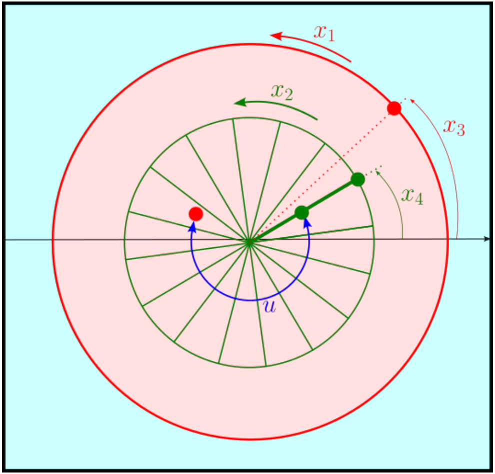
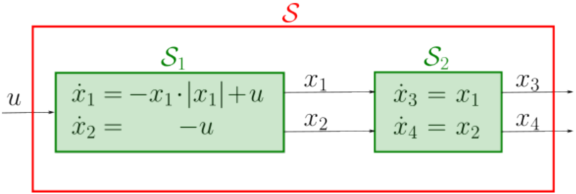
Swim Disk

Septembre 2024

1. Présentation


x_1
x_2
x_3
x_4
u


$\mathcal{S}$
$\mathcal{S}_1$
$\mathcal{S}_2$
u
$\dot{x}_1 = -x_1 \cdot |x_1| + u$
$\dot{x}_2 = -u$
x_1
x_2
$\dot{x}_3 = x_1$
$\dot{x}_4 = x_2$
x_3
x_4

$$\mathscr{S}_1 : \begin{cases} \dot{x}_1 &= -x_1 \cdot |x_1| + u \\ \dot{x}_2 &= -u \end{cases}$$

The *small-time local controllability* can only be obtained for driftless states.
For $\mathscr{S}_1$, the driftless states have the form $\bar{\mathbf{x}} = (0, \bar{x}_2)$.

Linearization. The linearized system around a driftless states $\bar{\mathbf{x}}$ is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\mathbf{b}} \cdot u \cdot$$

Controllability matrix

$$\mathbf{C}_{\mathsf{com}} = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

Since $\operatorname{rank}(\mathbf{C}_{\mathsf{com}}) = 1$, we cannot conclude anything about the local accessibility.

2. Lie brackets for controllability

Dubins car

$$
\begin{aligned}
\dot{x}_1 &= u_1 \cos x_3 \\
\dot{x}_2 &= u_1 \sin x_3 \\
\dot{x}_3 &= u_2
\end{aligned}
$$

i.e.,

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{=\mathbf{f}_1} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{=\mathbf{f}_2} \cdot u_2$$

$$\begin{aligned} [\mathbf{f}_1, \mathbf{f}_2](\mathbf{x}) &= \frac{d\mathbf{f}_2}{d\mathbf{x}} \cdot \mathbf{f}_1 - \frac{d\mathbf{f}_1}{d\mathbf{x}} \cdot \mathbf{f}_2 \\ &= -\begin{pmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{pmatrix} \end{aligned}$$

$$\text{Rank}\begin{pmatrix} \cos x_3 & 0 & \sin x_3 \\ \sin x_3 & 0 & -\cos x_3 \\ 0 & 1 & 0 \end{pmatrix} = 3$$

The system is small-time locally controllable.

Interpretation of the Lie bracket: If we apply the cyclic sequence:

$$\begin{array}{ccccc} t \in [0,\delta] & t \in [\delta,2\delta] & t \in [2\delta,3\delta] & t \in [3\delta,4\delta] & \dots \\ u_1 = 1 & u_1 = 0 & u_1 = -1 & u_1 = 0 & \dots \\ u_2 = 0 & u_2 = 1 & u_2 = 0 & u_2 = -1 & \end{array}$$

then

$$\mathbf{x}(t+4\delta) = \mathbf{x}(t) + [\mathbf{f},\mathbf{g}](\mathbf{x}(t))\,\delta^2 + o\left(\delta^2\right).$$

Swim disk with cubic friction

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -x_1^3 \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\mathbf{g}} \cdot u \cdot$$

The system has drift

$$
\begin{aligned}
[\mathbf{f},\mathbf{g}](\mathbf{x}) &= \tfrac{d\mathbf{g}}{d\mathbf{x}}\cdot\mathbf{f}-\tfrac{d\mathbf{f}}{d\mathbf{x}}\cdot\mathbf{g}=\begin{pmatrix}-3x_1^2\\0\end{pmatrix}\\
[\mathbf{f},[\mathbf{f},\mathbf{g}]](\mathbf{x}) &= \tfrac{d[\mathbf{f},\mathbf{g}]}{d\mathbf{x}}\cdot\mathbf{f}-\tfrac{d\mathbf{f}}{d\mathbf{x}}\cdot[\mathbf{f},\mathbf{g}]=\begin{pmatrix}15x_1^4\\0\end{pmatrix}\\
[[\mathbf{f},\mathbf{g}],\mathbf{g}] &= \tfrac{d\mathbf{g}}{d\mathbf{x}}\cdot[\mathbf{f},\mathbf{g}]-\tfrac{d[\mathbf{f},\mathbf{g}]}{d\mathbf{x}}\cdot\mathbf{g}=\begin{pmatrix}6x_1\\0\end{pmatrix}\\
[[[\mathbf{f},\mathbf{g}],\mathbf{g}],\mathbf{g}] &= \tfrac{d\mathbf{g}}{d\mathbf{x}}\cdot[[\mathbf{f},\mathbf{g}],\mathbf{g}]-\tfrac{d[[\mathbf{f},\mathbf{g}],\mathbf{g}]}{d\mathbf{x}}\cdot\mathbf{g}=\begin{pmatrix}-6\\0\end{pmatrix}
\end{aligned}
$$

Swim disk with square friction

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -x_1^2 \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\mathbf{g}} \cdot u \cdot$$

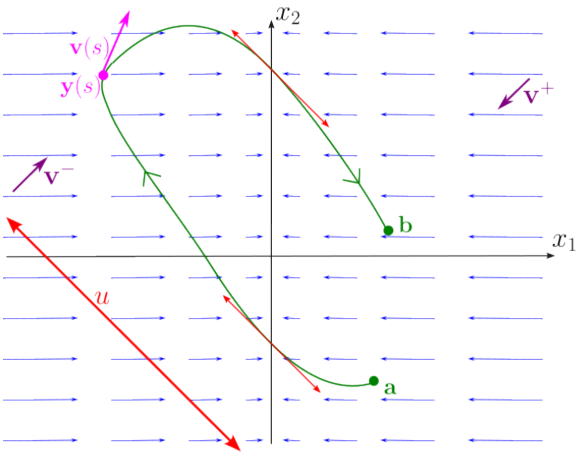
Again, the rank of Lie$(\mathbf{f}, \mathbf{g})$ is full, but the system is not controllable. Indeed

$$\dot{x}_1 + \dot{x}_2 = -x_1^2 \leq 0$$

We have bad Lie brackets.

3. Large cycles

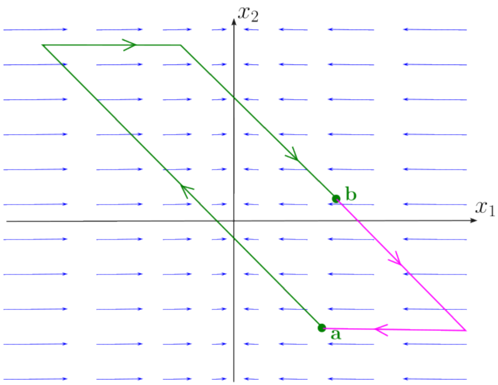
$$\mathscr{S}_1 : \begin{cases} \dot{x}_1 & = & -x_1 \cdot |x_1| + u \\ \dot{x}_2 & = & -u \end{cases}$$

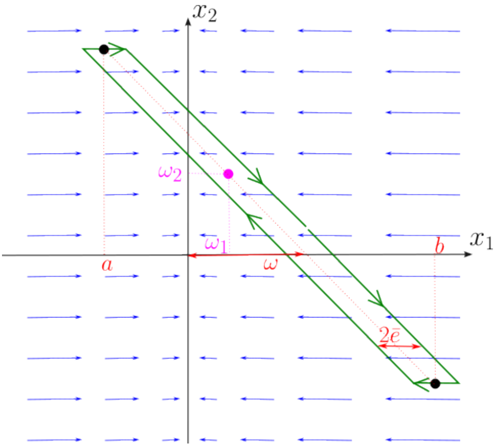

x_2
x_1
$\mathbf{v}(s)$
$\mathbf{y}(s)$
$\mathbf{v}^+$
$\mathbf{v}^-$
b
a
u

The system

$$\mathscr{S}_1 : \begin{cases} \dot{x}_1 &= -x_1 \cdot |x_1| + u \\ \dot{x}_2 &= -u \end{cases}$$

is globally accessible from any initial state.


x_2
x_1
b
a



Swim cycle

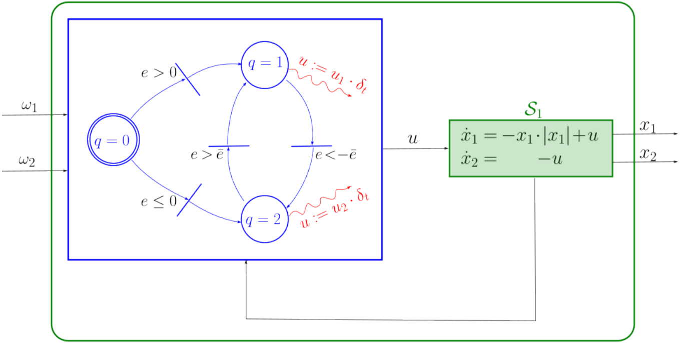
The parameters $(a, \omega, \bar{e})$ of the swim cycle corresponding to ω_1, ω_2, T, b are

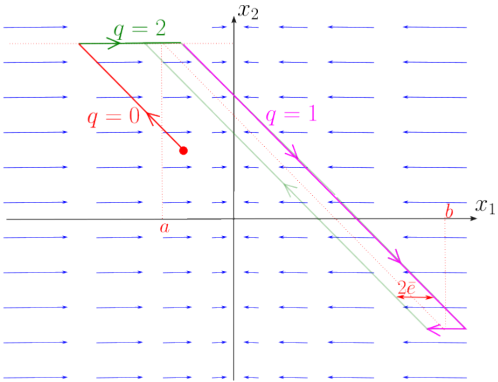
$$\omega = \omega_2 + \omega_1$$

$$a = \frac{-b^2 - b\sqrt{b^2 - 4(\omega_1 - b)\omega_1}}{2(b - \omega_1)}$$

$$\bar{e} = \frac{T}{\frac{2}{a^2} + \frac{2}{b^2}}$$

Controller


ω_1
ω_2
$q=0$
$e>0$
$q=1$
$u := u_1 \cdot \delta_t$
$e \leq 0$
$q=2$
$u := u_2 \cdot \delta_t$
$e > \bar{e}$
$e < -\bar{e}$
u
S_1
$\dot{x}_1 = -x_1 \cdot |x_1| + u$
$\dot{x}_2 = \quad -u$
x_1
x_2


x_2
$q = 2$
$q = 0$
$q = 1$
x_1
a
b
$2\bar{e}$

Proposition. The speed system $\mathscr{S}_1$ is fully $\mathscr{L}$-asymptotically controllable.

We take the $\mathscr{L}$ norm

$$\mathscr{L}(x) = \int_{-\infty}^{\infty} |x(\tau)|\, d\tau$$

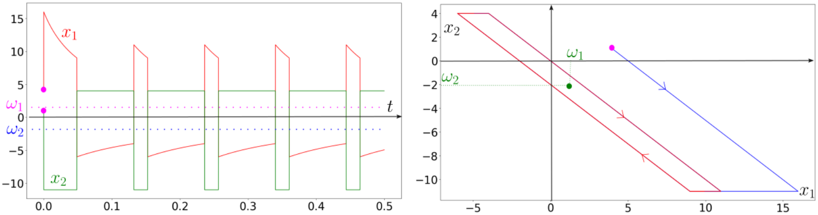
We define

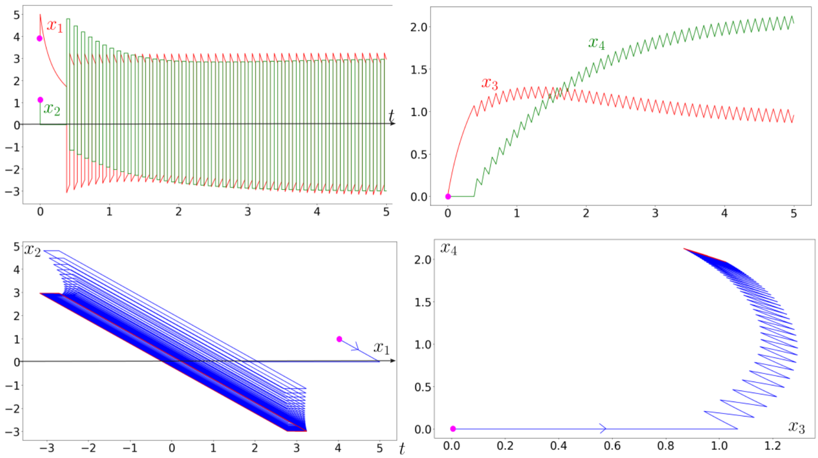
$$\begin{array}{lcll} x_{t_1\to\infty}(t) & = & 0 & \text{if } t \le t_1 \\ x_{t_1\to\infty}(t) & = & x(t) & \text{otherwise} \end{array}$$

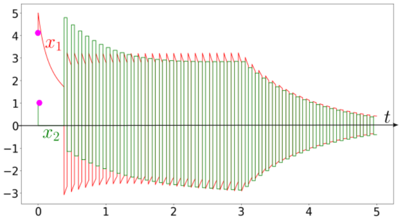
The function $x(t) : \mathbb{R} \to \mathbb{R}$ asymptotically converges to zero with respect to $\mathscr{L}$, we will write $x \overset{\mathscr{L}}{\to} \mathbf{0}$ if

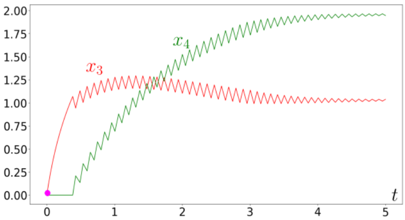
$$\forall \varepsilon > 0, \exists t_1, \mathscr{L}(x_{t_1 \to \infty}) \leq \varepsilon.$$

A system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ $\mathscr{L}$-asymptotically stable if for any $\mathbf{x}(0)$, each $x_i(t)$ satisfies $x_i \overset{\mathscr{L}}{\to} \mathbf{0}$.


15
10
5
0
−5
−10
x_1
x_2
ω_1
ω_2
t
0.0
0.1
0.2
0.3
0.4
0.5
4
2
0
−2
−4
−6
−8
−10
x_2
x_1
ω_1
ω_2
−5
0
5
10
15


x_1
x_2
t
x_3
x_4
x_2
x_1
t
x_4
x_3


5
4
3
2
1
0
−1
−2
−3
x_1
x_2
t
0
1
2
3
4
5


2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
x_4
x_3
t
0
1
2
3
4
5

References

1. Control with Lie Brackets [4] (Example 15.20), [1] page 121.
2. $\mathscr{L}$-asymptotically stability [3]
3. Controllability of linear systems [2]

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