Characterizing Sliding Surfaces of Cyber-Physical Systems

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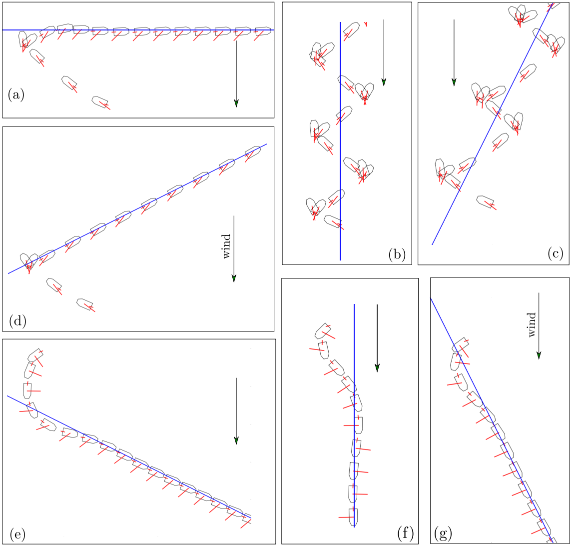
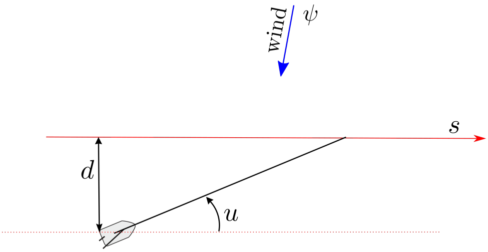
SNR 2020, Vienna (virtually)
https://youtu.be/ntiru1sxnxc

Easy-boat

https://youtu.be/bNqiwW4p6WE

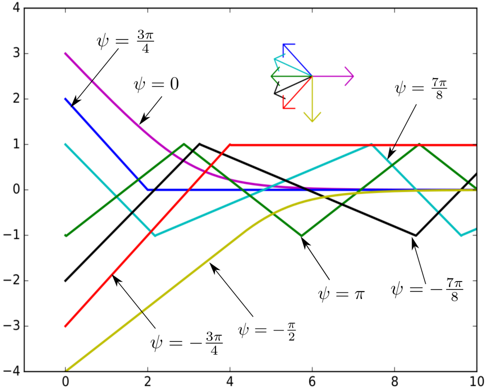

(a)
(b)
(c)
(d)
wind
(e)
(f)
(g)
wind


wind
ψ
s
d
u

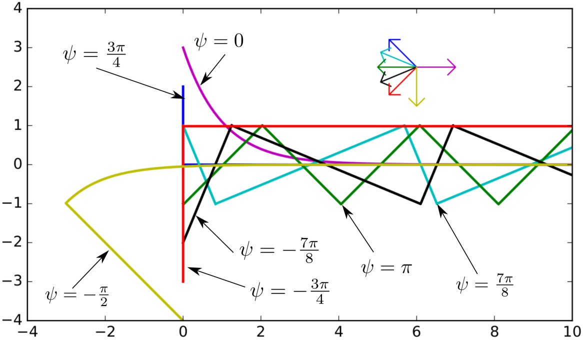
$$\begin{array}{rcl} \dot{d} & = & \sin u \\ \cos(\psi - u) + \cos\frac{\pi}{5} > 0 \end{array}$$

Controller in: (d, ψ, q) ; out:u
if $d^2 - 1 > 0$ then $q :=$ sign (d)
if $\cos(\psi + \text{atan}(d)) + \cos\frac{\pi}{4} \leq 0 \vee \left(d^2 \leq 1 \wedge \cos\psi + \cos\frac{\pi}{4} \leq 0\right)$
then $u := \pi + \psi - q\frac{\pi}{4}$.
else $u := -\text{atan}(d)$.

Simulation



Simulation in the (t, d)-space

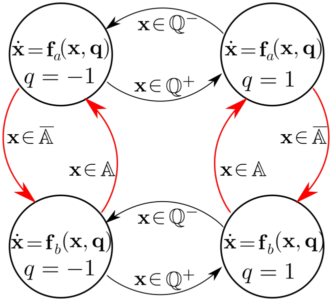


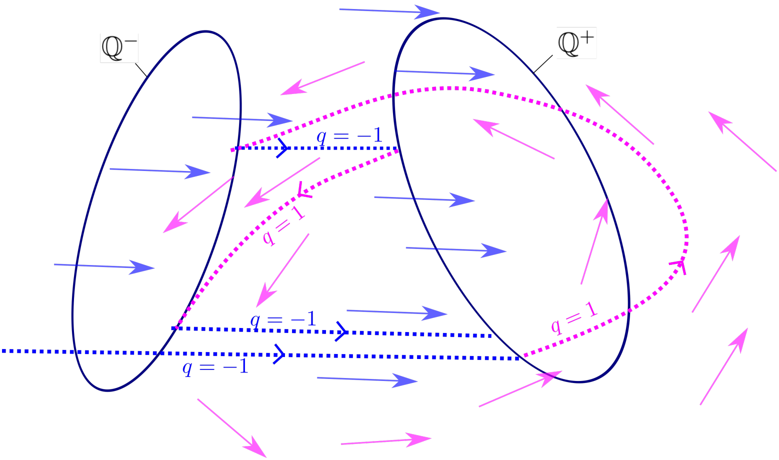
Simulation in the $(\int^t \cos u, d)$-space

Formalism

Given $\mathbb{Q}^-$, $\mathbb{Q}^+$ disjoint and closed, two smooth functions $\mathbf{f}_a, \mathbf{f}_b$. We define [3]

$$\mathcal{S}(\mathbb{A}) : \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, q) = \begin{cases} \mathbf{f}_a(\mathbf{x}, q) & \text{if } \mathbf{x} \in \mathbb{A} \\ \mathbf{f}_b(\mathbf{x}, q) & \text{if } \mathbf{x} \in \mathbb{B} = \overline{\mathbb{A}} \end{cases} \\ q = -1 \quad \text{as soon as } \mathbf{x} \in \mathbb{Q}^- \\ \quad = +1 \quad \text{as soon as } \mathbf{x} \in \mathbb{Q}^+ \end{cases}$$


$\dot{\mathbf{x}} = \mathbf{f}_a(\mathbf{x}, \mathbf{q})$
$q = -1$
$\dot{\mathbf{x}} = \mathbf{f}_a(\mathbf{x}, \mathbf{q})$
$q = 1$
$\dot{\mathbf{x}} = \mathbf{f}_b(\mathbf{x}, \mathbf{q})$
$q = -1$
$\dot{\mathbf{x}} = \mathbf{f}_b(\mathbf{x}, \mathbf{q})$
$q = 1$
$\mathbf{x} \in \mathbb{Q}^-$
$\mathbf{x} \in \mathbb{Q}^+$
$\mathbf{x} \in \overline{\mathbb{A}}$
$\mathbf{x} \in \mathbb{A}$
$\mathbf{x} \in \mathbb{A}$
$\mathbf{x} \in \overline{\mathbb{A}}$
$\mathbf{x} \in \mathbb{Q}^-$
$\mathbf{x} \in \mathbb{Q}^+$


$\mathbb{Q}^-$
$\mathbb{Q}^+$
$q = -1$
$q = 1$
$q = -1$
$q = -1$
$q = 1$

Sliding surface

The *sliding surface* $\mathbb{S}(\mathbb{A})$ for $\mathcal{S}(\mathbb{A})$ is the subset of $\partial\mathbb{A}$ such that the state can slide inside for a non degenerated interval of time. Since $\mathbb{Q}^-$, $\mathbb{Q}^+$ are disjoint, we have

$$\mathbb{S}(\mathbb{A}) = \mathbb{S}_{q=-1}(\mathbb{A}) \cup \mathbb{S}_{q=+1}(\mathbb{A})$$

We can thus assume that q is fixed and find the sliding surfaces for

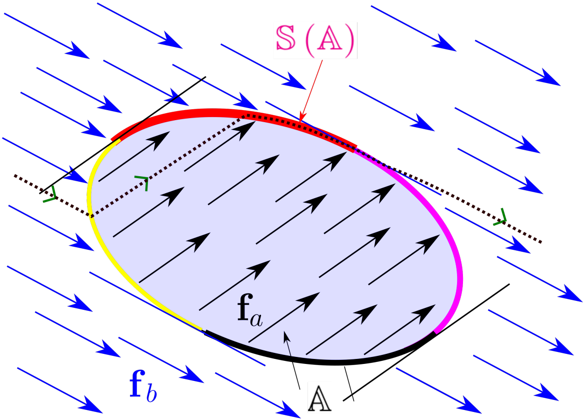
$$\mathcal{S}(\mathbb{A}): \quad \dot{\mathbf{x}} = \begin{cases} \mathbf{f}_a(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbb{A} \\ \mathbf{f}_b(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbb{B} = \overline{\mathbb{A}} \end{cases}$$

The *Lie derivative* of $c : \mathbb{R}^n \to \mathbb{R}$ with respect to $\mathbf{f}$ is

$$\mathcal{L}_{\mathbf{f}}^{c}\left(\mathbf{x}\right) = \frac{dc}{d\mathbf{x}}\left(\mathbf{x}\right) \cdot \mathbf{f}\left(\mathbf{x}\right).$$

If $\mathbb{A}{:}c\left(\mathbf{x}\right) \leq 0$, then

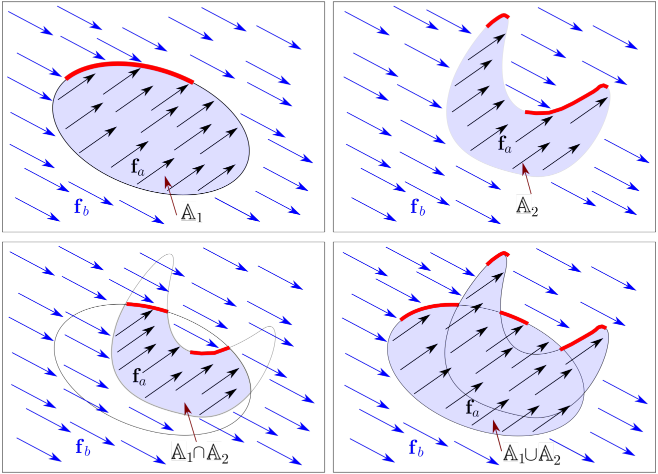
$$\mathbb{S}\left(\mathbb{A}\right) = \partial\mathbb{A} \cap \left\{\mathbf{x} \,|\, \mathcal{L}_a^c\left(\mathbf{x}\right) \geq 0 \;\wedge \mathcal{L}_b^c\left(\mathbf{x}\right) \leq 0\right\}.$$

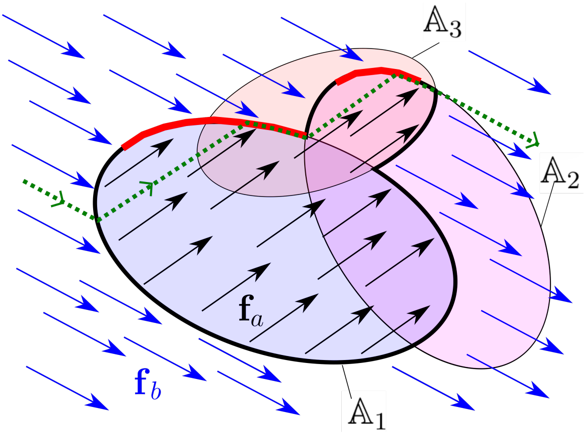


Sliding set $\mathbb{S}(\mathbb{A})$ (red) for $\mathbb{A} = \{\mathbf{x} | c(\mathbf{x}) \leq 0\}$

Proposition. If we have two closed sets $\mathbb{A}_1$ and $\mathbb{A}_2$. We have [3]

$$\begin{array}{lrcl} (i) & \mathbb{S}\left(\mathbb{A}_1 \cap \mathbb{A}_2\right) & = & \left(\mathbb{S}\left(\mathbb{A}_1\right) \cap \mathbb{A}_2\right) \cup \left(\mathbb{S}\left(\mathbb{A}_2\right) \cap \mathbb{A}_1\right) \\ (ii) & \mathbb{S}\left(\mathbb{A}_1 \cup \mathbb{A}_2\right) & = & \left(\mathbb{S}\left(\mathbb{A}_1\right) \cap \overline{\mathbb{A}_2}\right) \cup \left(\mathbb{S}\left(\mathbb{A}_2\right) \cap \overline{\mathbb{A}_1}\right) \end{array}$$


$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_1$
$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_2$
$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_1 \cap \mathbb{A}_2$
$\mathbf{f}_a$
$\mathbf{f}_b$
$\mathbb{A}_1 \cup \mathbb{A}_2$



$$\mathbb{S}\left(\mathbb{A}_1 \cup \left(\mathbb{A}_2 \cap \mathbb{A}_3\right)\right)$$

With easy-boat

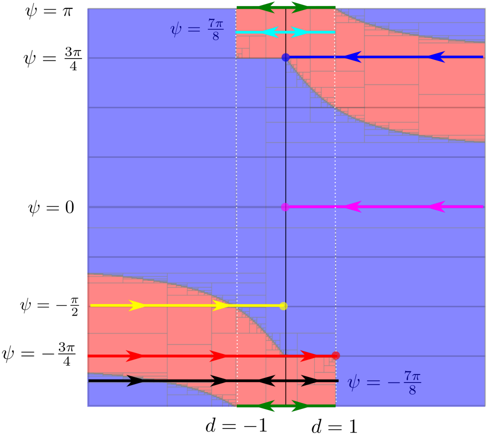
Three problems:

- Safety : the sailboat never goes against the wind. [1]
- Capture : The boat will be captured by its corridor [2]
- Characterize of the sliding surface.

We take $\mathbf{x} = (d, \psi)$,

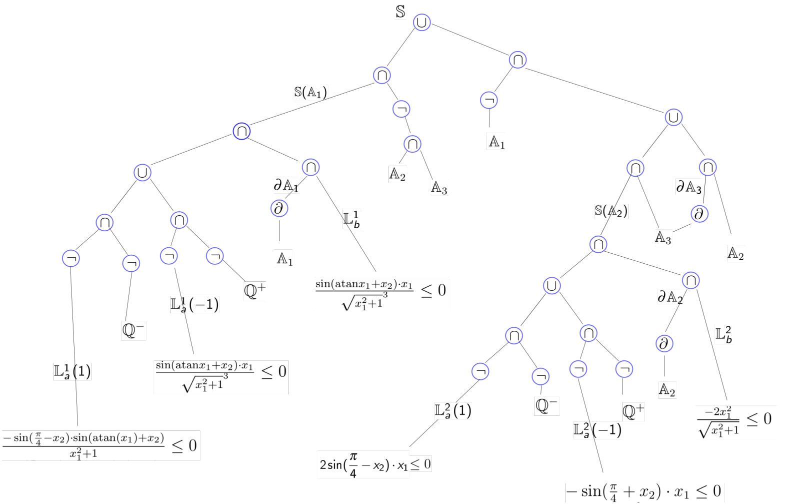
Function $\mathbf{f}(\mathbf{x}, q)$
if $\cos(x_2 + \mathsf{atan} x_1) + \cos\frac{\pi}{4} \leq 0 \vee \left(x_1^2 - 1 \leq 0 \wedge \cos x_2 + \cos\frac{\pi}{4} \leq 0\right)$
then $u := \pi + x_2 - q\frac{\pi}{4}$
else $u := -\mathsf{atan} x_1$.
Return $(\sin u, 0)$

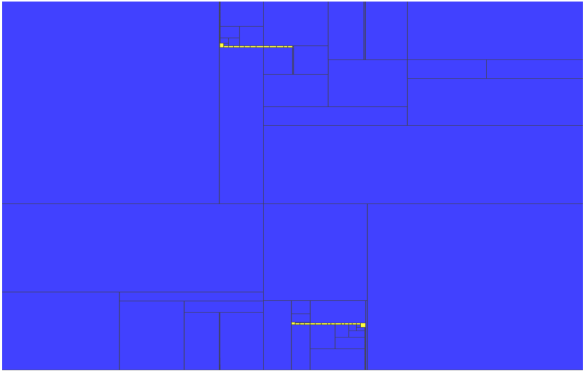
$$
\begin{aligned}
\mathbf{x} &= (d, \psi) \\
\mathbf{f}_a(\mathbf{x}, q) &= \begin{pmatrix} \sin\left(\pi + x_2 - q\frac{\pi}{4}\right) \\ 0 \end{pmatrix} \\
\mathbf{f}_b(\mathbf{x}) &= \begin{pmatrix} \sin\left(-\mathsf{atan} x_1\right) \\ 0 \end{pmatrix} \\
\mathbb{A}_1 &= \left\{\mathbf{x} \,\middle|\, \cos\left(x_2 + \mathsf{atan} x_1\right) + \cos\frac{\pi}{4} \leq 0\right\} \\
\mathbb{A}_2 &= \left\{\mathbf{x} \,\middle|\, x_1^2 - 1 \leq 0\right\} \\
\mathbb{A}_3 &= \left\{\mathbf{x} \,\middle|\, \cos x_2 + \cos\frac{\pi}{4} \leq 0\right\} \\
\mathbb{A} &= \mathbb{A}_1 \cup \left(\mathbb{A}_2 \cap \mathbb{A}_3\right) \\
\mathbb{Q}^- &= \{\mathbf{x} \,|\, x_1 + 1 \leq 0\} \\
\mathbb{Q}^+ &= \{\mathbf{x} \,|\, 1 - x_1 \leq 0\}
\end{aligned}
$$


$\psi = \pi$
$\psi = \frac{7\pi}{8}$
$\psi = \frac{3\pi}{4}$
$\psi = 0$
$\psi = -\frac{\pi}{2}$
$\psi = -\frac{3\pi}{4}$
$\psi = -\frac{7\pi}{8}$
$d = -1$
$d = 1$

$$
\begin{array}{rcccl}
\mathcal{L}_a^{c_1}(\mathbf{x},q) & = & \frac{dc_1}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_a(\mathbf{x},q) & = & \frac{-\sin(\frac{q\pi}{4}-x_2)\cdot\sin(\mathsf{atan}(x_1)+x_2)}{x_1^2+1} \\
\mathcal{L}_b^{c_1}(\mathbf{x}) & = & \frac{dc_1}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_b(\mathbf{x},q) & = & \frac{\sin(\mathsf{atan}x_1+x_2)\cdot x_1}{\sqrt{x_1^2+1}^3} \\
\mathcal{L}_a^{c_2}(\mathbf{x},q) & = & \frac{dc_2}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_a(\mathbf{x}) & = & 2\sin(\frac{q\pi}{4}-x_2)\cdot x_1 \\
\mathcal{L}_b^{c_2}(\mathbf{x}) & = & \frac{dc_2}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_b(\mathbf{x},q) & = & \frac{-2x_1^2}{\sqrt{x_1^2+1}} \\
\mathcal{L}_a^{c_3}(\mathbf{x},q) & = & \frac{dc_3}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_a(\mathbf{x},q) & = & 0 \\
\mathcal{L}_b^{c_3}(\mathbf{x}) & = & \frac{dc_3}{d\mathbf{x}}(\mathbf{x})\cdot\mathbf{f}_b(\mathbf{x},q) & = & 0
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{S}(\mathbb{A}_1) & = & \partial\mathbb{A}_1\cap\mathbb{L}_b^1\cap\left(\overline{\mathbb{L}_a^1(1)}\cap\overline{\mathbb{Q}^-}\cup\overline{\mathbb{L}_a^1(-1)}\cap\overline{\mathbb{Q}^+}\right) \\
\mathbb{S}(\mathbb{A}_2) & = & \partial\mathbb{A}_2\cap\mathbb{L}_b^2\cap\left(\overline{\mathbb{L}_a^2(1)}\cap\overline{\mathbb{Q}^-}\cup\overline{\mathbb{L}_a^2(-1)}\cap\overline{\mathbb{Q}^+}\right) \\
\mathbb{S}(\mathbb{A}_3) & = & \partial\mathbb{A}_3
\end{array}
$$


𝕊
𝕊(𝔸1)
𝔸1
𝔸2
𝔸3
∂𝔸1
𝕃1b
𝕃1a(−1)
𝕃1a(1)
ℚ+
ℚ−
sin(atanx1+x2)·x1 / √(x1²+1)³ ≤ 0
−sin(π/4−x2)·sin(atan(x1)+x2) / (x1²+1) ≤ 0
𝕊(𝔸2)
∂𝔸3
∂𝔸2
𝕃2b
𝕃2a(1)
𝕃2a(−1)
2sin(π/4 − x2) · x1 ≤ 0
− sin(π/4 + x2) · x1 ≤ 0
−2x1² / √(x1²+1) ≤ 0



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