A contractor which is minimal for narrow boxes

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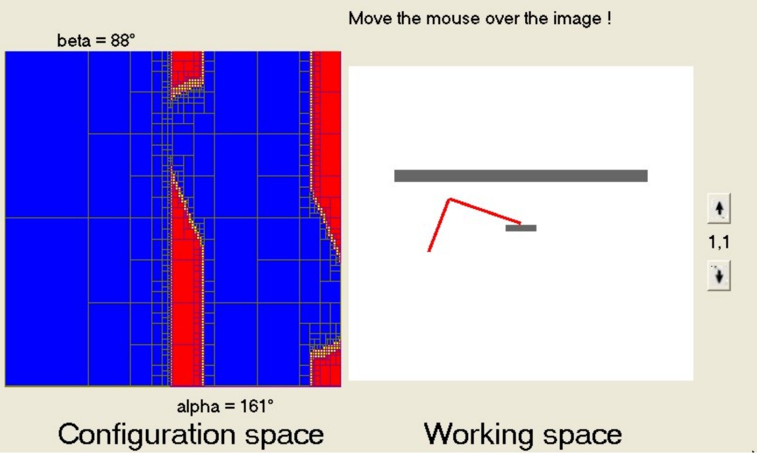
SWIM'23, Angers

1. Goodbye Nico

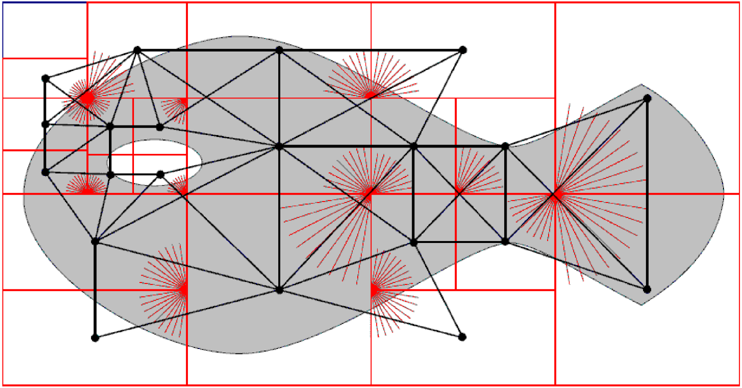


Configuration space

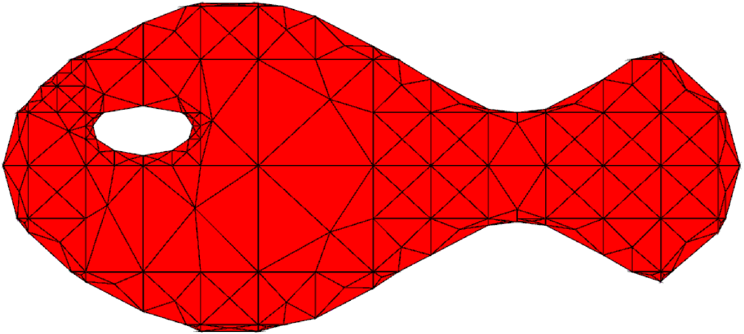
A star-spangled decomposition for the set

$$\mathbb{S} = \left\{ (x,y) \in \mathbb{R}^2 \mid \left(\begin{array}{c} x^2 + 4y^2 - 16 \\ 2\sin x - \cos y + y^2 - \frac{3}{2} \\ -(x + \frac{5}{2})^2 - 4(y - \frac{2}{5})^2 + \frac{3}{10} \end{array} \right) \leq 0 \right\},$$

is:



An extension of this approach has also been developed by N. Delanoue to compute a triangulation homeomorphic to $\mathbb{S}$.



Tips : With a microscope we can see everything.

2. Stability of a linear systems

Consider the *Palm* system

$$\dddot{x} + \sin(p_1 p_2) \cdot \ddot{x} + p_1^2 \cdot \dot{x} + p_1 p_2 \cdot x = 0$$

Its characteristic function is

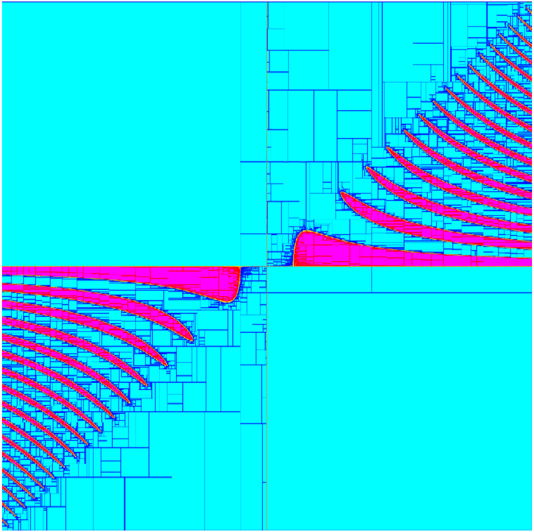
$$\theta(\mathbf{p}, s) = s^3 + \sin(p_1 p_2) \cdot s^2 + p_1^2 \cdot s + p_1 p_2$$

Stability domain

$$\mathbb{S} = \{\mathbf{p} \,|\, \theta(\mathbf{p}, s)\,\text{Hurwitz}\}.$$

We have

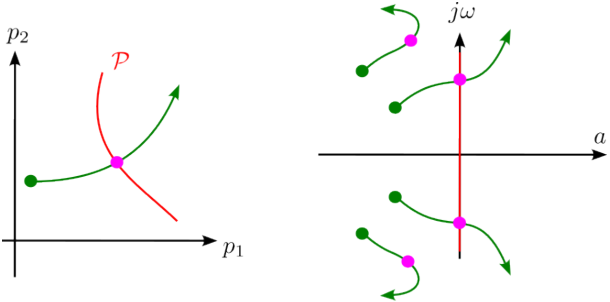
$$\mathbb{S} : \begin{cases} p_1 p_2 & \geq 0 \\ \sin(p_1 p_2) & \geq 0 \\ p_1^2 \sin(p_1 p_2) - p_1 p_2 & \geq 0 \end{cases}$$



Value set approach

The roots of $\theta(\mathbf{p},s)=0$ change continuously with $\mathbf{p}$.
We define the *value set*

$$\mathscr{P}=\{\mathbf{p}\,|\,\exists\omega>0,\,\theta(\mathbf{p},j\omega)=0\}.$$



Zero exclusion theorem

Cut off frequency. The roots of

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

are in the disk with center 0 and radius

$$\omega_c = 1 + \max\{\|a_0\|, \|a_1\|, \ldots, \|a_{n-1}\|\}$$

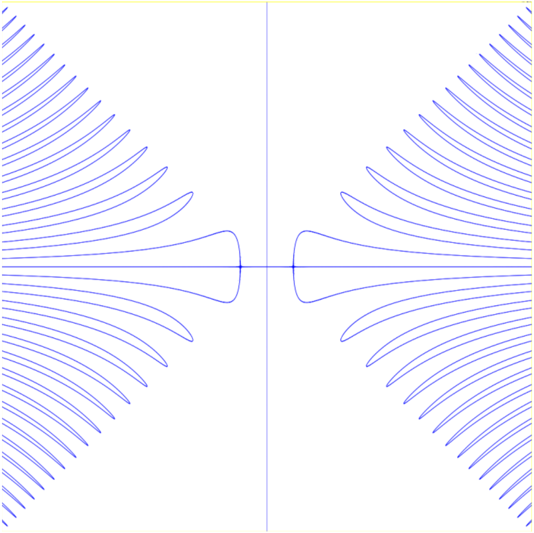
which is the Cauchy bound.

For

$$\theta(\mathbf{p}, s) = s^3 + \sin(p_1 p_2) \cdot s^2 + p_1^2 \cdot s + p_1 p_2$$

and $s = j\omega$, we get

$$\begin{array}{rc} & (j\omega)^3 + \sin(p_1 p_2) \cdot (j\omega)^2 + p_1^2 \cdot (j\omega) + p_1 p_2 = 0 \\ \Leftrightarrow & -j\omega^3 - \sin(p_1 p_2) \cdot \omega^2 + j p_1^2 \cdot \omega + p_1 p_2 = 0 \\ \Leftrightarrow & \left\{ \begin{array}{rcl} -\sin(p_1 p_2) \cdot \omega^2 + p_1 p_2 & = & 0 \\ -\omega^2 + p_1^2 & = & 0 \end{array} \right. \end{array}$$



Linear systems with delays

Periodic system

$$x(t+1) - x(t) = 0$$

The characteristic function is

$$\theta(s) = e^s - 1$$

The roots are

$$s = 2\pi kj, k \in \mathbb{N}$$

Turkulov system. Consider the system

$$\ddot{x}(t)+2\dot{x}(t-p_1)+x(t-p_2)=0$$

Its characteristic function is

$$\theta(\mathbf{p},s)=s^2+2se^{-sp_1}+e^{-sp_2}.$$

We define

$$\mathscr{P} = \{\mathbf{p} \,|\, \exists \omega > 0,\, \theta(\mathbf{p}, j\omega) = 0\}.$$

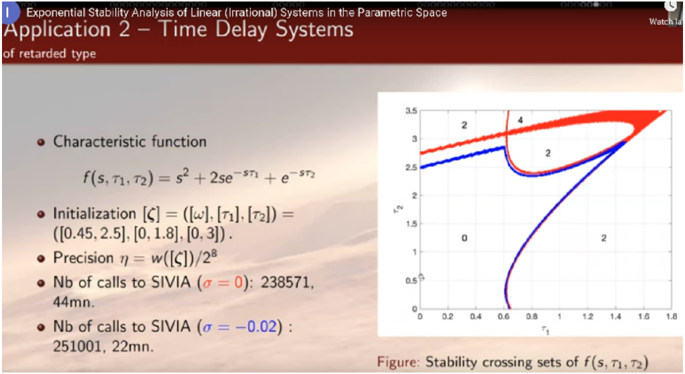
Now

$$\begin{aligned} & \theta(p_1, p_2, j\omega) \\ = \; & -\omega^2 + 2j\omega e^{-j\omega p_1} + e^{-j\omega p_2} \\ = \; & -\omega^2 + 2j\omega(\cos(\omega p_1) - j\sin(\omega p_1)) \\ & + \cos(\omega p_2) - j\sin(-\omega p_2) \\ = \; & -\omega^2 + 2\omega\sin(\omega p_1) + \cos(\omega p_2) \\ & + j\cdot(2\omega\cos(\omega p_1) - \sin(\omega p_2)) \end{aligned}$$

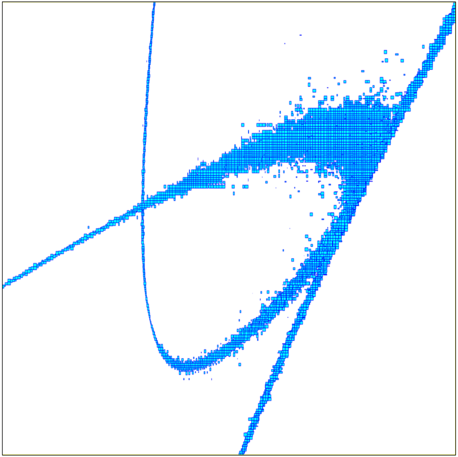
We have

$$\begin{aligned} & \theta(p_1,p_2,j\omega)=0 \\ \Leftrightarrow \quad & \underbrace{\begin{pmatrix} -\omega^2+2\omega\sin(\omega p_1)+\cos(\omega p_2) \\ 2\omega\cos(\omega p_1)-\sin(\omega p_2) \end{pmatrix}}_{\mathbf{f}(p_1,p_2,\omega)}=\mathbf{0} \end{aligned}$$

With $[p_1] = [0, 2.5]$, $[p_2] = [1, 4]$, $[\omega] = [0, 10]$, with a Matlab implementation, with a forward-backward contractor, and $\varepsilon = 2^{-8}$, [5] got:



https://youtu.be/DaR2NZZIV10?t=2453



$\varepsilon = 2^{-8}$, Codac [9] generated 43173 boxes.

We still have a *Clustering effect*

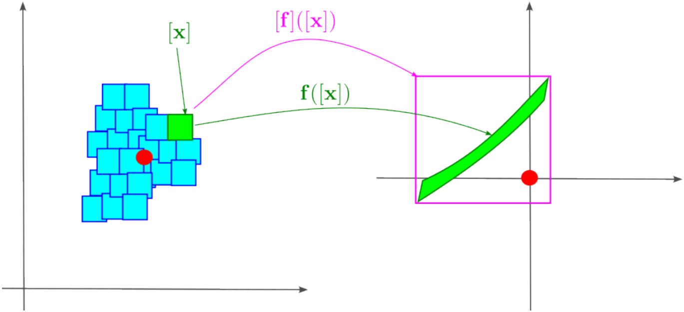
Hey !

With a microscope you can see everything

3. Minimal contractors

Given a function $\mathbf{f}:\mathbb{R}^n \mapsto \mathbb{R}^p$. An inclusion function for $\mathbf{f}$ is minimal if

$$[\mathbf{f}]([\mathbf{x}]) = [\![\{\mathbf{y} = \mathbf{f}(\mathbf{x}) \,|\, \mathbf{x} \in [\mathbf{x}]\}]\!].$$

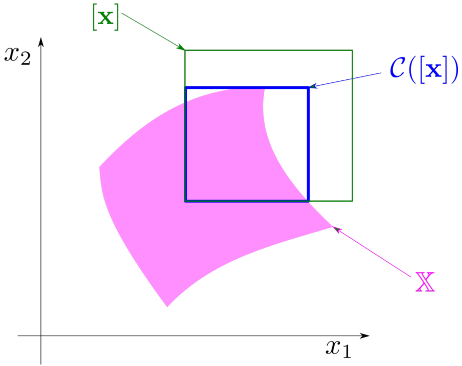


With a minimal inclusion, the clustering effect may exist, when solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$

A *contractor* associated to the set $\mathbb{X} \subset \mathbb{R}^n$ is a function $\mathscr{C} : \mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$\begin{array}{cc} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contraction)} \\ [\mathbf{x}] \cap \mathbb{X} \subset \mathscr{C}([\mathbf{x}]) & \text{(consistency)} \end{array}$$

It is *minimal* if $\mathscr{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \mathbb{X}]\!]$.


$[\mathbf{x}]$
x_2
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$
x_1

Tree matrices

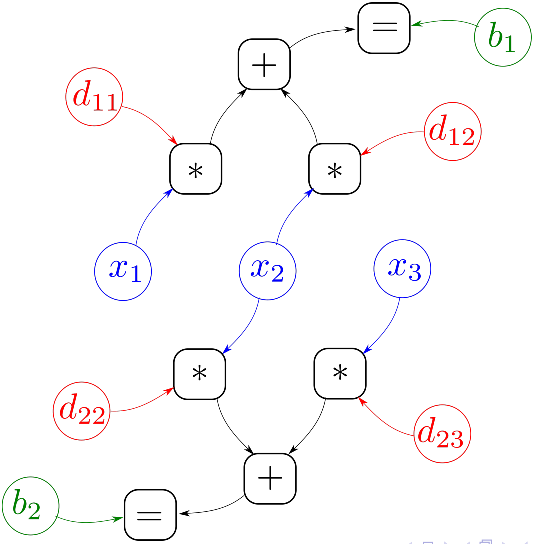
Consider the interval linear system:

$$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & d_{22} & d_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

where

$$d_{ij} \in [d_{ij}], x_j \in [x_j], b_i \in [b_i]$$

The optimal contraction can be obtained by a simple interval propagation.


=
b_1
+
d_{11}
d_{12}
*
*
x_1
x_2
x_3
*
*
d_{22}
d_{23}
+
b_2
=

No cycle for:

$$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & d_{23} & 0 \\ 0 & 0 & d_{33} & d_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

A matrix $\mathbf{D}$ such that $\mathbf{D} \cdot \mathbf{x} = \mathbf{b}$ has no cycle is a *tree matrix*.

We a Gauss Jordan transformation:

$$\mathbf{Ax} = \mathbf{c} \Leftrightarrow \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{Q} \cdot \mathbf{c}$$

we may get a tree matrix: $\mathbf{D} = \mathbf{Q} \cdot \mathbf{A}$.

Simplex contractor

For the linear system

$$\mathbf{A}\mathbf{x} = \mathbf{c}, \mathbf{x} \in [\mathbf{x}], \mathbf{c} \in [\mathbf{c}]$$

we can use the simplex algorithm to build the minimal contractor. Guarantee can be obtained with an inflation [8]

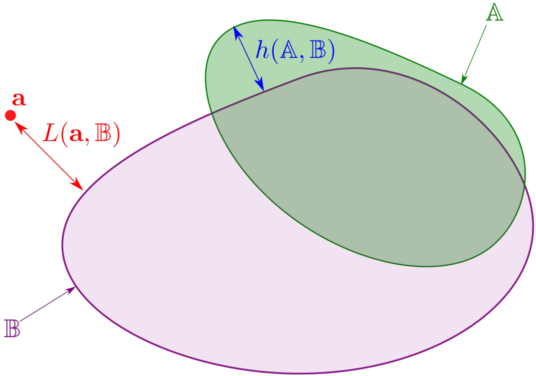
4. Asymptotic minimality

Proximity. Denote by $L(\mathbf{a},\mathbf{b})$ a distance between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{R}^n$ induced by the L-norm (L_∞ or L_2).
The *proximity* of $\mathbb{A}$ to $\mathbb{B}$ is

$$h(\mathbb{A},\mathbb{B}) = \sup_{\mathbf{a}\in\mathbb{A}} L(\mathbf{a},\mathbb{B})$$

where

$$L(\mathbf{a},\mathbb{B}) = \inf_{\mathbf{b}\in\mathbb{B}} L(\mathbf{a},\mathbf{b}).$$



Proximity of $\mathbb{A}$ to $\mathbb{B}$

Definition. The pessimism of an inclusion function $[\mathbf{f}]$ is

$$\eta([\mathbf{x}]) = h([\mathbf{f}]([\mathbf{x}]), [\![\mathbf{f}([\mathbf{x}])]\!])$$

Definition. An inclusion function $[\mathbf{f}]$ is of order j if

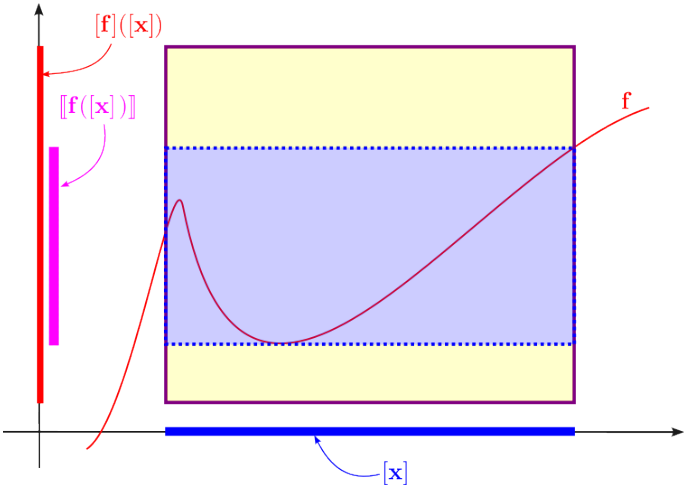
$$\eta([\mathbf{x}]) = o(w^j([\mathbf{x}]))$$

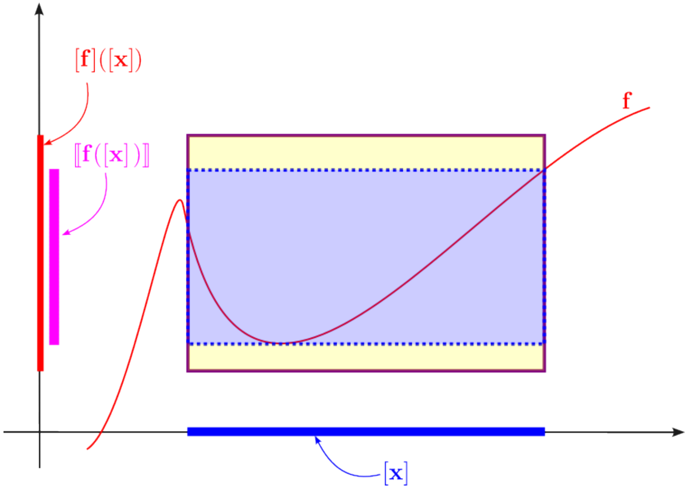
Definition. $[\mathbf{f}]$ is convergent if it is of order $j = 0$:

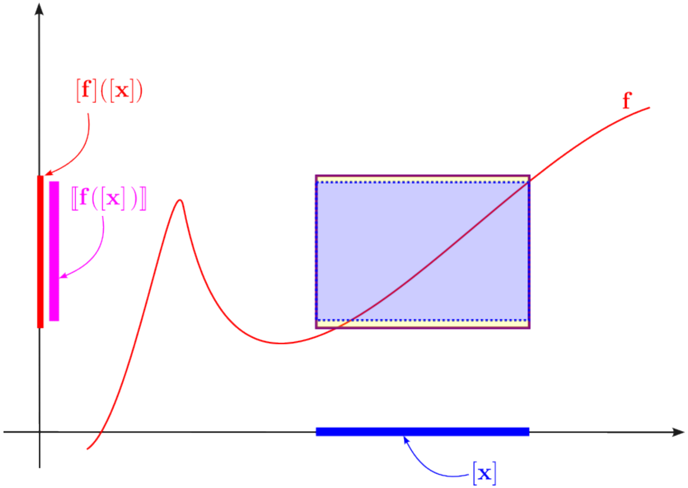
$$\eta([\mathbf{x}]) = o(w^0([\mathbf{x}])) = O(w([\mathbf{x}]))$$

Definition. $[\mathbf{f}]$ is asymptotically minimal if it is of order $j = 1$:

$$\eta([\mathbf{x}]) = o(w([\mathbf{x}]))$$


[f]([x])
[[f([x])]]
f
[x]


[f]([x])
[[f([x])]]
f
[x]


[f]([x])
[[f([x])]]
f
[x]

Proposition. The centered form

$$[\mathbf{f}]([\mathbf{x}]) = \mathbf{f}(\mathbf{m}) + [\mathbf{f}'] ([\mathbf{x}]) \cdot ([\mathbf{x}] - \mathbf{m})$$

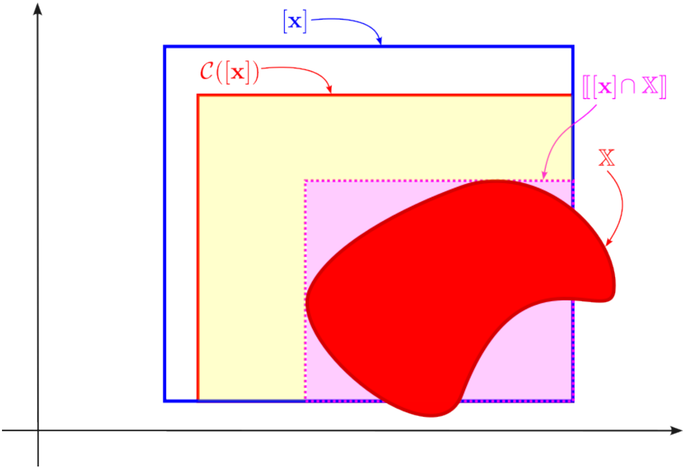
where $\mathbf{m} = \text{center}([\mathbf{x}])$ is asymptotically minimal.

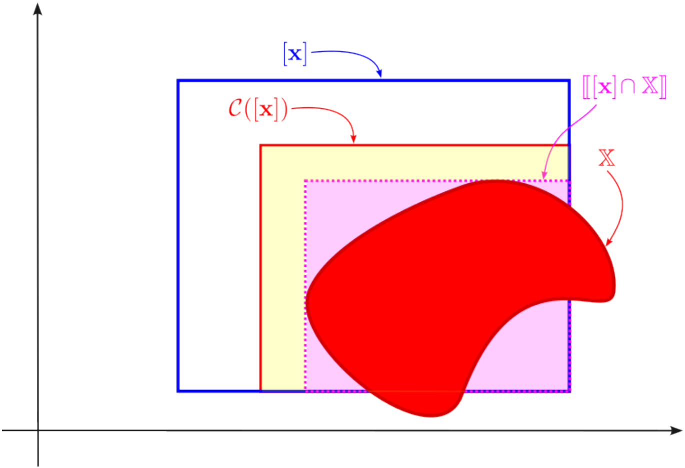
Definition. The *pessimism* of a contractor $\mathscr{C}$ for $\mathbb{X}$ at $[\mathbf{x}]$ is

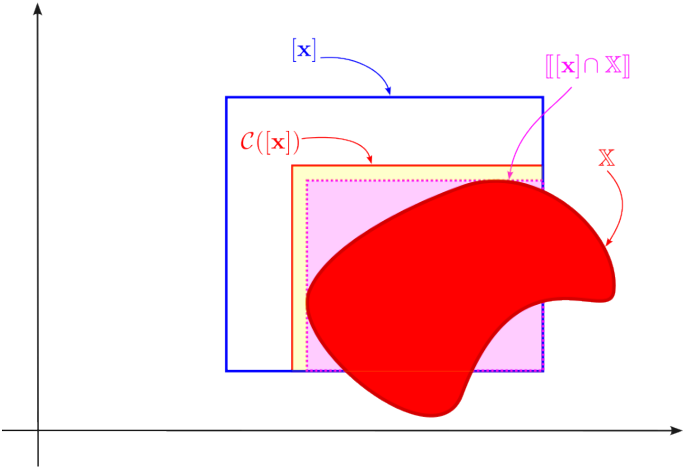
$$\eta([\mathbf{x}]) = h(\mathscr{C}([\mathbf{x}]), [\![[\mathbf{x}] \cap \mathbb{X}]\!])$$

Definition. A contractor $\mathscr{C}$ for $\mathbb{X}$ is of order j if

$$\eta([\mathbf{x}]) = o(w^j([\mathbf{x}]))$$


[x]
C([x])
[[x] ∩ 𝕏]
𝕏


[x]
C([x])
[[x] ∩ X]
X

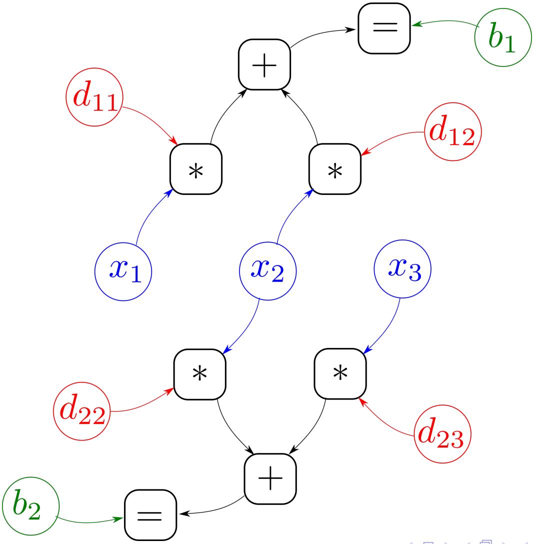

[x]
[[x] ∩ X]
C([x])
X

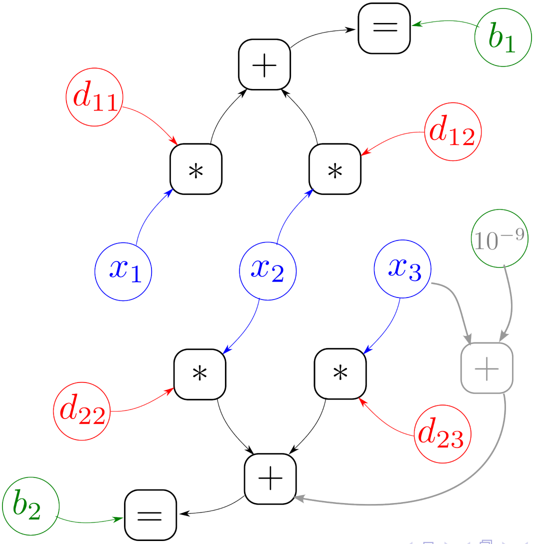
Proposition. Consider a set $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$. Take $[\mathbf{x}]$ with center $\mathbf{m}$. Define $\mathbf{Q}$ s.t. $\mathbf{Q} \cdot \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$ is a tree matrix.
An interval propagation on;

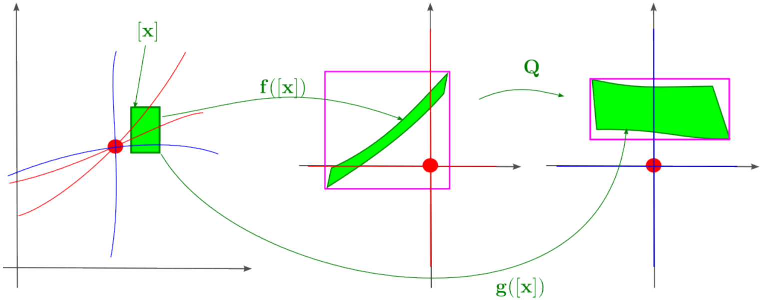
$$\mathbf{Q} \cdot \mathbf{f}(\mathbf{m}) + \mathbf{Q} \cdot \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) = \mathbf{0}$$
$$\mathbf{A} \in [\tfrac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}])$$
$$\mathbf{x} \in [\mathbf{x}]$$

yields an asymptotically minimal contractor for $\mathbb{X}$.

Proof. ...


=
b_1
+
d_{11}
d_{12}
*
*
x_1
x_2
x_3
*
*
d_{22}
d_{23}
+
b_2
=


=
b_1
+
d_{11}
d_{12}
*
*
10^{-9}
x_1
x_2
x_3
*
*
+
d_{22}
d_{23}
+
b_2
=


[x]
f([x])
Q
g([x])

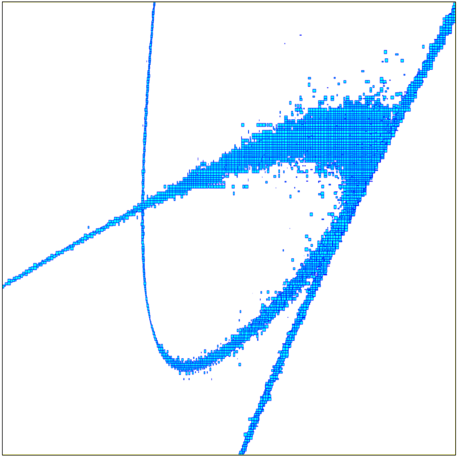
Centered contractor

Input: $\mathbf{f}, [\mathbf{x}]$

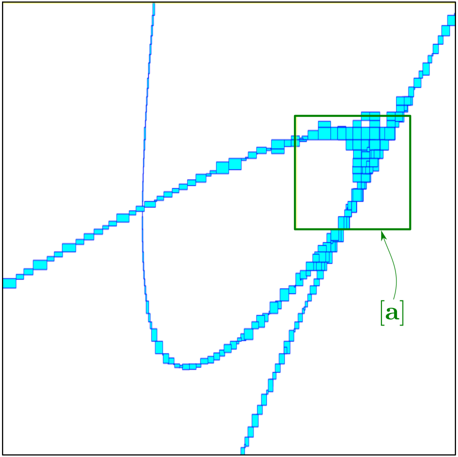
$\mathbf{m} = \text{center}([\mathbf{x}])$
Compute the Gauss-Jordan matrix $\mathbf{Q}$ for $\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$
Define $\mathbf{g}(\mathbf{x}) = \mathbf{Q} \cdot \mathbf{f}(\mathbf{x})$
For $i \in \{1, \ldots, p\}$
  For $j \in \{1, \ldots, n\}$
    $[\mathbf{a}] = [\frac{\partial g_i}{\partial \mathbf{x}}]([\mathbf{x}])$
    $[s] = \sum_{k \neq j} [a_k] \cdot ([x_k] - m_k)$
    $[x_j] = [x_j] \cap (-g_i(\mathbf{m}) - [s])$
Return $[\mathbf{x}]$

```
def GaussJordan(A):
    n=A.shape[0]
    m=A.shape[1]
    P,L,U = lu(A)
    Q=inv(P@L)
    for i in range(n-1, 0, -1):
        p=m-n
        K=U[i,i+p]*np.eye(n)
        K[0:i,i]=-U[0:i,i+p]
        Q=K@Q
        U=Q@A
    return Q
```

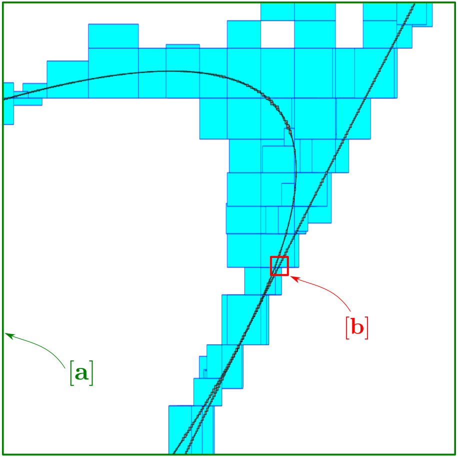
5. Results



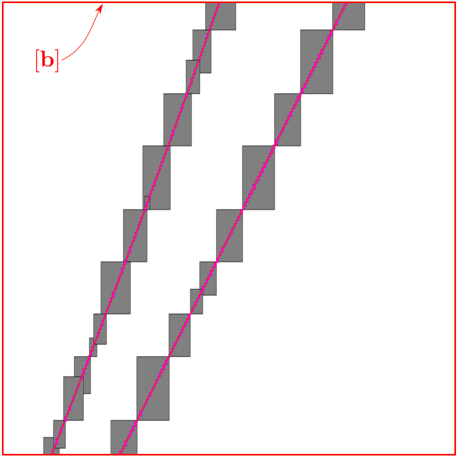
With a forward-backward contractor and $\varepsilon = 2^{-8}$



With the centered contractor $\varepsilon = 2^{-4}$



Blue: $\varepsilon = 2^{-4}$; Thin: $\varepsilon = 2^{-8}$



Gray: $\varepsilon = 2^{-8}$; Magenta: $\varepsilon = 2^{-12}$

Contributions

Notion of asymptotic minimal contractor
Link between the preconditioning and acyclic constraint networks
Better results than the basic affine arithmetic
No use of guaranteed linear programming

Perspectives

Compare with modern affine-arithmetic approaches
Improve the tree preconditioning
Use linear programming with an order 1 inflation
Implement in codac.io

References

- Thesis of Nico (2006) : [1]
- Counting the connected components of a subset of $\mathbb{R}^n$[2]
- Stability : [3]
- Palm system : [4]
- Propagation in a tree is perfect : [6]
- Turkulov system [10]
- Centered form [7]

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