A new type of intervals for solving problems involving partially defined functions

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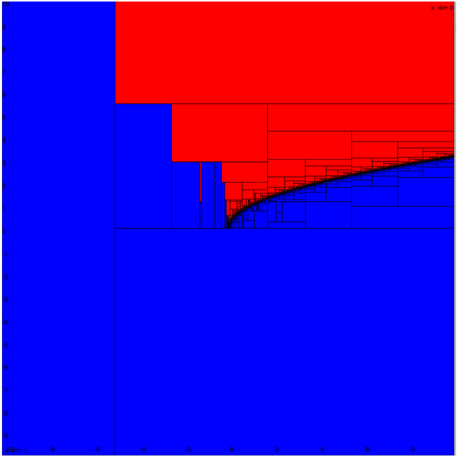
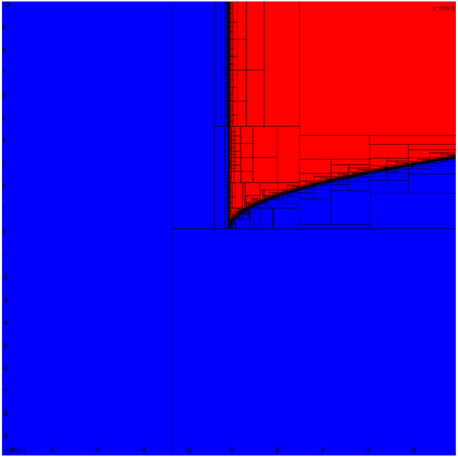
Motivation

Consider the set

$$\mathbb{S} = \{(x,y) \,|\, y - \sqrt{2x - x} \geq 0\}$$

```
from codac import *
f = Function("x","y","y-sqrt(2*x-x)")
S = SepFwdBwd(f,Interval(0,oo))
SIVIA(IntervalVector([[-10,10],[-10,10]]),S,0.01)
```


x: dim 0
dim 1


x: dim 0
y: dim 1
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-8
-6
-4
-2
0
2
4
6
8

Consider the set inversion problem [4]

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}).$$

A set inversion algorithm relies on the two tests:

$$\begin{array}{ccc} \mathbf{f}([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \mathbf{f}([\mathbf{x}]) \cap \mathbb{X} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset \end{array}$$

Consider the set inversion problem [4]

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}).$$

A set inversion algorithm should rely on the two tests:

$$\begin{array}{rcl} \mathbf{f}([\mathbf{x}]) \subset \mathbb{Y} \text{ and } [\mathbf{x}] \subset \text{dom}(\mathbf{f}) & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \mathbf{f}([\mathbf{x}]) \cap \mathbb{X} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset \end{array}$$

How to check that $[\mathbf{x}] \subset \mathrm{dom}(\mathbf{f})$ for functions ?

For

$$f(x) = \log(\frac{\sin x}{x})$$

We have

$$\mathrm{dom} f = \{x | x \cdot \sin x > 0\}$$

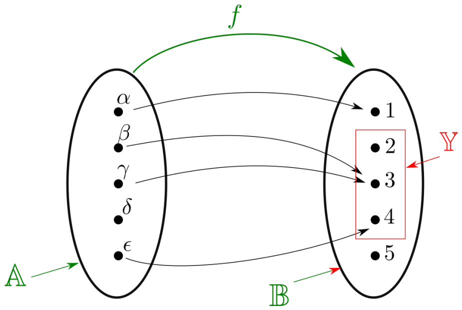
For

$$f(x_1, x_2) = \log\left(\frac{1 + \sin\sqrt{x_1^2 - \log(x_2)}}{1 - \sqrt{e^{x_1} - x_2}}\right)$$

$$\mathrm{dom} f : \left\{ \begin{array}{c} e^{x_1} - x_2 \geq 0 \\ x_1^2 - \log(x_2) \geq 0 \\ \left(1 - \sqrt{e^{x_1} - x_2}\right) \cdot \left(1 + \sin\sqrt{x_1^2 - \log(x_2)}\right) > 0 \end{array} \right.$$

Our goal is to show that the test $[\mathbf{x}] \subset \mathrm{dom}(\mathbf{f})$ can be done automatically using a new interval arithmetic.
We include the quantity NaN *(not a number : $\imath$)* [1], [2], [3].

Not a number



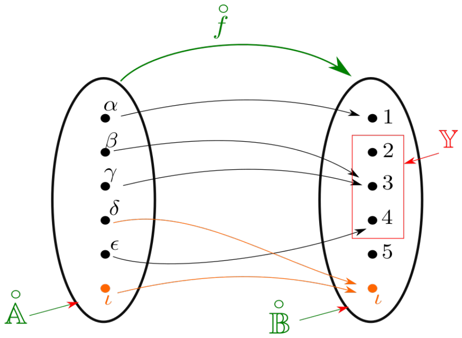
We have

$$\begin{aligned} f^{-1}(\mathbb{Y}) &= \{\beta,\gamma,\varepsilon\} \\ f^{-1}(\overline{\mathbb{Y}}) &= \{\alpha\} \\ \mathrm{dom} f &= \{\alpha,\beta,\gamma,\varepsilon\} \end{aligned}$$

Now, since $f(\{\gamma,\delta\}) \subset \mathbb{Y}$, the inner test will classify δ inside $f^{-1}(\mathbb{Y})$ which is wrong.

We define the extended function of f as:

$$\mathring{f}(x) = \begin{cases} f(x) & \text{if } x \subset \operatorname{dom} f \\ \imath & \text{otherwise} \end{cases}$$



We now have functions that are defined everywhere.

With real number

Consider the extended set of reals:

$$\mathring{\mathbb{R}} = \mathbb{R} \cup \{\imath\}$$

Operations on real number can be extended to $\mathring{\mathbb{R}}$ as follows:

$$\begin{aligned} f(x) &= \imath \qquad \text{if } x \notin \text{dom}(f) \\ f(\imath) &= \imath \\ \imath \diamond x &= \imath \end{aligned}$$

Intervals in $\mathbb{R} \cup \{\imath\}$

On a lattice $(\mathbb{A}, \leq_{\mathbb{A}})$, we can define intervals, interval hull, contractors.
Examples of lattices are real numbers, integers, trajectories, graphs, etc.
To be able to use interval methods, the lattice structure is required.
We show here that it is not strictly necessary by considering union of lattices.

$\mathbb{R}$ is a lattice:

$$1 \leq 2$$
$$2 \wedge 3 = 2$$
$$2 \vee 3 = 3$$

We can thus define the set of intervals of $\mathbb{R}$

$$\mathbb{IR} \supset \{[1,2],[1,\infty],[1,1],\emptyset\}$$

$\{\imath\}$ is a lattice:

$$\imath \leq \imath$$
$$\imath \wedge \imath = \imath$$
$$\imath \vee \imath = \imath$$

We can thus define intervals of $\{\imath\}$

$$\mathbb{I}\{\imath\} = \{[\imath,\imath],\emptyset\}$$

$\mathbb{R} \cup \{\iota\}$ is not a lattice :

$$1 \wedge \iota = ?$$

Intervals in union of lattices

Consider two lattices $(\mathbb{A}, \leq_{\mathbb{A}})$ and $(\mathbb{B}, \leq_{\mathbb{B}})$ that are disjoint.
We define intervals of $\mathbb{C} = \mathbb{A} \cup \mathbb{B}$ as subsets $\mathbb{C}$ which have the form

$$[c] = [a] \cup [b]$$

where $[a] \in \mathbb{IA}$ and $[b] \in \mathbb{IB}$.

Example. Take $\mathbb{A} = \mathbb{R}$ and $\mathbb{B} = \{a, b, c, \ldots, z\}$. Both are lattices. Examples of intervals for $\mathbb{C} = \mathbb{A} \cup \mathbb{B}$ are

$$
\begin{aligned}
[c_1] &= [2,5] \\
[c_2] &= [e,h] \\
[c_3] &= [2,5] \cup [e,h] \\
[c_4] &= [4,9] \cup [g,i] \\
[c_5] &= \emptyset \\
[c_6] &= \mathbb{A} \cup \mathbb{B}
\end{aligned}
$$

The set of intervals of $\mathbb{C}$ is closed under intersection.

As a consequence, contractor methods can be used.

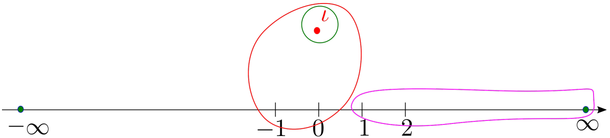
Extended intervals

The set $\mathring{\mathbb{R}}$ can be equipped with a partial order $\leq_{\mathring{\mathbb{R}}}$ relation derived from $\mathbb{R}$:

$$\iota \leq_{\mathring{\mathbb{R}}} \iota$$
$$a \in \mathbb{R}, b \in \mathbb{R} \quad \text{then} \quad a \leq_{\mathring{\mathbb{R}}} b \text{ iff } a \leq_{\mathbb{R}} b$$

Examples of intervals of $\mathring{\mathbb{R}}$ are

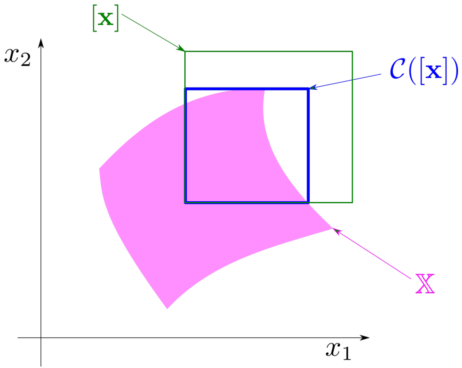
$$[1, \infty]$$
$$[-1, 0] \cup \{\imath\}$$
$$\{\imath\}$$
$$\emptyset$$



Interval arithmetic

$$
\begin{array}{rcl}
[1,2]+[2,3] & = & [3,5] \\
[1,2]+[\imath] & = & [\imath] \\
[1,2]+([2,3]\cup[\imath]) & = & [3,5]\cup[\imath] \\
\log([-1,1]) & = & [-\infty,0]\cup[\imath] \\
\frac{1}{[0,0]} & = & [\imath]
\end{array}
$$

Extended contractors



Classical contractor $\mathscr{C}$ for the set $\mathbb{X}$

Extended Contractor for the square root

Consider the constraint

$$y = \sqrt{x}$$

where all variables belong to $\mathring{\mathbb{R}}$.

We have

$$\left\{ \begin{array}{c} y = \sqrt{x} \\ x \in \mathring{\mathbb{R}} \\ y \in \mathring{\mathbb{R}} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{lc} & y = \sqrt{x}, x \in \mathbb{R}^{+}, y \in \mathbb{R} \\ \text{or} & x < 0, y = \imath \\ \text{or} & x = \imath, y = \imath \end{array} \right.$$

$C_{\sqrt{\cdot}}([x],[y])$	$y \in \mathbb{R}$	$y = \imath$
$x \in \mathbb{R}^+$	$\begin{pmatrix} [x] \cap [y]^2 \\ [y] \cap \sqrt{[x]}) \end{pmatrix}$	$\begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}$
$x \in \mathbb{R}^-$	$\begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}$	$\begin{pmatrix} [x] \\ \{\imath\} \end{pmatrix}$
$x = \imath$	$\begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}$	$\begin{pmatrix} \{\imath\} \\ \{\imath\} \end{pmatrix}$

For instance

$$\begin{aligned} & C_{\sqrt{\cdot}}([-4,4] \cup [\imath],[1,5]) \\ = \quad & \underbrace{C_{\sqrt{\cdot}}([0,4],[1,5])}_{[1,4]\times[1,2]} \cup \underbrace{C_{\sqrt{\cdot}}([-4,0],[1,5])}_{\emptyset} \cup \underbrace{C_{\sqrt{\cdot}}([\imath],[1,5])}_{\emptyset} \\ = \quad & [1,4] \times [1,2] \end{aligned}$$

Contractor for the addition

Consider the constraint

$$z = x + y$$

where all variables belong to $\mathring{\mathbb{R}}$. Note that in $\mathring{\mathbb{R}}$, we do not have

$$z = x + y \Leftrightarrow x = z - y$$

Indeed, take $x = 1, y = \imath, z = \imath$.

Our constraint can be decomposed as

$$\left\{\begin{array}{c} z = x + y \\ x \in \mathring{\mathbb{R}} \\ y \in \mathring{\mathbb{R}} \\ y \in \mathring{\mathbb{R}} \end{array}\right. \Leftrightarrow \left\{\begin{array}{lc} & z = x + y,\ x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R} \\ \text{or} & x = \imath, y \in \mathbb{R}, z = \imath \\ \text{or} & x \in \mathbb{R}, y = \imath, z = \imath \\ \text{or} & x = \imath, y = \imath, z = \imath \end{array}\right.$$

		$z \in \mathbb{R}$	$z = \imath$
$x \in \mathbb{R}$	$y \in \mathbb{R}$	$\begin{pmatrix} [x] \cap ([z]-[y]) \\ [y] \cap ([z]-[x]) \\ [z] \cap ([x]+[y]) \end{pmatrix}$	$\begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \end{pmatrix}$
$x \in \mathbb{R}$	$y = \imath$	$\begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \end{pmatrix}$	$\begin{pmatrix} [x] \\ \{\imath\} \\ \{\imath\} \end{pmatrix}$
$x = \imath$	$y \in \mathbb{R}$	$\begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \end{pmatrix}$	$\begin{pmatrix} \{\imath\} \\ [y] \\ \{\imath\} \end{pmatrix}$
$x = \imath$	$y = \imath$	$\begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \end{pmatrix}$	$\begin{pmatrix} \{\imath\} \\ \{\imath\} \\ \{\imath\} \end{pmatrix}$

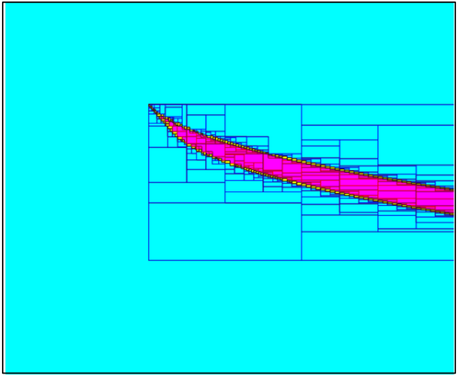
Contractor table for $z = x + y$

Illustration

Consider the set

$$\{(x,y) \,|\, y + \sqrt{x+y} \in [1,2]\}$$

We get



An implementation is given here:
https://replit.com/@aulin/iota

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