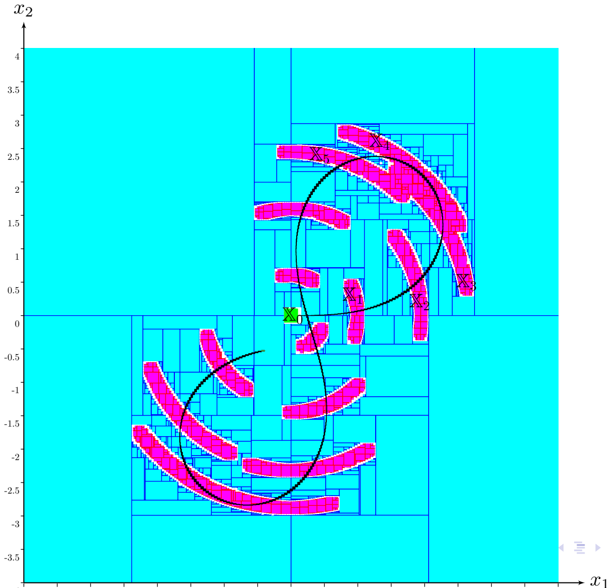
The symmetry is

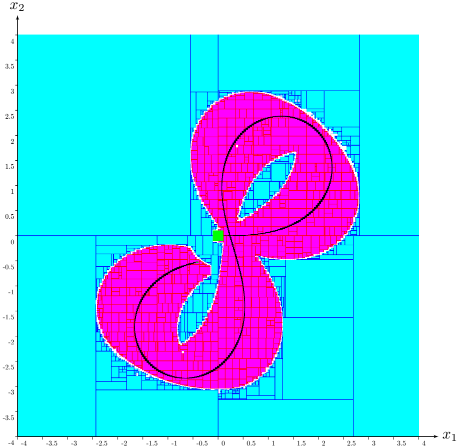
$$\mathbf{g}_{\mathbf{p}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \mathbf{R}_{p_3} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ x_3 + p_3 \\ x_4 \end{pmatrix} \circ \mathbf{\Phi}_{p_4}(\mathbf{x})$$

corresponds to the direct Euclidian group $E^{+}(2)$.

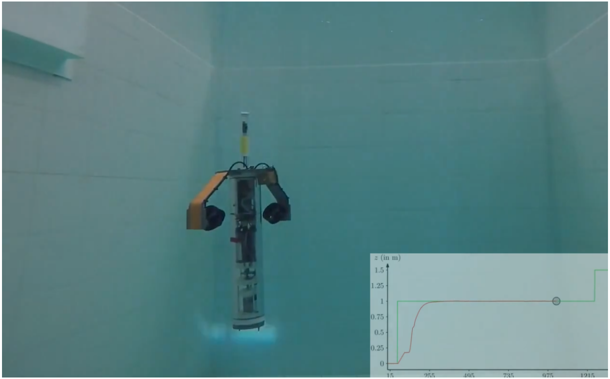
We get

$$
\begin{aligned}
\mathbf{\Phi}_t(\mathbf{x}) &= \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{0})} \circ \mathbf{a}(t) \\
&= \left(\begin{array}{c} \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) + \mathbf{R}_{x_3 - a_3(x_4)} \cdot \left(\begin{array}{c} a_1(t+x_4) - a_1(x_4) \\ a_2(t+x_4) - a_2(x_4) \end{array} \right) \\ x_3 + a_3(t+x_4) - a_3(x_4) \\ a_4(t+x_4) \end{array} \right)
\end{aligned}
$$


x_2
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
x_1
$\mathbb{X}_0$
$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$



$$\operatorname*{Proj}_{(x_1,x_2)} \bigcup_{t\in[0,14]} \mathbb{X}_t$$



Pseudo-Lagrangian float for building a map of underwater currents [6]

Youtube

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