Symmetries for Interval Analysis

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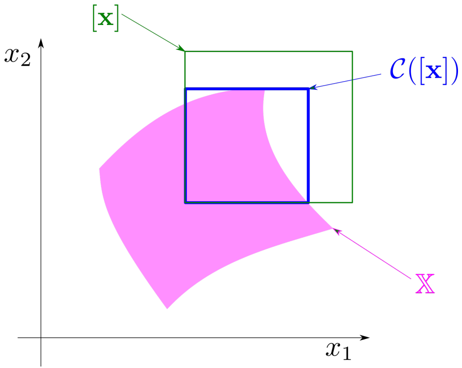
Symmetries for contractors

Building contractors relies on three concepts:

- Monotonicity
- Symmetries
- Compositions
- ~~Interval computation~~


$[\mathbf{x}]$
x_2
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$
x_1

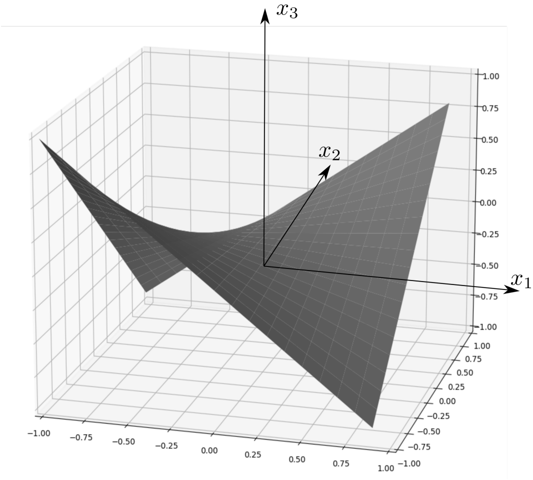
Multiplication

We consider the product

$$x_3 = x_1 \cdot x_2$$

Equivalently

$$\mathbb{X} = \{(x_1, x_2, x_3) \mid x_1 \cdot x_2 = x_3\}$$


x_3
x_2
x_1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

We have

$$x_1 \cdot x_2 = x_3 \Leftrightarrow (-x_1) \cdot x_2 = -x_3$$

We say that $x_1 \cdot x_2 = x_3$ is invariant by the symmetry

$$\sigma_1 : \begin{cases} x_1 & \mapsto & -x_1 \\ x_2 & \mapsto & x_2 \\ x_3 & \mapsto & -x_3 \end{cases}$$

Equivalently, $x_1 \cdot x_2 = x_3$ is said to be invariant by

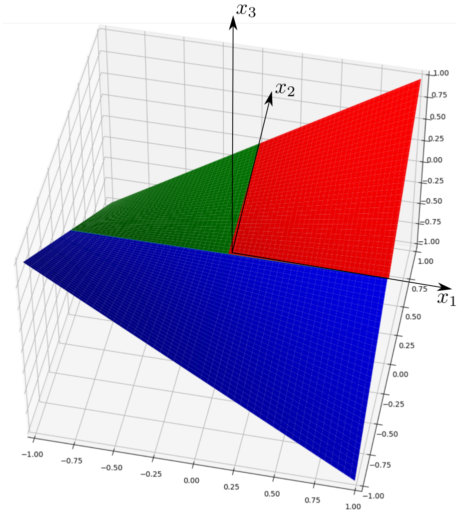
$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We also have

$$x_1 \cdot x_2 = x_3 \Leftrightarrow x_1 \cdot (-x_2) = -x_3$$

invariant with respect to

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$


x_3
x_2
x_1
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

Due to the monotonicity, the minimal contractor for the box

$$[\mathbf{x}] \subset [\mathbf{a}] = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ .$$

associated to $x_1 \cdot x_2 = x_3$ is

$$\mathscr{C}_0 \begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap [\frac{x_3^-}{x_2^+}, \frac{x_3^+}{x_2^-}] \\ [x_2] \cap [\frac{x_3^-}{x_1^+}, \frac{x_3^+}{x_1^-}] \\ [x_3] \cap [x_1^- \cdot x_2^-, x_1^+ \cdot x_2^+] \end{pmatrix}.$$

We will see later that

$$\mathbb{X} = \sigma_2 \bullet \sigma_1 \bullet \mathbb{X}_0$$

where

$$\mathbb{X}_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \,|\, x_1 \cdot x_2 = x_3\}$$

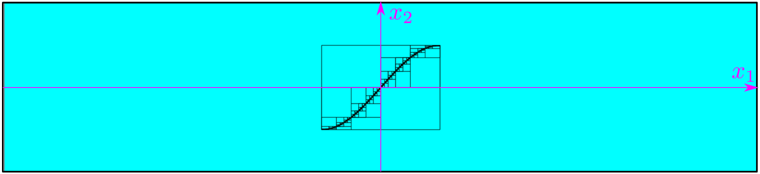
We will see later that

$$\mathscr{C} = \sigma_2 \bullet \sigma_1 \bullet \mathscr{C}_0$$

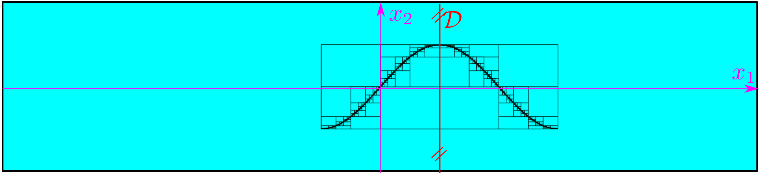
is an optimal contractor for $\mathbb{X}$ as soon as $\mathscr{C}_0$ is a minimal contractor for $\mathbb{X}_0$.

Sine

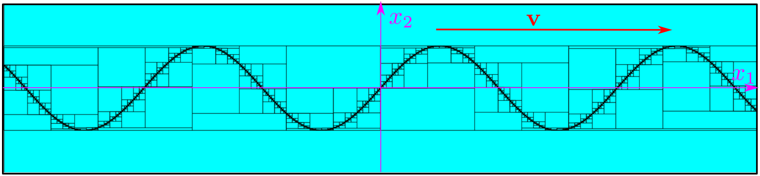
$$x_2 = \sin x_1$$



$\mathscr{C}_0$



$$\sigma_D \bullet \mathscr{C}_0$$



$$\mathscr{C}_{\text{sin}} = \sigma_{\mathbf{v}} \bullet \sigma_D \bullet \mathscr{C}_0$$

Sinc

$$x_2 = \frac{\sin x_1}{x_1}$$

We have no symmetries available

$$x_2 = \frac{\sin x_1}{x_1} \Leftrightarrow \left\{ \begin{array}{rcl} x_1 x_2 & = & a \\ \sin x_1 & = & a \end{array} \right.$$

We use the contractor composition.
We loose the minimality

Rotate constraint

We want a procedure to generate automatically a minimal contractor for the *rotate* constraint

$$\mathbb{X}:\left\{\begin{array}{lcl} x_3x_1 - x_4x_2 - x_5 & = & 0 \\ x_4x_1 + x_3x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array}\right.$$

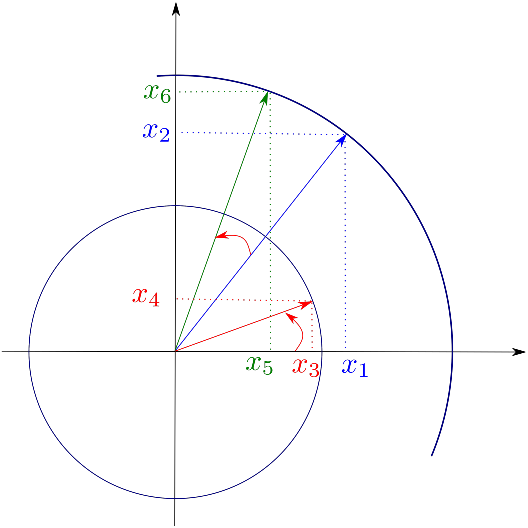
The automatic generation allows us to avoid coding errors
The code is shorter and more clear
The code is more generic

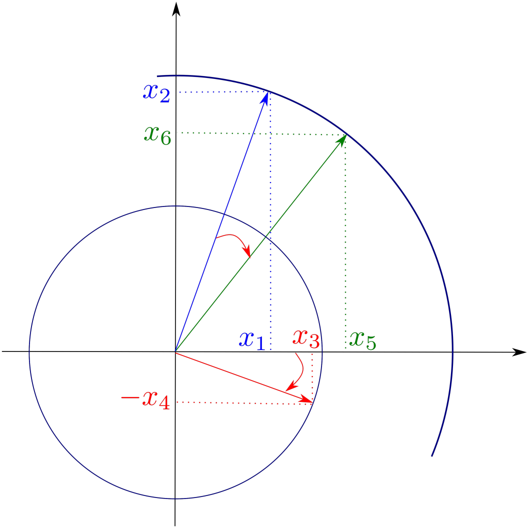
A symmetry for *rotate* is

$$\sigma : \begin{cases} x_1 & \mapsto & x_5 \\ x_2 & \mapsto & x_6 \\ x_3 & \mapsto & x_3 \\ x_4 & \mapsto & -x_4 \\ x_5 & \mapsto & x_1 \\ x_6 & \mapsto & x_2 \end{cases}$$

Indeed

$$\left\{\begin{array}{ccc} x_3x_1 - x_4x_2 - x_5 & = & 0 \\ x_4x_1 + x_3x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{ccc} x_3x_5 + x_4x_6 - x_1 & = & 0 \\ -x_4x_5 + x_3x_6 - x_2 & = & 0 \\ x_3^2 + (-x_4)^2 - 1 & = & 0 \end{array}\right.$$


x_6
x_2
x_4
x_5
x_3
x_1


x_2
x_6
x_1
x_3
x_5
$-x_4$

Hyperoctohedral symmetries

The hyperoctahedral group B_n is the group of symmetries of the unit hypercube [4] of $\mathbb{R}^n$.
It contains $2^n \cdot n!$ elements.

For $n = 2$, we have $2^2 \cdot 2! = 8$ elements:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\sigma_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_7 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We write equivalently

$$\sigma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } \sigma : \begin{cases} \mathbb{R}^2 & \mapsto & \mathbb{R}^2 \\ (x_1, x_2) & \mapsto & (x_2, -x_1) \end{cases}$$

	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_1	σ_1	σ_0	σ_6	σ_4	σ_3	σ_7	σ_2	σ_5
σ_2	σ_2	σ_5	σ_0	σ_7	σ_6	σ_1	σ_4	σ_3
σ_3	σ_3	σ_4	σ_7	σ_0	σ_1	σ_6	σ_5	σ_2
σ_4	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2	σ_7	σ_6
σ_5	σ_5	σ_2	σ_4	σ_6	σ_7	σ_3	σ_0	σ_1
σ_6	σ_6	σ_7	σ_1	σ_5	σ_2	σ_0	σ_3	σ_4
σ_7	σ_7	σ_6	σ_3	σ_2	σ_5	σ_4	σ_1	σ_0

Multiplication table

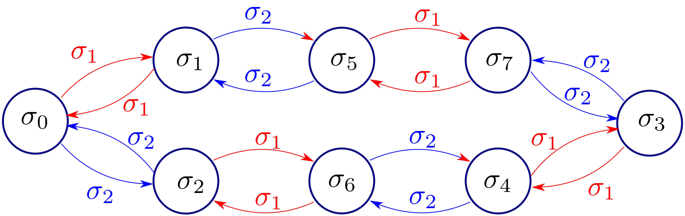
The two elements σ_1, σ_2 are generators of the group B_2 or equivalently, we write $B_2 = \langle \sigma_1, \sigma_2 \rangle$.

Compute for instance $\sigma_6 \circ \sigma_7$, we get

$$\sigma_6 \circ \sigma_7 = \underbrace{\sigma_1 \circ \sigma_2}_{\sigma_6} \circ \underbrace{\sigma_1 \circ \underbrace{\sigma_2 \circ \sigma_1}_{\sigma_5}}_{\sigma_7} = \sigma_4$$

	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_1	σ_1	σ_0	σ_6	σ_4	σ_3	σ_7	σ_2	σ_5
σ_2	σ_2	σ_5	σ_0	σ_7	σ_6	σ_1	σ_4	σ_3
σ_3	σ_3	σ_4	σ_7	σ_0	σ_1	σ_6	σ_5	σ_2
σ_4	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2	σ_7	σ_6
σ_5	σ_5	σ_2	σ_4	σ_6	σ_7	σ_3	σ_0	σ_1
σ_6	σ_6	σ_7	σ_1	σ_5	σ_2	σ_0	σ_3	σ_4
σ_7	σ_7	σ_6	σ_3	σ_2	σ_5	σ_4	σ_1	σ_0

To prove that $\{\sigma_1, \sigma_2\}$ is a generator pair of B_2, we build the Cayley graph.



Cayley graph associated to B_2

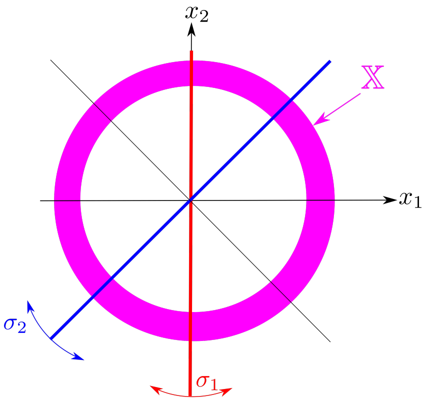
We see that $\sigma_6 \circ \sigma_7 = \underbrace{\sigma_1 \circ \sigma_2}_{\sigma_6} \circ \underbrace{\sigma_1 \circ \sigma_2 \circ \sigma_1}_{\sigma_7} = \sigma_4$

Hyperoctohedral symmetries of a set

We define the *stabiliser* of B_n with respect to $\mathbb{X} \subset \mathbb{R}^n$ as

$$B_n(\mathbb{X}) = \{\sigma \in B_n \,|\, \sigma(\mathbb{X}) = \mathbb{X}\}.$$

$B_n(\mathbb{X})$ is a subgroup of B_n.


x_2
$\mathbb{X}$
x_1
σ_2
σ_1

Checking that a symmetry is a stabilizer

Checking that $\sigma \in B_n(\mathbb{X})$ *i.e.,* $\sigma(\mathbb{X}) = \mathbb{X}$ amounts to check that two polynomial equalities are equivalent.

For

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

the equality $\sigma(\mathbb{X}) = \mathbb{X}$ rewrites into

$$\left\{ \begin{array}{ccc} x_3 x_1 - x_4 x_2 - x_5 & = & 0 \\ x_4 x_1 + x_3 x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ccc} x_3 x_5 + x_4 x_6 - x_1 & = & 0 \\ -x_4 x_5 + x_3 x_6 - x_2 & = & 0 \\ x_3^2 + (-x_4)^2 - 1 & = & 0 \end{array} \right.$$

```
from sympy import *
x1,x2,x3,x4,x5,x6=symbols('x1 x2 x3 x4 x5 x6')
S1=[x3*x1-x4*x2-x5,x4*x1+x3*x2-x6,x3**2+x4**2-1]
S2=[x3*x5+x4*x6-x1,-x4*x5+x3*x6-x2,x3**2+(-x4)**2-1]
G1=groebner(S1,x1,x2,x3,x4,x5,x6,order='lex')
G2=groebner(S2,x1,x2,x3,x4,x5,x6,order='lex')
print(G1==G2)
```

We have indeed the form

$$\left\{\begin{array}{lcl} P_1(\mathbf{x}) & = & 0 \\ P_2(\mathbf{x}) & = & 0 \\ P_3(\mathbf{x}) & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} Q_1(\mathbf{x}) & = & 0 \\ Q_2(\mathbf{x}) & = & 0 \\ Q_3(\mathbf{x}) & = & 0 \end{array}\right.$$

where $\mathbf{x} = (x_1, \ldots, x_6)$.

To prove the equivalence, we check that for all j,

$$Q_j(\mathsf{x}) = \sum_i A_{ij}(\mathsf{x}) \cdot P_i(\mathsf{x})$$

and that for all i,

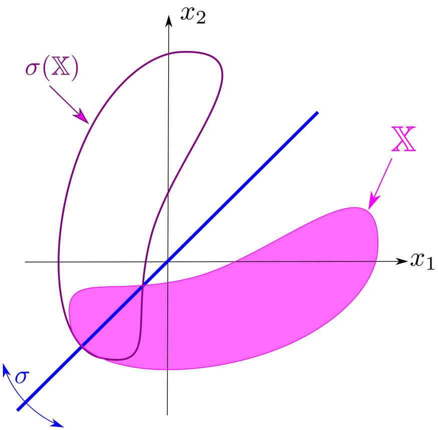
$$P_i(\mathsf{x}) = \sum_j B_{ij}(\mathsf{x}) \cdot Q_j(\mathsf{x}).$$

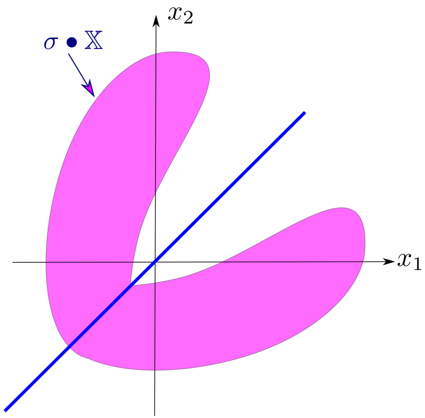
Equivalently, we check that the Grobner basis, computed by the Buchberger's algorithm, for $\{P_1, P_2, P_3\}$ and for $\{Q_1, Q_2, Q_3\}$ are the same.

Acts

For $\sigma \in B_n$, we define the *act* operator:

$$\sigma \bullet \mathbb{X} = \mathbb{X} \cup \sigma(\mathbb{X}).$$


x_2
$\sigma(\mathbb{X})$
$\mathbb{X}$
x_1
σ

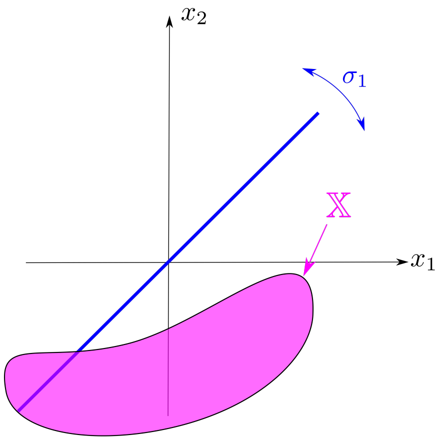

x_2
$\sigma \bullet \mathbb{X}$
x_1

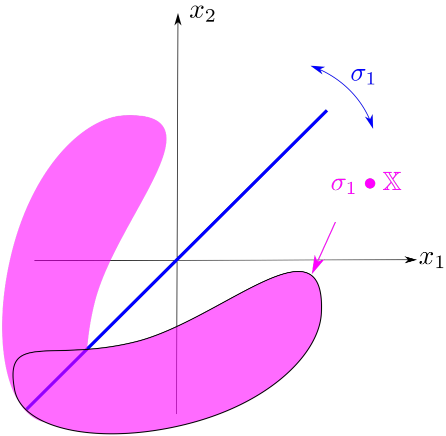
Semi-group action

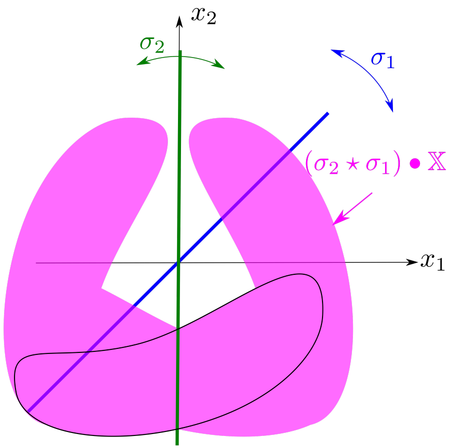
Given $\{\sigma_1, \sigma_2, \cdots\}$ in B_n, we define the operator $\star$ as follows

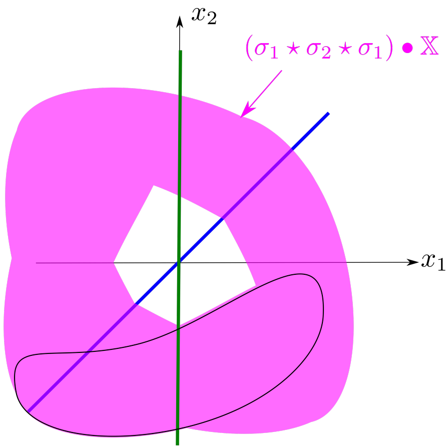
$$\begin{array}{rcl} (\sigma_2 \star \sigma_1) \bullet \mathbb{X} & = & \sigma_2 \bullet (\sigma_1 \bullet \mathbb{X}) \\ (\sigma_3 \star \sigma_2 \star \sigma_1) \bullet \mathbb{X} & = & \sigma_3 \bullet (\sigma_2 \bullet (\sigma_1 \bullet \mathbb{X})) \\ & \vdots & \end{array}$$

The algebraic structure $(\Sigma, \star)$ corresponds to a semi-group action [3].


x_2
σ_1
$\mathbb{X}$
x_1


x_2
σ_1
$\sigma_1 \bullet \mathbb{X}$
x_1


x_2
σ_2
σ_1
$(\sigma_2 \star \sigma_1) \bullet \mathbb{X}$
x_1


x_2
$(\sigma_1 \star \sigma_2 \star \sigma_1) \bullet \mathbb{X}$
x_1

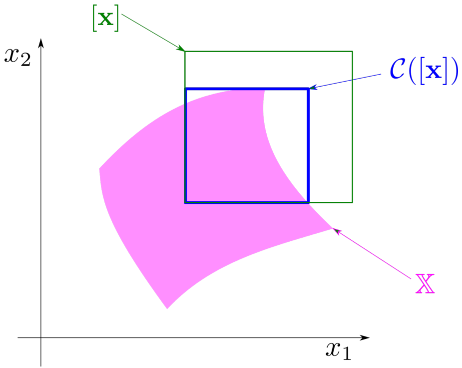
Contractor Acts

A *contractor* $\mathscr{C}$ for a set $\mathbb{X} \subset \mathbb{R}^n$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$\begin{array}{ll} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ [\mathbf{x}] \subset [\mathbf{y}] \ \Rightarrow \ \mathscr{C}([\mathbf{x}]) \subset \mathscr{C}([\mathbf{y}]). & \text{(monotonicity)} \\ \mathscr{C}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(consistency)} \end{array}$$

There exists a is a minimal contractor for $\mathbb{X}$, given by

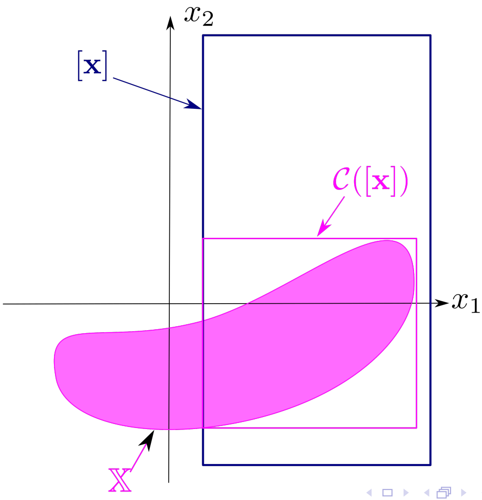
$$\mathscr{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \mathbb{X}]\!]$$

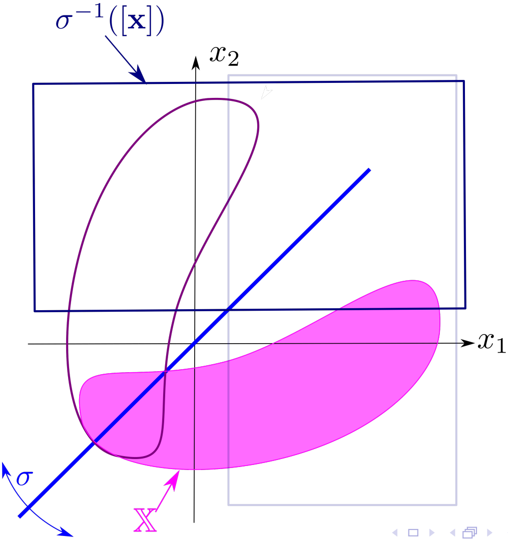


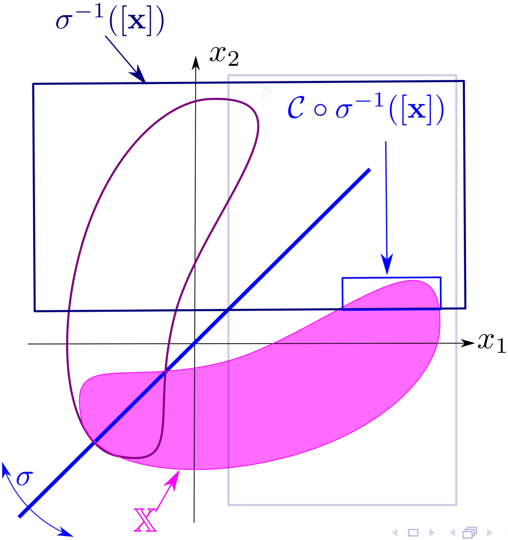
Minimal contractor $\mathscr{C}$ for the set $\mathbb{X}$

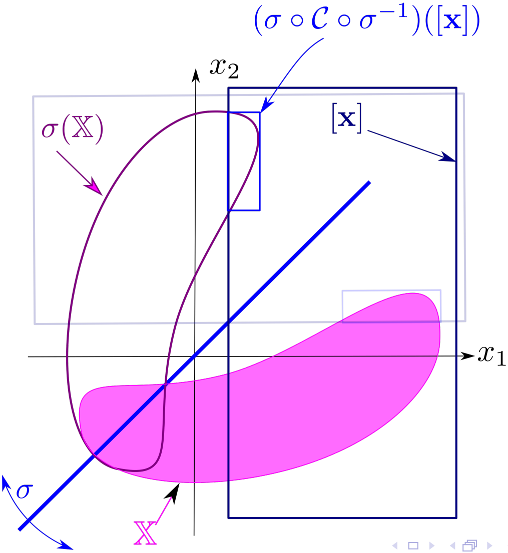
If $\mathscr{C}$ is a contractor in $\mathbb{R}^n$, and $\sigma \in B_n$, we define the *contractor act* of σ on $\mathscr{C}$ as

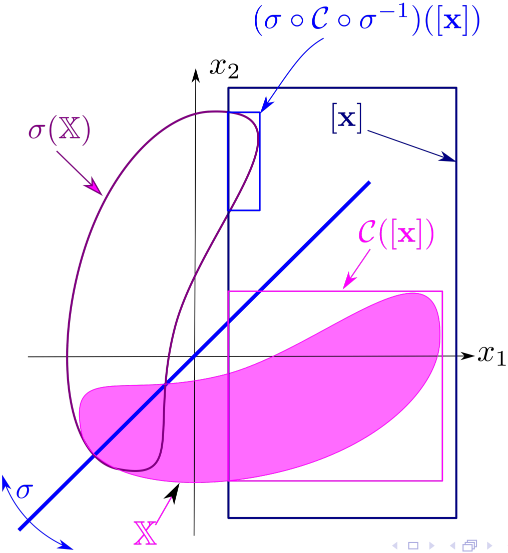
$$\sigma \bullet \mathscr{C} = \mathscr{C} \sqcup \sigma \circ \mathscr{C} \circ \sigma^{-1}.$$

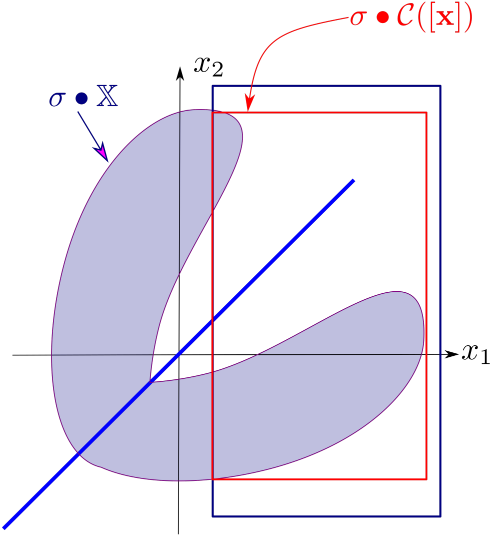

x_2
$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
x_1
$\mathbb{X}$


$\sigma^{-1}([\mathbf{x}])$
x_2
x_1
σ
$\mathbb{X}$


$\sigma^{-1}([\mathbf{x}])$
x_2
$\mathcal{C} \circ \sigma^{-1}([\mathbf{x}])$
x_1
σ
$\mathbb{X}$


$(\sigma \circ \mathcal{C} \circ \sigma^{-1})([\mathbf{x}])$
x_2
$\sigma(\mathbb{X})$
$[\mathbf{x}]$
x_1
σ
$\mathbb{X}$


$(\sigma \circ \mathcal{C} \circ \sigma^{-1})([\mathbf{x}])$
x_2
$\sigma(\mathbb{X})$
$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
x_1
σ
$\mathbb{X}$


$\sigma \bullet \mathcal{C}([\mathbf{x}])$
x_2
$\sigma \bullet \mathbb{X}$
x_1

Proposition. If $\mathscr{C}$ is a minimal contractor for $\mathbb{X}$ and if $\sigma \in B_n$ then $\sigma \bullet \mathscr{C}$ is a minimal contractor for $\sigma \bullet \mathbb{X}$, *i.e.*,

$$\sigma \bullet \mathscr{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \sigma \bullet \mathbb{X}]\!].$$

Corollary. If $\mathscr{C}$ is a minimal contractor for $\mathbb{X}$ and if $\sigma_1, \dots, \sigma_k$ are in B_n then $(\sigma_k \star \cdots \star \sigma_1) \bullet \mathscr{C}$ is a minimal contractor for $(\sigma_k \star \cdots \star \sigma_1) \bullet \mathbb{X}$, i.e.,

Square constraint

We consider the set

$$\mathbb{X}: x_1^2 - x_2 = 0.$$

Over $[a] = \mathbb{R}^+ \times \mathbb{R}^+$, the minimal contractor is

$$\mathscr{C}_0 \left(\begin{array}{c} [x_1] \\ [x_2] \end{array} \right) = \left(\begin{array}{c} [x_1] \cap [\sqrt{x_2^-}, \sqrt{x_2^+}] \\ [x_2] \cap [x_1^{-2}, x_1^{+2}] \end{array} \right).$$

The stabilizers for $\mathbb{X}$ are

$$B_2(\mathbb{X}) = \left\{ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

.
Indeed

$$x_1^2 - x_2 = 0 \Leftrightarrow (-x_1)^2 - x_2 = 0$$

ValidSequence($\{\sigma_1\}, [\mathbf{a}]$) returns True. Thus $\sigma_1 \bullet \mathscr{C}_0$ is the minimal contractor for $\mathbb{X}$.

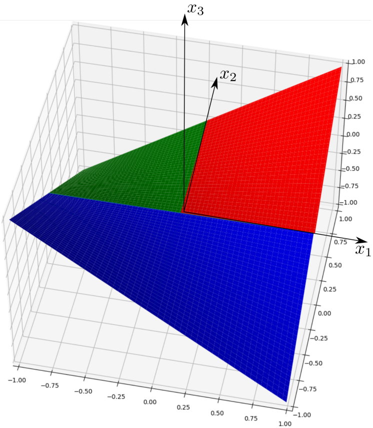
Product constraint

We consider the constraint

$$\mathbb{X}: \ x_1 x_2 = x_3.$$

i.e.

$$\mathbb{X} = \{\mathbf{x} = (x_1, x_2, x_3) | x_1 x_2 = x_3\}$$



Product constraint: $x_1 x_2 = x_3$

A minimal contractor over

$$[\mathbf{a}] \quad = \quad \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.$$

for the constraint $x_1 x_2 = x_3$ is:

$$\mathscr{C}_0 \begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap [\frac{x_3^-}{x_2^+}, \frac{x_3^+}{x_2^-}] \\ [x_2] \cap [\frac{x_3^-}{x_1^+}, \frac{x_3^+}{x_1^-}] \\ [x_3] \cap [x_1^- \cdot x_2^-, x_1^+ \cdot x_2^+] \end{pmatrix}.$$

B_3 has $2^3 * 3! = 48$ elements. One of them is

$$\sigma_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since

$$x_1 x_2 - x_3 = 0 \Leftrightarrow (-x_1) \cdot (-x_2) - x_3 = 0$$

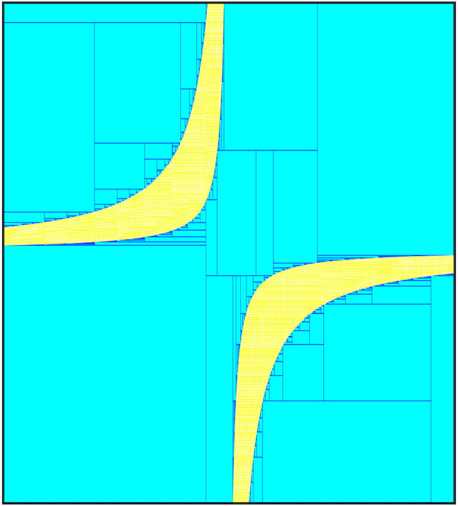
σ_0 is a stabilizer.

Other stabilizers are

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The algorithm FindMinimalSequence($\{\sigma_0, \sigma_1, \sigma_2\}, [\mathbf{a}]$) finds two expressions of minimal length for the minimal contractor for $\mathbb{X}$: $(\sigma_2 \star \sigma_0) \bullet \mathscr{C}_0$ and $(\sigma_0 \star \sigma_2) \bullet \mathscr{C}_0$.

Consider the set of all $\mathbf{x} \in \mathbb{R}^2$, such that $x_1 \cdot x_2 \in [-9, -2]$. We take the contractor $(\sigma_2 \star \sigma_0) \bullet \mathscr{C}_0$ inside a paver.



Rotate constraint

We consider the constraint

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\mathbf{R}_\theta} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}}$$

If $x_2 = 0$ then, we get the classical Polar constraint [5] :

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta \\ x_1 \sin\theta \end{pmatrix}.$$

We rewrite the constraint as

$$\mathbb{X}:\begin{cases} x_3x_1 - x_4x_2 - x_5 & = & 0 \\ x_4x_1 + x_3x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{cases}$$

Generator

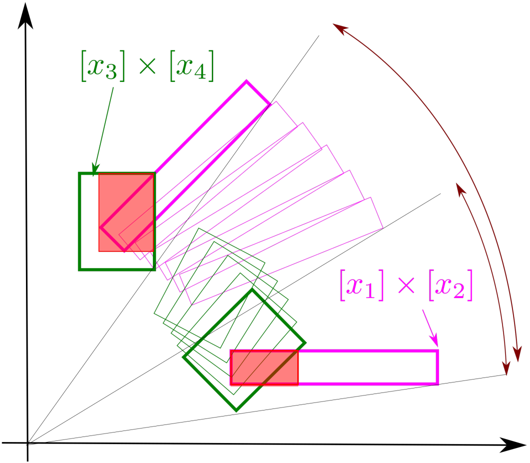
We take

$$[\mathbf{a}] \quad = \quad \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \quad .$$

To get a minimal contractor on $[\mathbf{a}]$, we need to build

$$\mathscr{C}_0 : [\mathbf{x}] \to [\![[\mathbf{x}] \cap [\mathbf{a}] \cap \mathbb{X}]\!].$$

A possibility is to use the contractor based on the monotonicity given in [2] [1].


$[x_3] \times [x_4]$
$[x_1] \times [x_2]$

Computing the stabilizers of *Rotate*

We have

$$\mathrm{card}\,(B_6) = 2^6 \cdot 6! = 46080$$

Take one of them, say

$$\sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We check that σ_0 is a stabilizer i.e.,

$$\left\{\begin{array}{ccc} x_3 x_1 - x_4 x_2 - x_5 & = & 0 \\ x_4 x_1 + x_3 x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{ccc} x_3 x_5 + x_4 x_6 - x_1 & = & 0 \\ -x_4 x_5 + x_3 x_6 - x_2 & = & 0 \\ x_3^2 + (-x_4)^2 - 1 & = & 0 \end{array}\right.$$

Other elements of $B_6(\mathbb{X})$ could be found, at least four of them should be added to be able to generate B_6. For instance

$$\begin{array}{rlcc}
\sigma_1 : & (x_1,x_2,x_3,x_4,x_5,x_6) & \mapsto & (x_5,x_6,x_3,-x_4,x_1,x_2) \\
\sigma_2 : & (x_1,x_2,x_3,x_4,x_5,x_6) & \mapsto & (x_2,-x_1,-x_4,x_3,x_5,x_6) \\
\sigma_3 : & (x_1,x_2,x_3,x_4,x_5,x_6) & \mapsto & (x_1,-x_2,x_3,-x_4,x_5,-x_6) \\
\sigma_4 : & (x_1,x_2,x_3,x_4,x_5,x_6) & \mapsto & (-x_1,-x_2,-x_3,-x_4,x_5,x_6) \\
\sigma_5 : & (x_1,x_2,x_3,x_4,x_5,x_6) & \mapsto & (-x_1,x_2,x_3,-x_4,-x_5,x_6)
\end{array}$$

There exists 0 sequence of length 4 among the 4!=144 which generates $B_6(\mathbb{X})$.
There exist 54 sequences of length 5 among the 5!=720 existing ones which generates $B_6(\mathbb{X})$.
One of them is $\sigma_5 \star \sigma_4 \star \sigma_3 \star \sigma_2 \star \sigma_1$.
A minimal contractor is given by $(\sigma_5 \star \sigma_4 \star \sigma_3 \star \sigma_2 \star \sigma_1) \bullet \mathscr{C}_0$.

```
def s1(X1,X2,X3,X4,X5,X6):return X5,X6,X3,-X4,X1,X2
def s2(X1,X2,X3,X4,X5,X6):return X2,-X1,-X4,X3,X5,X6
def _s2(X1,X2,X3,X4,X5,X6):return -X2,X1,X4,-X3,X5,X6
def s3(X1,X2,X3,X4,X5,X6):return X1,-X2,X3,-X4,X5,-X6
def s4(X1,X2,X3,X4,X5,X6):return -X1,-X2,-X3,-X4,X5,X6
def s5(X1,X2,X3,X4,X5,X6):return -X1,X2,X3,-X4,-X5,X6
```

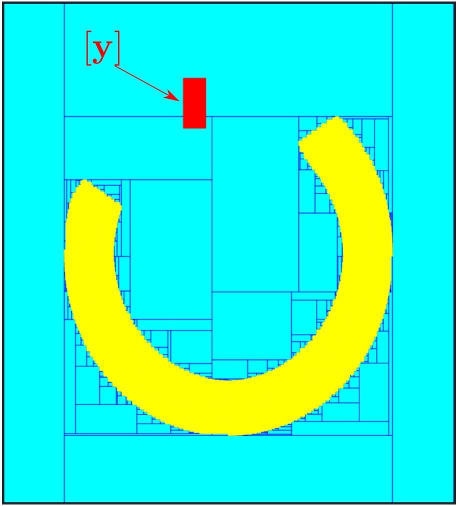
```
def Crot(X1,X2,X3,X4,X5,X6):
   def C0(X1,X2,X3,X4,X5,X6): ...
   def A(C,s,_s):
      return lambda X1,X2,X3,X4,X5,X6 :
         union_tuple(C(X1,X2,X3,X4,X5,X6),
             s(*C(*_s(X1,X2,X3,X4,X5,X6))))
   return A(A(A(A(A(C0,s1,s1),s2,_s2),s3,s3),
            s4,s4),s5,s5)(X1,X2,X3,X4,X5,X6)
```

Illustration

Consider the set of all $\mathbf{x} \in \mathbb{R}^2$, such that

$$\mathbf{y} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \mathbf{x}$$

with $\theta \in [1,5]$, $\mathbf{y} \in [\mathbf{y}] = [-4,-2] \times [10,14]$. We use the contractor $(\sigma_4 \star \cdots \star \sigma_0) \bullet \mathscr{C}_0$ inside a paver.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