A Modified Twin Arithmetic to Characterize Uncertain Sets

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1. Interval functions

Thin function

A thin function is $\mathbf{F} : \mathbb{IR}^n \to \mathbb{IR}^m$ satisfies

$$
\begin{array}{ccccc}
[\mathbf{a}] \subset [\mathbf{b}] & \Rightarrow & \mathbf{F}([\mathbf{a}]) \subset \mathbf{F}([\mathbf{b}]) & & \text{(monotonicity)} \\
d([\mathbf{a}],[\mathbf{b}]) \to 0 & \Rightarrow & d(\mathbf{F}([\mathbf{a}],\mathbf{F}[\mathbf{b}]) \to 0 & & \text{(continuity)} \\
w([\mathbf{x}]) = 0 & \Rightarrow & w(\mathbf{F}([\mathbf{x}])) = 0 & & \text{(thin)}
\end{array}
$$

Example.

$$\mathbf{F}([\mathbf{x}]) = \begin{pmatrix} [x_1] + \sin([x_2] + 2) \\ [x_1] \cdot [x_2] + [x_1] \end{pmatrix}.$$

Thick functions

A thick function is $\mathbf{F} : \mathbb{IR}^n \to \mathbb{IR}^m$ satisfies

$$\begin{array}{ccccc} [\mathbf{a}] \subset [\mathbf{b}] & \Rightarrow & \mathbf{F}([\mathbf{a}]) \subset \mathbf{F}([\mathbf{b}]) & & \text{(monotonicity)} \\ d([\mathbf{a}],[\mathbf{b}]) \to 0 & \Rightarrow & d(\mathbf{F}([\mathbf{a}],\mathbf{F}[\mathbf{b}]) \to 0 & & \text{(continuity)} \end{array}$$

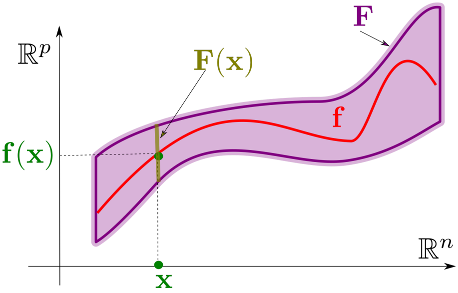
We do not assume that $\mathbf{F}$ is thin, i.e.,

$$w([\mathbf{x}]) = 0 \not\Rightarrow w(\mathbf{F}([\mathbf{x}])) = 0.$$

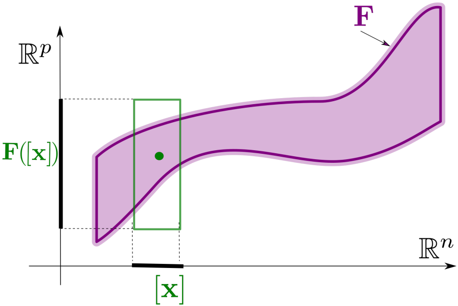
Example

$$\mathbf{F}([\mathbf{x}]) = \begin{pmatrix} [x_1] + [1,2] \cdot \sin([x_2] + [2,3]) \\ [x_1] \cdot [x_2] \cdot [1,3]^2 + [x_1] \cdot [4,5] \end{pmatrix}.$$

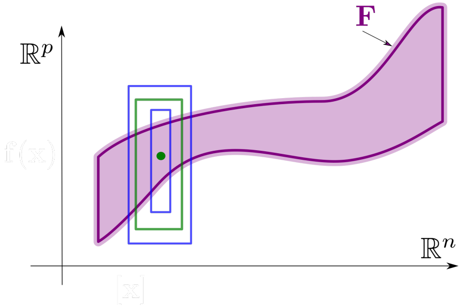
- In practice, $\mathbf{F}$ has a closed form with respect to the classicial interval operators.
- A thick function $\mathbf{F}$ is used to approximate an uncertain function $\mathbf{f}:\mathbb{R}^n \rightarrow \mathbb{R}^m$.



$$\mathbf{f} \in \mathbf{F}$$



$$\mathbf{f}([\mathbf{x}]) \subset \mathbf{F}([\mathbf{x}])$$

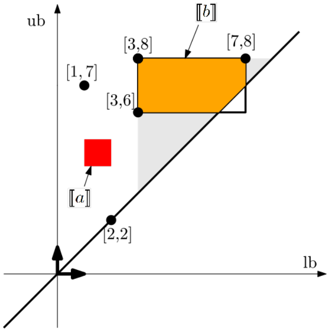


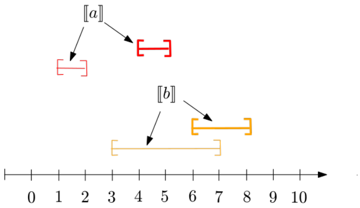
F is inclusion monotonic

2. Twins

A *twin* (or *thick interval*) $[\![x]\!]$ (see, *e.g.*, Nesterov et. al. [12],[7], Sainz et.al. [13], Chabert et.al. [1]) is a subset of $\mathbb{IR}$

$$\begin{aligned} [\![x]\!] &= [\![[x^-],[x^+]]\!] \\ &= \left\{[x^-,x^+] \in \mathbb{IR} \mid x^- \in [x^-] \text{ and } x^+ \in [x^+]\right\}. \end{aligned}$$



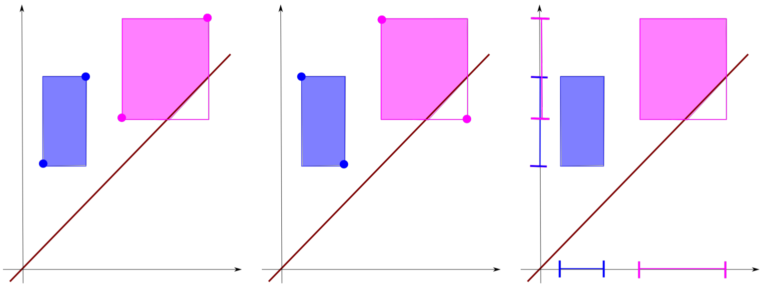


Endpoints diagrams [9]

Different ways to represent twins:

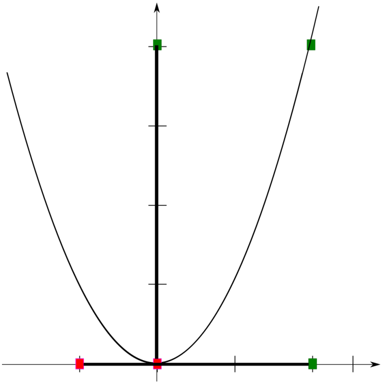
1. as an interval of the set of intervals with respect to $\leq$, where $[a] \leq [b] \Leftrightarrow a^- \leq b^-$ and $a^+ \leq b^+$
2. as an interval of the set of intervals with respect to $\subset$ [13] where $[a] \subset [b] \Leftrightarrow b^- \leq a^-$ and $a^+ \leq b^+$)
3. or as a vector of two intervals containing the lower bound and the upper bound respectively.

We chosed Option 3, whereas [5] has chosen Options 1 and 2.

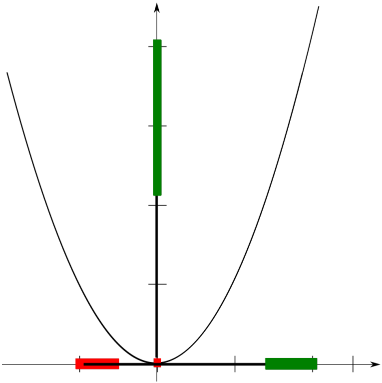


Different ways to represent twins

The square of a twin



$$x \in [x] \Rightarrow x^2 \in [x]^2$$



$$[x] \in [\![x]\!] \Rightarrow [x]^2 \in [\![x]\!]^2$$

Assume that

$$[x] \in [\![x]\!] = [\![\,[x^-]\,,[x^+]\,]\!] = [\![[-3,-1]\,,[-2,2]]\!]$$

what can we say about $[y] = [x]^2$? What could represent $[\![x]\!]^2$?

Since

$$[\![x]\!] = [\![[-3,-1],[-2,2]]\!]$$

we have

$$\begin{array}{ccc} [-1,0] \in [\![x]\!] & \Rightarrow & [-1,0]^2 = [0,1] \in [\![x]\!]^2 \\ [-3,-2] \in [\![x]\!] & \Rightarrow & [-3,-2]^2 = [4,9] \in [\![x]\!]^2 \end{array}$$

Thus, we want $[\![[0,4],[1,9]]\!] \subset [\![[-3,-1],[-2,2]]\!]^2$.

The square of an interval [10]

$$
\begin{array}{rcl}
[x]^2 & = & [x^-, x^+]^2 \\
& = & \{x^2 | x \in [x]\} \\
& = & \left\{ \begin{array}{cc} [\min(x^{-2}, x^{+2}), \max(x^{-2}, x^{+2})] & \text{if } 0 \notin [x] \\ [0, \max(x^{-2}, x^{+2})] & \text{if } 0 \in [x] \end{array} \right.
\end{array}
$$

A closed form for interval square is

$$[x]^2 = [\max(0, \mathrm{sign}(x^- \cdot x^+)) \cdot \min(x^{-2}, x^{+2}), \max(x^{-2}, x^{+2})].$$

Assume that

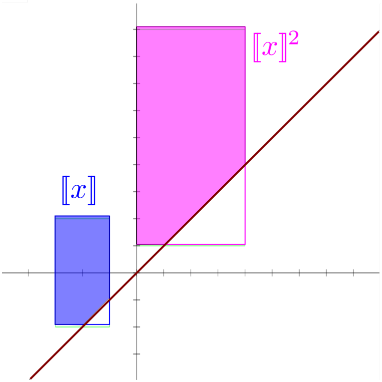
$$[x] \in [\![[x^-], [x^+]]\!] = [\![[-3,-1], [-2,2]]\!]$$

what can we say about $[y] = [x]^2$?

Since $x^- \in [x^-] = [-3,-1]$ and $x^+ \in [x^+] = [-2,2]$, we have

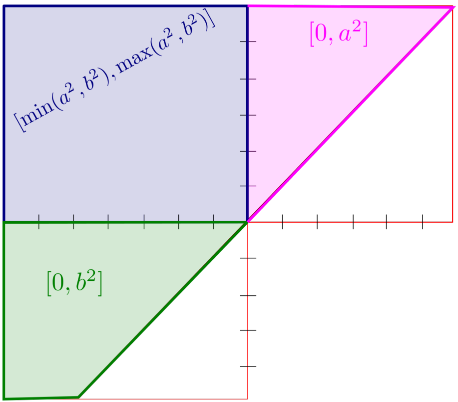
$$\begin{array}{rcl} y^- & = & \max(0, \text{sign}(x^- \cdot x^+)) \cdot \min(x^{-2}, x^{+2}) \\ & \in & \max(0, \text{sign}([x^-] \cdot [x^+])) \cdot \min([x^-]^2, [x^+]^2) \\ & = & \max(0, \text{sign}([-3,-1] \cdot [-2,2])) \cdot \min([-3,-1]^2, [-2,2]^2) \\ & = & [0,1] \cdot [0,4] = [0,4] \\ y^+ & = & \max(x^{-2}, x^{+2}) \\ & \in & \max([x^-]^2, [x^+]^2) \\ & = & \max([1,9],[0,4]) = [1,9] \end{array}$$

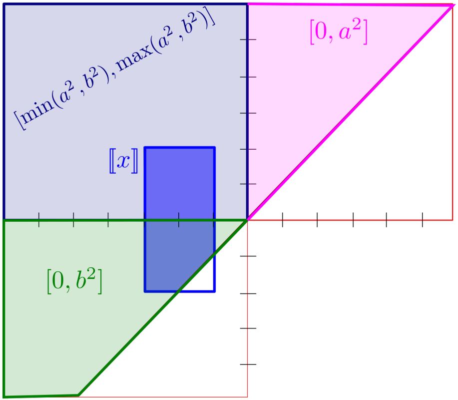
Therefore $[x]^2 \in [\![[0,4], [1,9]]\!]$.



Square of a twin

$$
\begin{aligned}
[a,b]^2 &= [0,b^2] && \text{if } a \geq 0 \\
&= [0,a^2] && \text{if } b \geq 0 \\
&= [\min(a^2,b^2),\max(a^2,b^2)] && \text{if } a < 0 < b
\end{aligned}
$$


$[\min(a^2,b^2),\max(a^2,b^2)]$
$[0,a^2]$
$[0,b^2]$


$[\min(a^2,b^2),\max(a^2,b^2)]$
$[0,a^2]$
$[\![x]\!]$
$[0,b^2]$

Thus

$$
\begin{aligned}
[x]^2 &\in [\![[-3,-1],[-2,2]]\!]^2 \\
&= [\![[-3,-1],[-2,0]]\!]^2 \cup [\![[-3,-1],[0,2]]\!]^2 \\
&= \left[\!\!\left[0,[-2,0]^2\right]\!\!\right] \\
&\quad \cup \left[\!\!\left[\min([-3,-1]^2,[0,2]^2),\max([-3,-1]^2,[0,2]^2)\right]\!\!\right] \\
&= [\![0,[0,4]]\!] \cup [\![\min([1,9],[0,4]),\max([1,9],[0,4])]\!] \\
&= [\![0,[0,4]]\!] \cup [\![[0,4],[1,9]]\!] \\
&= [\![[0,4],[1,9]]\!]
\end{aligned}
$$

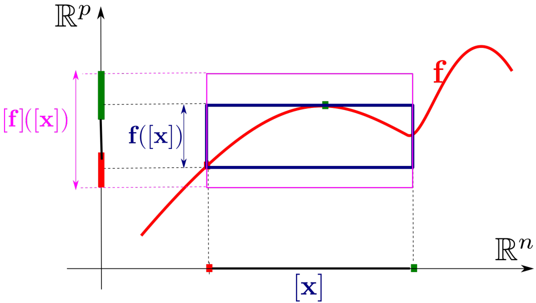
Comparison with classical twin arithmetic

For us

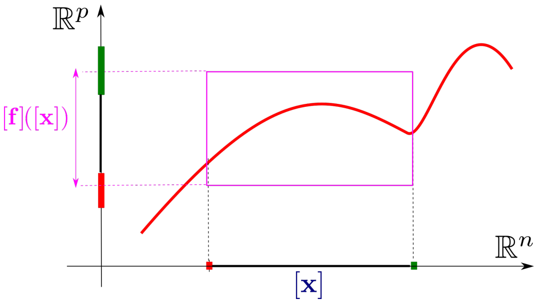
$$\begin{array}{rcl} x \in [x] & \Rightarrow & x^2 \in [x]^2 \\ [x] \in [\![x]\!] & \Rightarrow & [x]^2 \in [\![x]\!]^2 \end{array}$$

If $[\![x]\!]$ is degenerated, $[\![x]\!]^2$ is also degenerated.

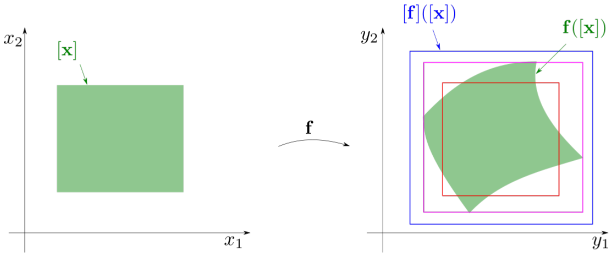
The objectives of the two twin arithmetics are different.



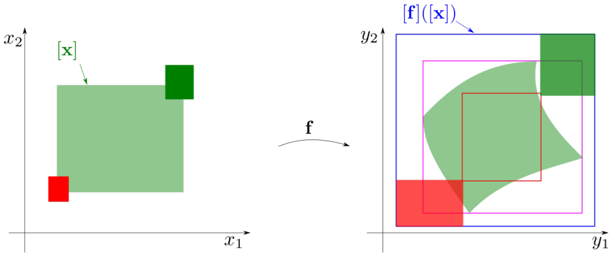
Classical twin arithmetic wants to find an inner and outer approximation of $\mathbf{f}([\mathbf{x}])$



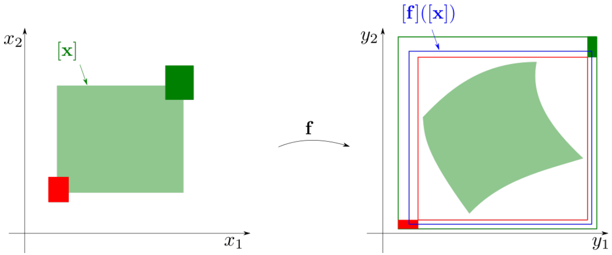
Our twin arithmetic wants to find an inner and outer approximation of $[\mathbf{f}]([\mathbf{x}]), [\mathbf{x}] \in [\![\mathbf{x}]\!]$



Goal of classical twin arithmetic



Goal of classical twin arithmetic



Goal of our twin arithmetic

Modified twin arithmetic

If $\diamond \in \{+,-,\cdot,\dots\}$, we define (similar to Guardenes et.al. [5])

$$[\![x]\!] \diamond [\![y]\!] \quad = \quad [\![\{[x] \diamond [y] \mid [x] \in [\![x]\!], [y] \in [\![y]\!]\}]\!].$$

where $[\![A]\!]$ denotes the smallest twin that contains the set of intervals A.

Thus, if $[\![a]\!] = [\![[a^-],[a^+]]\!]$ and $[\![b]\!] = [\![[b^-],[b^+]]\!]$,

$$
\begin{array}{rcl}
[\![a]\!] + [\![b]\!] & = & [\![[a^-]+[b^-],[a^+]+[b^+]]\!] \\
[\![a]\!] - [\![b]\!] & = & [\![[a^-]-[b^+],[a^+]-[b^-]]\!] \\
[\![a]\!] \cdot [\![b]\!] & = & [\![\min([a^-]\cdot[b^-],[a^-]\cdot[b^+],[a^+]\cdot[b^-],[a^+]\cdot[b^+])\,, \\
 & & \max([a^-]\cdot[b^-],[a^-]\cdot[b^+],[a^+]\cdot[b^-],[a^+]\cdot[b^+])]\!]
\end{array}
$$

If $f \in \{\text{sqr}, \sin, \cos, \dots\}$, we define

$$f(\llbracket x \rrbracket) = \llbracket \{ f([x]) \mid [x] \in \llbracket x \rrbracket, [y] \in \llbracket y \rrbracket \} \rrbracket.$$

For instance, if $\llbracket a \rrbracket = \llbracket [a^-], [a^+] \rrbracket$, we have

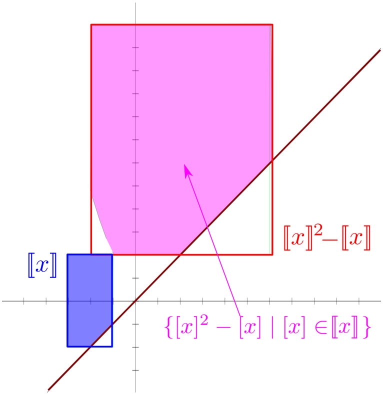
$$\begin{array}{rcl} \exp(\llbracket a \rrbracket) & = & \llbracket \exp([a^-]), \exp([a^+]) \rrbracket \\ \text{sqr}(\llbracket a \rrbracket) & = & \Big\llbracket \max(0, \text{sign}([a^-] \cdot [a^+])) \cdot \min(([a^-]^2, [a^+]^2) \\ & & , \ \max([a^-]^2, [a^+]^2) \Big\rrbracket \end{array}$$

Fundamental theorem

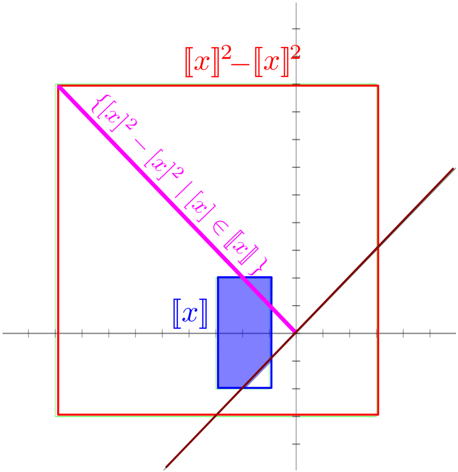
Denote by $[\![\mathbf{F}]\!]$ the twin extension of $\mathbf{F}$. We have

$$\left\{\begin{array}{c} \mathbf{x} \in [\mathbf{x}] \\ \mathbf{f} \in \mathbf{F} \end{array}\right. \Rightarrow \mathbf{f}(\mathbf{x}) \in \mathbf{F}(\mathbf{x}) \in [\![\mathbf{F}]\!]([\mathbf{x}])$$

This property is the main motivation for our twin arithmetic.



$F([x]) = [x]^2 - [x]$ and $[\![x]\!] = [\![[-3,-1], [-2,2]]\!]$, then we get:
$[\![F]\!]([\![x]\!]) = [\![[-2,6],[2,12]]\!]$



$F([x]) = [x]^2 - [x]^2$ and $[\![x]\!] = [\![[-3,-1],[-2,2]]\!]$

An online Python program associated to the previous examples can be found here:
https://replit.com/@aulin/twins

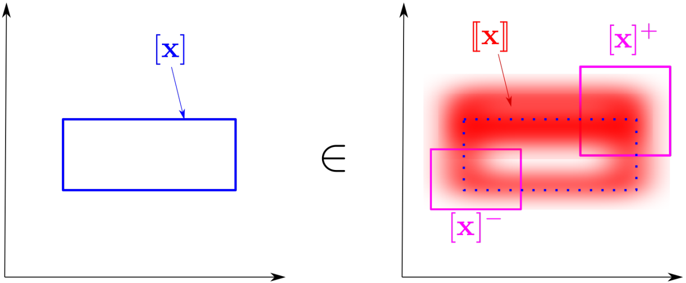
3. Thick set inversion

Thick box

A *thick box* $[\![\mathbf{x}]\!]$ is

$$[\![\mathbf{x}]\!] = \left\{ \left[\mathbf{x}^-, \mathbf{x}^+\right] \in \mathbb{IR}^n \mid \mathbf{x}^- \in \left[\mathbf{x}^-\right], \, \mathbf{x}^+ \in \left[\mathbf{x}^+\right] \right\}$$

where $[\mathbf{x}^-]$ and $[\mathbf{x}^+]$ are boxes of $\mathbb{R}^n$.



The box $[\mathbf{x}]$ (thin) belongs to the thick box $[\![\mathbf{x}]\!] = [\![[\mathbf{x}^-], [\mathbf{x}^+]]\!]$

Thick set inversion

Given a set $\mathbb{Y}$ and a thick function $\mathbf{F} : \mathbb{IR}^n \to \mathbb{IR}^m$.
We want to find $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$ where $\mathbf{f} \in \mathbf{F}$

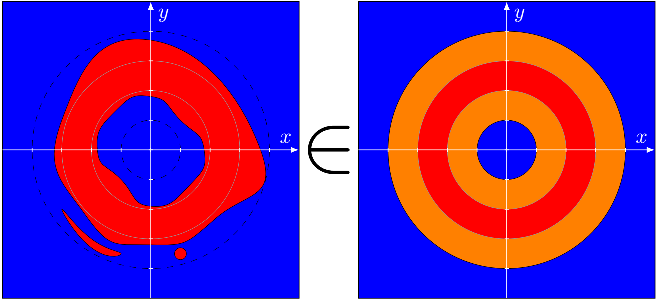
If

$$\begin{array}{rcl} \mathbb{X}^{\subset} & = & \{\mathbf{x} \in \mathbb{R}^n | \mathbf{F}(\mathbf{x}) \subset \mathbb{Y}\} \\ \mathbb{X}^{\supset} & = & \{\mathbf{x} \in \mathbb{R}^n | \mathbf{F}(\mathbf{x}) \cap \mathbb{Y} \neq \emptyset\} \\ \partial\mathbb{X} & = & \mathbb{X}^{\supset} \backslash \mathbb{X}^{\subset} \qquad \text{(penumbra)} \end{array}$$

We have

$$\mathbb{X} \in [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!].$$

The pair $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!]$ corresponds to a thick set (Desrochers et.al. [2], Dubois et. al. [3]).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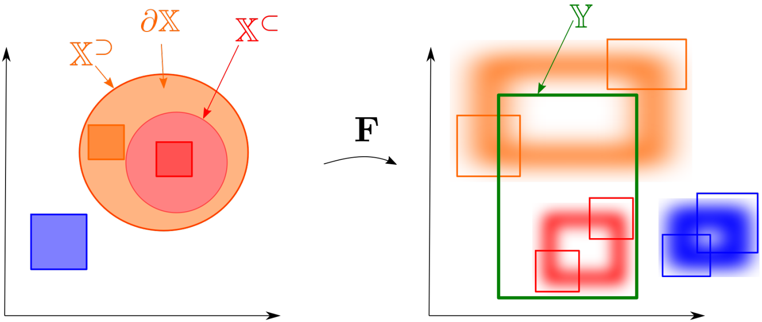


Principle of thick inversion algorithm

Localization

A robot is known to be at a distance less than $20m$ from 3 landmarks $\mathbf{m}(i), i \in \{1,2,3\}$, We know that

$$\begin{array}{rcc} \mathbf{m}(1) & \in & [-1,3] \times [1,5] \\ \mathbf{m}(2) & \in & [8,12] \times [-3,1] \\ \mathbf{m}(3) & \in & [8,12] \times [4,8] \end{array}$$

We build the thick function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{IR}^3$

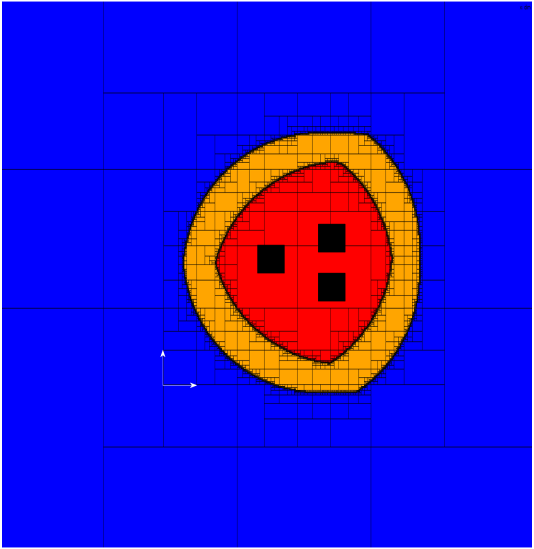
$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} (x_1 - [-1,3])^2 + (x_2 - [1,5])^2 \\ (x_1 - [8,12])^2 + (x_2 - [-3,1])^2 \\ (x_1 - [8,12])^2 + (x_2 - [4,8])^2 \end{pmatrix}$$

The twin extention is

$$[\![\mathbf{F}]\!]([\![\mathbf{x}]\!]) = \begin{pmatrix} ([\![x_1]\!] - [-1,3])^2 + ([\![x_2]\!] - [1,5])^2 \\ ([\![x_1]\!] - [8,12])^2 + ([\![x_2]\!] - [-3,1])^2 \\ ([\![x_1]\!] - [8,12])^2 + ([\![x_2]\!] - [4,8])^2 \end{pmatrix}$$

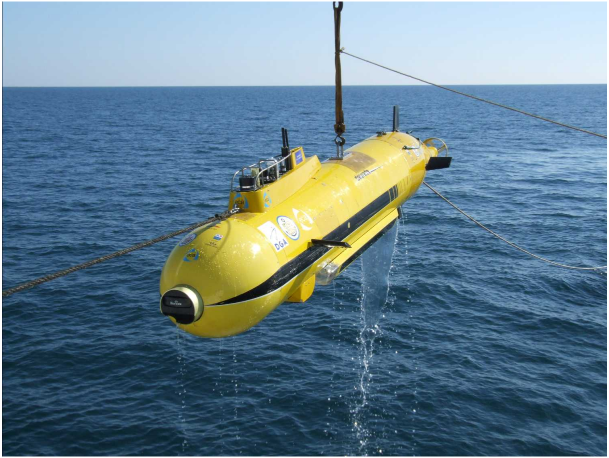
Take

$$\mathbb{Y} = [0,400]^{\times 3}.$$



An online Python program : https://replit.com/@aulin/twindisks

Underwater exploration



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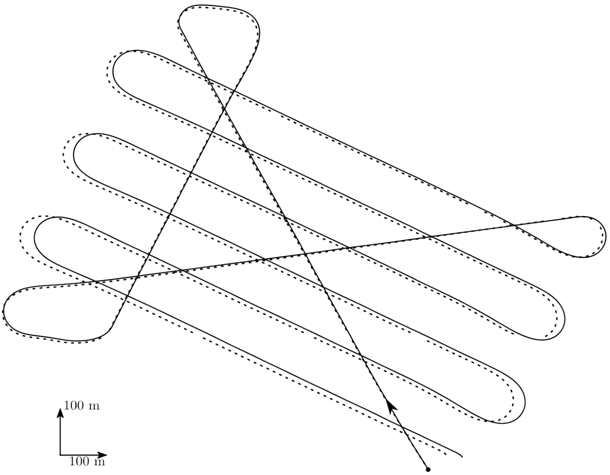
100 m
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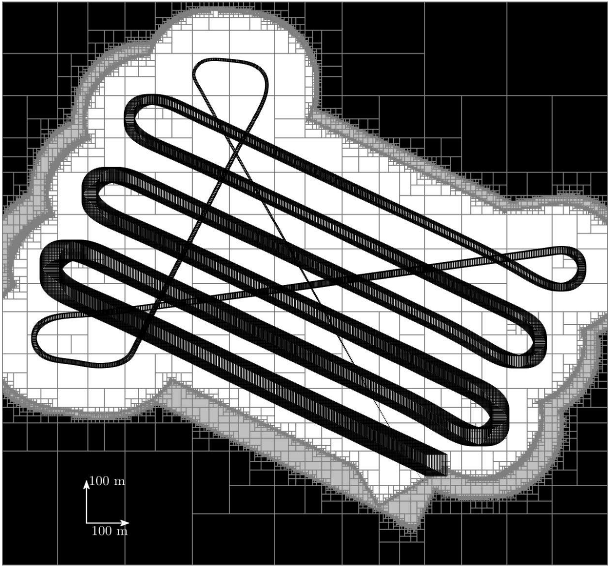
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Conclusion

Twins [12][8] , modal intervals [4], etc are used characterize from the inside and the outside sets of the form

$$\{y \in \mathbb{R} \,|\, \forall \mathbf{p} \in [\mathbf{p}], \exists \mathbf{x} \in [\mathbf{x}],\, y = f(\mathbf{x}, \mathbf{p})\}$$

in the case where $f\colon \mathbb{R}^n \to \mathbb{R}$.
A special case is the computation of the range of a function:

$$\{y \in \mathbb{R} \,|\, \exists \mathbf{x} \in [\mathbf{x}], y = \mathbf{f}(\mathbf{x})\}.$$

Extensions to the case where $f\colon \mathbb{R}^n \to \mathbb{R}^m$ are proposed [6][11] [14] exist and need bisections to get an accurate result.
Our arithmetic aims at characterizing the image $[\mathbf{y}]$ of a box $[\mathbf{x}]$ by an thick function:

$$\{[\mathbf{y}] \mid \exists \mathbf{x} \in [\mathbf{x}], [\mathbf{y}] = \mathbf{F}(\mathbf{x})\}.$$

The function $\mathbf{F}$ represents an unknown function $\mathbf{f}$. The uncertainty in not represented by a parametric model depending on $\mathbf{p}$.

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