Interval Integration of Triangular Systems

Luc Jaulin

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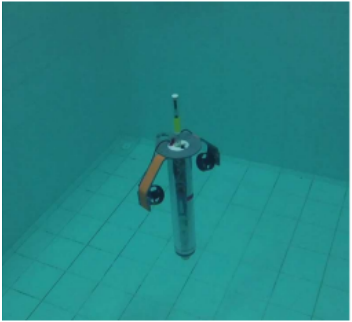
1. Dead reckoning



Dead reckoning (no exteroceptive measurements)

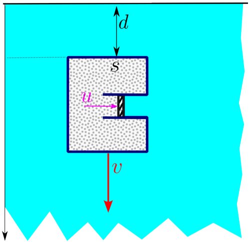
We have

- a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- an uncertain input $\mathbf{u}(t) \in [\mathbf{u}]$
- an initial state vector $\mathbf{x}(0) \in \mathbb{X}_0$

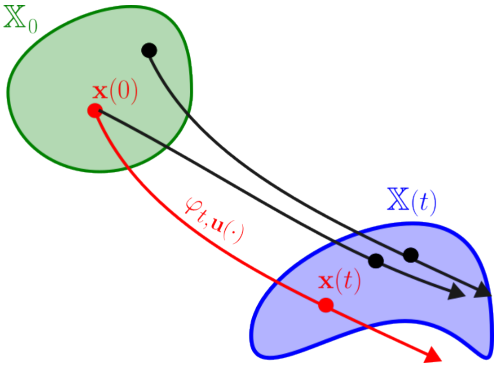
Compute in real time the set

$$\mathbb{X}(t) = \left\{ \mathbf{a} | \, \exists \mathbf{x}(0) \in \mathbb{X}_0, \exists \mathbf{u}(\cdot) \in [\mathbf{u}], \mathbf{a} = \varphi_{t,\mathbf{u}(\cdot)}(\mathbf{x}(0)) \right\}$$

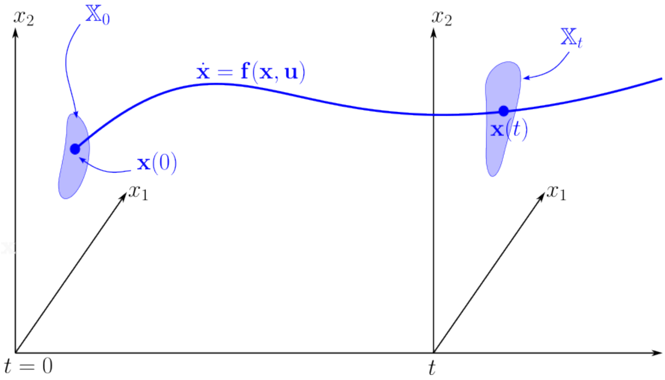
where $\varphi_{t,\mathbf{u}(\cdot)}$ is the flow.

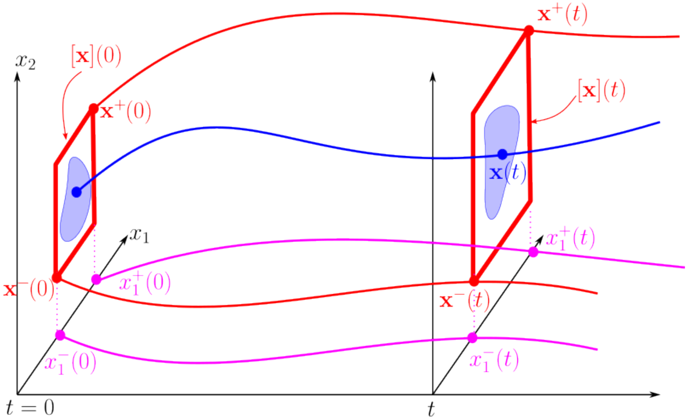


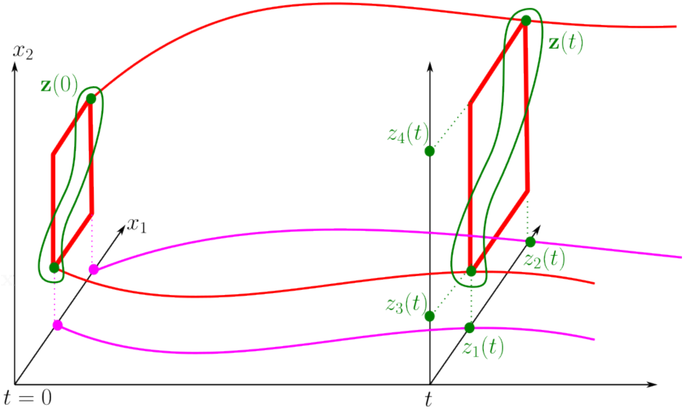
$$\begin{cases} \dot{s} & = & u \\ \dot{v} & = & \frac{s}{1+s} - \frac{1}{1+s} v \cdot \mid v \mid \\ \dot{d} & = & v \end{cases}$$


$\mathbb{X}_0$
$\mathbf{x}(0)$
$\varphi_{t,\mathbf{u}(\cdot)}$
$\mathbb{X}(t)$
$\mathbf{x}(t)$

2. Interval differential equation


x_2
$\mathbb{X}_0$
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
x_2
$\mathbb{X}_t$
$\mathbf{x}(t)$
$\mathbf{x}(0)$
x_1
x_1
$t = 0$
t


$\mathbf{x}^+(t)$
$[\mathbf{x}](0)$
x_2
$\mathbf{x}^+(0)$
$[\mathbf{x}](t)$
$\mathbf{x}(t)$
x_1
$x_1^+(t)$
$\mathbf{x}^-(0)$
$x_1^+(0)$
$\mathbf{x}^-(t)$
$x_1^-(0)$
$x_1^-(t)$
$t = 0$
t


x_2
$\mathbf{z}(0)$
x_1
$\mathbf{z}(t)$
$z_4(t)$
$z_2(t)$
$z_3(t)$
$z_1(t)$
$t = 0$
t

Define

$$\mathbf{z} = (x_1^-, x_1^+, \ldots, x_n^-, x_n^+).$$

We want an *interval differential equation* (IDE)

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z})$$

such that

$$\mathbb{X}_t \subset [\mathbf{x}](t) = \underbrace{[x_1^-(t), x_1^+(t)] \times \cdots \times [x_n^-(t), x_n^+(t)]}_{\simeq \mathbf{z}(t)}.$$

An IDE is :

$$\underbrace{\begin{pmatrix} \dot{\mathbf{x}}^-(t) \\ \dot{\mathbf{x}}^+(t) \end{pmatrix}}_{\dot{\mathbf{z}}(t)} = \underbrace{\begin{pmatrix} \mathbf{f}^-(\mathbf{x}^-(t),\mathbf{x}^+(t),\mathbf{u}^-(t),\mathbf{u}^+(t)) \\ \mathbf{f}^+(\mathbf{x}^-(t),\mathbf{x}^+(t),\mathbf{u}^-(t),\mathbf{u}^+(t)) \end{pmatrix}}_{\mathbf{g}(\mathbf{z}(t),\mathbf{u}^-(t),\mathbf{u}^+(t))}$$

We write

$$[\dot{\mathbf{x}}](t) = [\![\mathbf{f}]\!]([\mathbf{x}](t),[\mathbf{u}](t))$$

where

$$\begin{array}{rcl} [\dot{\mathbf{x}}](t) & = & [\dot{\mathbf{x}}^-(t),\dot{\mathbf{x}}^+(t)] \\ [\mathbf{x}](t) & = & [\mathbf{x}^-(t),\mathbf{x}^+(t)] \\ [\mathbf{u}](t) & = & [\mathbf{u}^-(t),\mathbf{u}^+(t)] \\ [\![\mathbf{f}]\!] & = & [\mathbf{f}^-,\mathbf{f}^+] \end{array}$$

The IDE is *self consistent* if

$$\left.\begin{array}{rcll} \mathbf{x}^-(0) & \leq & \mathbf{x}^+(0) & \\ \mathbf{u}^-(t_1) & \leq & \mathbf{u}^+(t_1) & ,\forall t_1 \leq t \end{array}\right\} \Rightarrow \mathbf{x}^-(t) \leq \mathbf{x}^+(t), \forall t$$

The IDE is an *interval enclosure* of the system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ if

$$\left.\begin{array}{c} \mathbf{x}(0) = \mathbf{x}_0 \in [\mathbf{x}_0] \\ \mathbf{u}(t) \in [\mathbf{u}](t), \forall t \end{array}\right\} \Rightarrow \mathbf{x}(t) \in [\mathbf{x}](t), \forall t$$

Proposition. Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

or equivalently

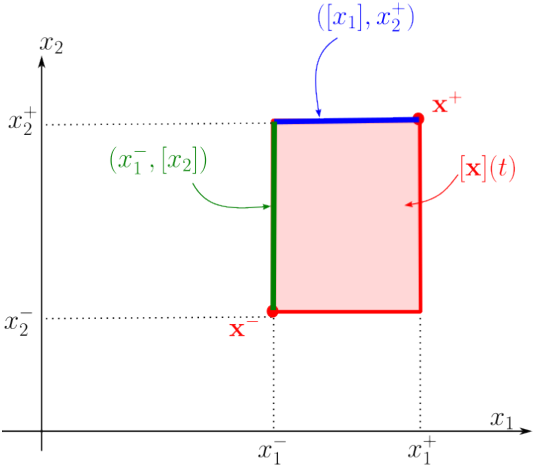
$$\begin{array}{ccc} \dot{x}_1 & = & f_1(x_1, x_2, \ldots, x_n, \mathbf{u}) \\ \vdots & & \vdots \\ \dot{x}_n & = & f_n(x_1, x_2, \ldots, x_n, \mathbf{u}) \end{array}$$

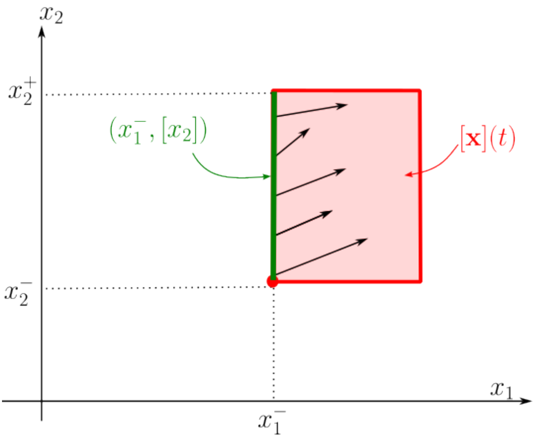
An interval enclosure is

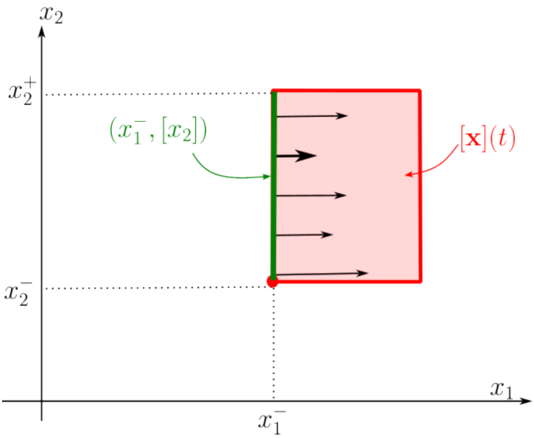
$$\underbrace{\begin{pmatrix} \dot{x}_1^- \\ \dot{x}_1^+ \\ \dot{x}_2^- \\ \dot{x}_2^+ \\ \vdots \\ \dot{x}_n^- \\ \dot{x}_n^+ \end{pmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{pmatrix} \text{lb}(& [f_1](x_1^-, [x_2], [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ \text{ub}(& [f_1](x_1^+, [x_2], [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ \text{lb}(& [f_2]([x_1], x_2^-, [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ \text{ub}(& [f_2]([x_1], x_2^+, [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ & \vdots & \\ \text{lb}(& [f_n]([x_1], [x_2], [x_3], \dots, x_n^-, [\mathbf{u}]) &) \\ \text{ub}(& [f_n]([x_1], [x_2], [x_3], \dots, x_n^+, [\mathbf{u}]) &) \end{pmatrix}}_{\mathbf{g}(\mathbf{z},[\mathbf{u}])}$$

An interval enclosure is

$$\underbrace{\begin{pmatrix} \dot{x}_1^- \\ \dot{x}_1^+ \\ \dot{x}_2^- \\ \dot{x}_2^+ \\ \vdots \\ \dot{x}_n^- \\ \dot{x}_n^+ \end{pmatrix}}_{[\dot{\mathbf{x}}]} = \underbrace{\begin{pmatrix} \mathrm{lb}(& [f_1](x_1^-, [x_2], [x_3], \ldots, [x_n], [\mathbf{u}]) &) \\ \mathrm{ub}(& [f_1](x_1^+, [x_2], [x_3], \ldots, [x_n], [\mathbf{u}]) &) \\ \mathrm{lb}(& [f_2]([x_1], x_2^-, [x_3], \ldots, [x_n], [\mathbf{u}]) &) \\ \mathrm{ub}(& [f_2]([x_1], x_2^+, [x_3], \ldots, [x_n], [\mathbf{u}]) &) \\ & \vdots & \\ \mathrm{lb}(& [f_n]([x_1], [x_2], [x_3], \ldots, x_n^-, [\mathbf{u}]) &) \\ \mathrm{ub}(& [f_n]([x_1], [x_2], [x_3], \ldots, x_n^+, [\mathbf{u}]) &) \end{pmatrix}}_{[\![\mathbf{f}]\!]([\mathbf{x}],[\mathbf{u}])}$$


$([x_1], x_2^+)$
x_2
x_2^+
$\mathbf{x}^+$
$(x_1^-, [x_2])$
$[\mathbf{x}](t)$
x_2^-
$\mathbf{x}^-$
x_1
x_1^-
x_1^+

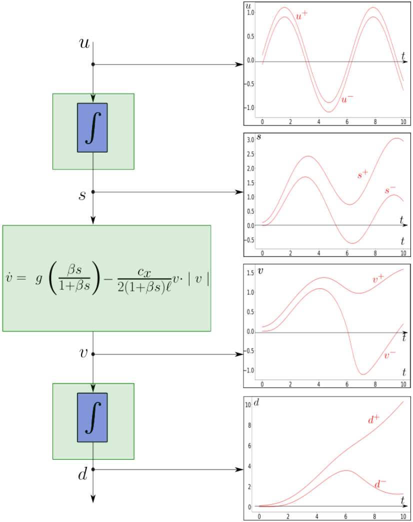

x_2
x_2^+
x_2^-
$(x_1^-, [x_2])$
$[\mathbf{x}](t)$
x_1
x_1^-



The left face should not go faster than the bold arrow

$$
\begin{aligned}
\dot{s}^- &= u^- \\
\dot{s}^+ &= u^+ \\
\dot{v}^- &= \mathsf{lb}\left(\frac{[s]}{1+[s]} - \frac{1}{1+[s]} \cdot v^- \cdot \mid v^- \mid\right) \\
\dot{v}^+ &= \mathsf{ub}\left(\frac{[s]}{1+[s]} - \frac{1}{1+[s]} \cdot v^+ \cdot \mid v^+ \mid\right) \\
\dot{d}^- &= v^- \\
\dot{d}^+ &= v^+
\end{aligned}
$$

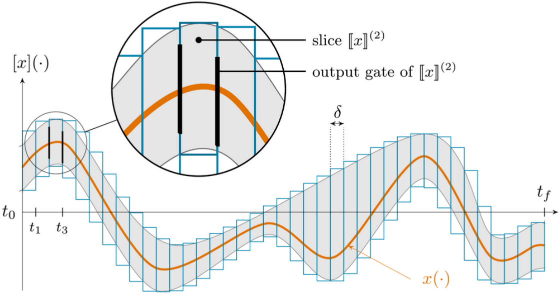
$$\underbrace{\begin{pmatrix} \dot{s}^- \\ \dot{s}^+ \\ \dot{v}^- \\ \dot{v}^+ \\ \dot{d}^- \\ \dot{d}^+ \end{pmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{pmatrix} u^- \\ u^+ \\ \frac{s^-}{1+s^-} - \max\begin{pmatrix} \frac{v^- \cdot |v^-|}{1+s^-} \\ \frac{v^- \cdot |v^-|}{1+s^+} \end{pmatrix} \\ \frac{s^+}{1+s^+} - \min\begin{pmatrix} \frac{v^+ \cdot |v^+|}{1+s^-} \\ \frac{v^+ \cdot |v^+|}{1+s^+} \end{pmatrix} \\ v^- \\ v^+ \end{pmatrix}}_{\mathbf{g}(\mathbf{z}, u^-, u^+)}$$


u
s
$\dot{v} = g\left(\frac{\beta s}{1+\beta s}\right) - \frac{cx}{2(1+\beta s)\ell} v \cdot |v|$
v
d
u^+
u^-
s^+
s^-
v^+
v^-
d^+
d^-
t

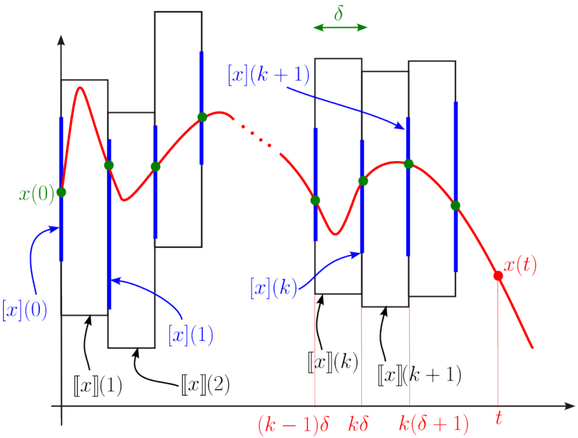
- Due to the operator nondifferentiability, integrating the IDE with guarantee is not trivial.
- Lohner or Taylor based method cannot be used.

3. Toward a safe and fast dead reckoning

We want an integration method for a differential inclusion which is guaranteed, real-time, fast.



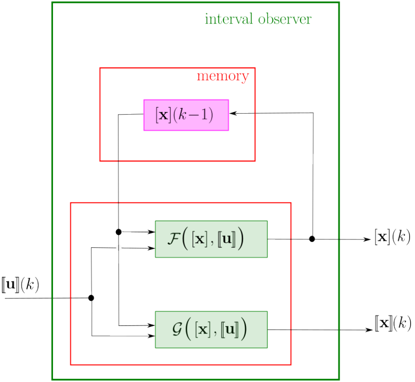
A tube $[x](t)$ which encloses the trajectory $x(t)$



A tube $[x](t)$ which encloses the trajectory $x(t)$

We want a dead reckoning of the form

$$\begin{cases} [\mathbf{x}](k) & = \mathscr{F}\left([\mathbf{x}](k-1), [\![\mathbf{u}]\!](k)\right) \\ [\![\mathbf{x}]\!](k) & = \mathscr{G}\left([\mathbf{x}](k-1), [\![\mathbf{u}]\!](k)\right) \end{cases}$$



The memory is made with gates only

Triangular systems

Consider the triangular system

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \mathbf{u}) \\
\dot{x}_2 &= f_2(x_1, x_2, \mathbf{u}) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, \mathbf{u})
\end{aligned}
$$

Assume that at time k, we have

$$\begin{array}{rcl} \mathbf{u}([k-1,k]) & \subset & [\![\mathbf{u}]\!](k) \\ x_1([k-1,k]) & \subset & [\![x_1]\!](k) \\ \vdots & & \vdots \\ x_{i-1}([k-1,k]) & \subset & [\![x_{i-1}]\!](k) \\ x_i(k-1) & \in & [x_i](k-1) \end{array}$$

We want to compute $[x_i](k)$, and $[\![x_i]\!](k)$ such that

$$\begin{array}{rcl} x_i([k-1,k]) & \subset & [\![x_i]\!](k) \\ x_i(k) & \in & [x_i](k) \end{array}$$

We have

$$\dot{x}_i = f_i(x_i, \underbrace{x_1, \cdots, x_{i-1}, \mathbf{u}}_{\mathbf{U}})$$

with

$$\begin{array}{ccc} \mathbf{U}([k-1,k]) & \subset & [\![\mathbf{U}]\!](k) \\ x_i(k-1) & \in & [x_i](k-1) \end{array}$$

This is a *scalar* differential inclusion which is comfortable, since we may have an analytical solution.

Scalar interval flow

$$\dot{v}(t) = f(v(t), \mathbf{u}(t))$$
$$v(0) = v_0 \in [v_0]$$
$$\mathbf{u}(t) \in [\![\mathbf{u}]\!] \subset \mathbb{R}^n$$

The signal $\mathbf{u}(t)$ is inside the box $[\![\mathbf{u}]\!] \subset \mathbb{R}^m$.

Given $\delta > 0$, an *interval flow* for $\dot{v}(t) = f(v(t), \mathbf{u}(t))$ is

$$\Phi_f : \begin{array}{ccc} \mathbb{R} \times \mathbb{IR} \times \mathbb{IR}^m & \rightarrow & \mathbb{IR} \times \mathbb{IR} \\ (\delta, [v_0], [\![\mathbf{u}]\!]) & \rightarrow & ([v], [\![v]\!]) \end{array}$$

with

$$\left.\begin{array}{c} v(0) \in [v_0] \\ \forall t \in [0, \delta], \mathbf{u}(t) \in [\![\mathbf{u}]\!] \\ \dot{v}(t) = f(v(t), \mathbf{u}(t)) \\ ([v], [\![v]\!]) = \Phi_f(\delta, [v_0], [\![\mathbf{u}]\!]) \end{array}\right\} \Rightarrow \left\{ \begin{array}{c} v(\delta) \in [v] \\ \forall t \in [0, \delta], v(t) \in [\![v]\!] \end{array}\right.$$

Example. The integrator

$$\begin{cases} \dot{v}(t) = u(t) \\ v(0) = v_0 \in [v_0] \\ u(t) \in [\![u]\!] = [u^-, u^+] \end{cases}$$

An interval flow is

$$\Phi_f(\delta, [v_0], [\![\mathbf{u}]\!]) = \begin{pmatrix} [v_0] + \delta [\![u]\!] \\ [v_0] + [0, \delta] \cdot [\![u]\!] \end{pmatrix}$$

Example. First order linear system

$$\begin{cases} \dot{v}(t) = av(t) + u(t) \\ v(0) = v_0 \in [v_0] \\ u(t) \in [\![u]\!] = [u^-, u^+] \end{cases}$$

An interval flow is

$$\Phi_f(\delta, [v_0], [\![u]\!]) = \begin{pmatrix} e^{a\delta}\left([v_0] - \frac{[\![u]\!]}{a}\left(e^{-a\delta} - 1\right)\right) \\ e^{a[0,\delta]}\left([v_0] - \frac{[\![u]\!]}{a}\left(e^{-a[0,\delta]} - 1\right)\right) \end{pmatrix}$$

Example. Riccati system

$$\begin{cases} \dot{v}(t) = u_1(t) - u_2(t)v^2(t) \\ v(0) = v_0 \in [v_0] \subset \mathbb{R}^+ \\ \mathbf{u}(t) \in [\![\mathbf{u}]\!] \end{cases}$$

The solution of the Riccati equation $\dot{v} = a - bv^2$ for $a \geq 0, b > 0$ is

$$v(t) \quad = \quad \frac{\left(e^{2\sqrt{ab}t}-1\right)\sqrt{ab}+\left(e^{2\sqrt{ab}t}+1\right)v_0 b}{\left(e^{2\sqrt{ab}t}+1\right)\sqrt{ab}+\left(e^{2\sqrt{ab}t}-1\right)v_0 b}$$

Thus

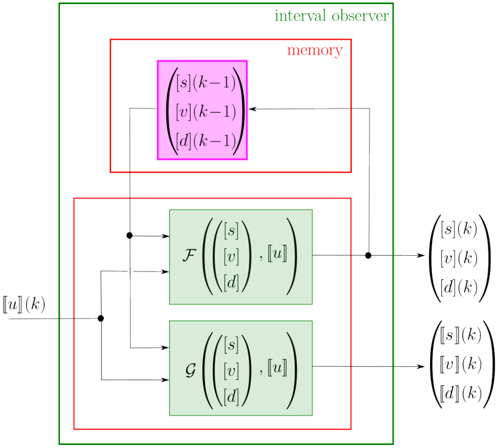
$$\Phi_f(\delta, [v_0], [\![\mathbf{u}]\!]) = \ldots$$

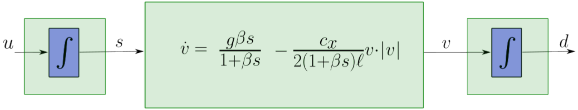
4. Online dead reckoning

We want an interval estimator of the form

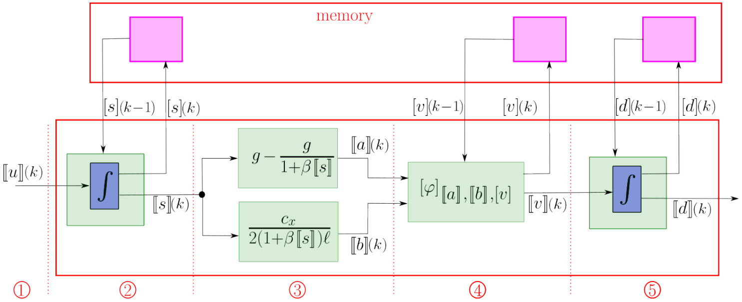
$$\begin{pmatrix} [s](k) \\ [v](k) \\ [d](k) \end{pmatrix} = \mathscr{F}\left(\begin{pmatrix} [s](k-1) \\ [v](k-1) \\ [d](k-1) \end{pmatrix}, [\![u]\!](k)\right)$$

$$\begin{pmatrix} [\![s]\!](k) \\ [\![v]\!](k) \\ [\![d]\!](k) \end{pmatrix} = \mathscr{G}\left(\begin{pmatrix} [s](k-1) \\ [v](k-1) \\ [d](k-1) \end{pmatrix}, [\![u]\!](k)\right)$$


interval observer
memory
$\begin{pmatrix}[s](k-1)\\ [v](k-1)\\ [d](k-1)\end{pmatrix}$
$\mathcal{F}\left(\begin{pmatrix}[s]\\ [v]\\ [d]\end{pmatrix}, [\![u]\!]\right)$
$\mathcal{G}\left(\begin{pmatrix}[s]\\ [v]\\ [d]\end{pmatrix}, [\![u]\!]\right)$
$[\![u]\!](k)$
$\begin{pmatrix}[s](k)\\ [v](k)\\ [d](k)\end{pmatrix}$
$\begin{pmatrix}[\![s]\!](k)\\ [\![v]\!](k)\\ [\![d]\!](k)\end{pmatrix}$

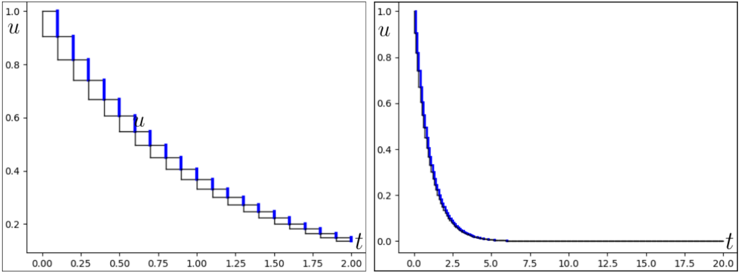


The float is a triangular system


memory
[s](k−1)
[s](k)
[v](k−1)
[v](k)
[d](k−1)
[d](k)
[[u]](k)
[[s]](k)
$g - \frac{g}{1+\beta[[s]]}$
$\frac{c_x}{2(1+\beta[[s]])\ell}$
[[a]](k)
[[b]](k)
$[\varphi]_{[[a]],[[b]],[v]}$
[[v]](k)
[[d]](k)
①
②
③
④
⑤

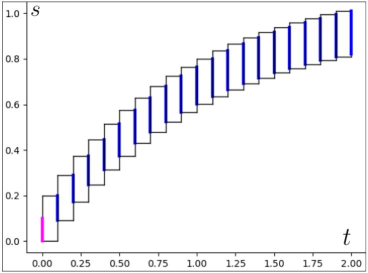
In:	$[d_0], [v_0], [s_0]$
Init	$[d] = [d_0]; [v] = [v_0]; [s] = [s_0]$
Main loop	For $k = 1$ to $k_{\max}$
Step 1	Read $[\![u]\!] = [\![u]\!](k)$
Step 2	$[\![s]\!] = [s] + [\![u]\!] \cdot [0, \delta]$
	$[s] = [s] + [\![u]\!] \cdot \delta$
Step 3	$[\![a]\!] = g \cdot \left(1 - \frac{1}{1+\beta [\![s]\!]}\right)$
	$[\![b]\!] = \frac{c_x}{2(1+\beta [\![s]\!])\ell}$
Step 4	$[\![v]\!] = [\varphi]_{[\![a]\!],[\![b]\!],[v]}([0, \delta])$
	$[v] = [\varphi]_{[\![a]\!],[\![b]\!],[v]}(\delta)$
Step 5	$[\![d]\!] = [d] + [\![v]\!] \cdot [0, \delta]$
	$[d] = [d] + [\![v]\!] \cdot \delta$
	Write$(k, [d], [\![d]\!], [v], [\![v]\!], [s], [\![s]\!])$

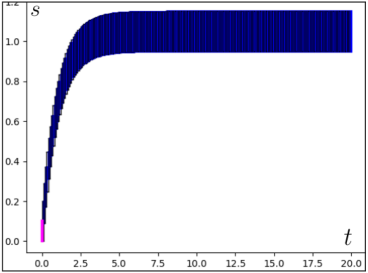
.


1.0
u
0.8
0.6
u
0.4
0.2
t
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
1.0
u
0.8
0.6
0.4
0.2
0.0
t
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

Tube for $u(t)$

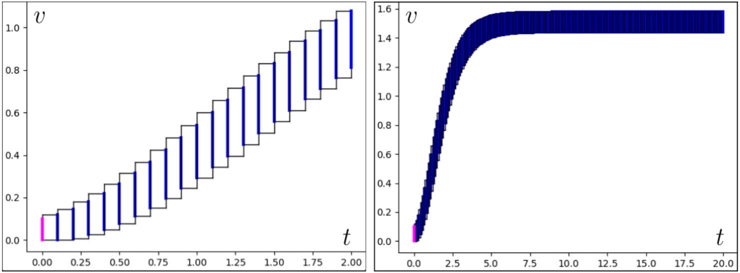
.


s
t
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00


s
t
1.0
0.8
0.6
0.4
0.2
0.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

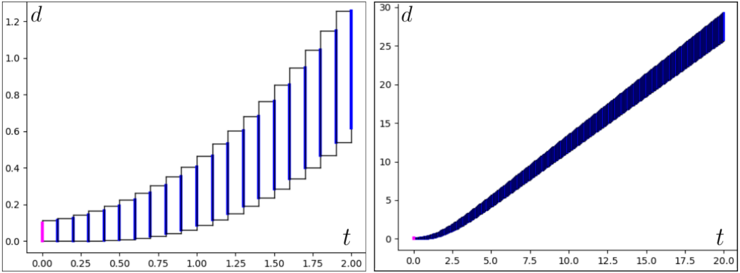
Tube for $s(t)$

.


v
t
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
1.6
1.4
1.2
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

Tube for $v(t)$

.


d
t
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
30
25
20
15
10
5
0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

Tube for $d(t)$

Conclusion

- For strict triangular systems, a basic interval propagation solves the dead reckoning approach easily
- When non strict triangular systems scalar interval flows are needed

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