Minkowski operations of sets with application to robot localization

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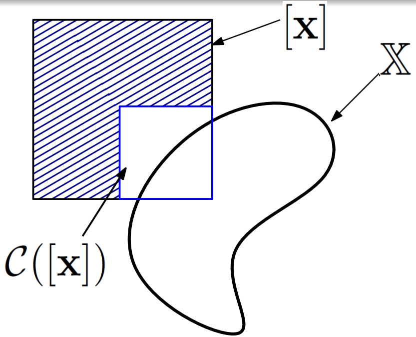
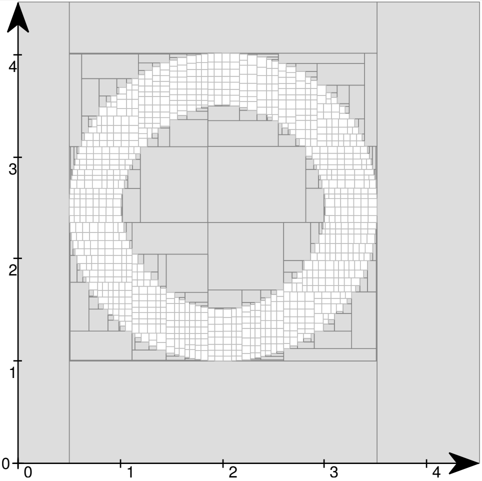
B. Desrochers, L. Jaulin

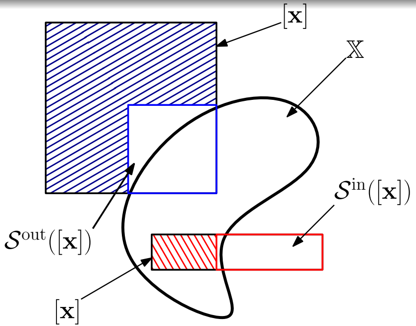
Contractors

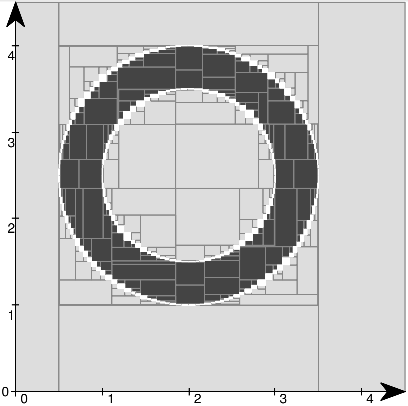
$$\mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] \qquad \text{(contractance)}$$
$$[\mathbf{x}] \subset [\mathbf{y}] \ \Rightarrow \ \mathscr{C}([\mathbf{x}]) \subset \mathscr{C}([\mathbf{y}]) \qquad \text{(monotonicity)}$$


$[\mathbf{x}]$
$\mathbb{X}$
$\mathcal{C}([\mathbf{x}])$


4
3
2
1
0
0
1
2
3
4

Separators

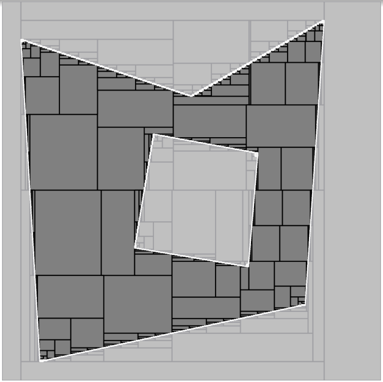

$[\mathbf{x}]$
$\mathbb{X}$
$\mathcal{S}^{\text{in}}([\mathbf{x}])$
$\mathcal{S}^{\text{out}}([\mathbf{x}])$
$[\mathbf{x}]$


4
3
2
1
0
0
1
2
3
4

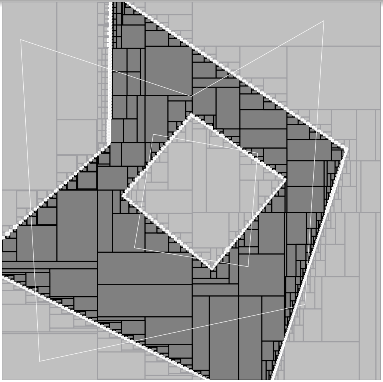
Algebra

If $\mathscr{S}_i = \left\{\mathscr{S}_i^{\text{in}}, \mathscr{S}_i^{\text{out}}\right\}, i \geq 1$, are separators, we define

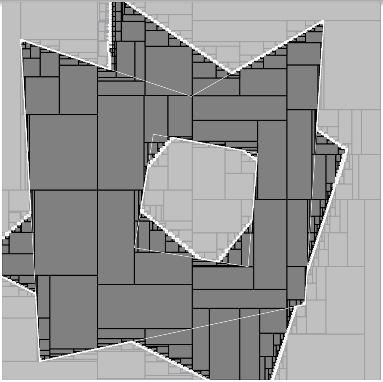
$$\begin{array}{rclc} \mathscr{S}_1 \cap \mathscr{S}_2 & = & \left\{\mathscr{S}_1^{\text{in}} \cup \mathscr{S}_2^{\text{in}}, \mathscr{S}_1^{\text{out}} \cap \mathscr{S}_2^{\text{out}}\right\} & \text{(intersection)} \\ \mathscr{S}_1 \cup \mathscr{S}_2 & = & \left\{\mathscr{S}_1^{\text{in}} \cap \mathscr{S}_2^{\text{in}}, \mathscr{S}_1^{\text{out}} \cup \mathscr{S}_2^{\text{out}}\right\} & \text{(union)} \\ \mathscr{S}_1 \backslash \mathscr{S}_2 & = & \mathscr{S}_1 \cap \overline{\mathscr{S}_2}. & \text{(difference)} \end{array}$$



Set $\mathbb{M}$



$\mathrm{Rot}(\mathbb{M})$



$$\mathrm{Rot}(\mathbb{M}) \cup \mathbb{M}$$

Registration problem

Consider the set:

$$\mathbb{P} = \{\mathbf{p} \in \mathbb{R}^p \mid \mathbf{f}(\mathbb{A}, \mathbf{p}) \subset \mathbb{B}\}.$$

where $\mathbf{f}:\mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^m$.
Finding $\mathbb{P}$ (if small) is generally performs using registration methods (as ICP).
The set $\mathbf{f}(\mathbb{A}, \mathbf{p})$ is the *registered set*.

We have

$$\begin{array}{rl} & \mathbf{f}(\mathbb{A},\mathbf{p}) \subset \mathbb{B} \\ \Leftrightarrow & \forall \mathbf{a} \in \mathbb{A}, \mathbf{f}(\mathbf{a},\mathbf{p}) \in \mathbb{B} \\ \Leftrightarrow & \neg\exists \mathbf{a} \in \mathbb{A}, \mathbf{f}(\mathbf{a},\mathbf{p}) \in \overline{\mathbb{B}} \\ \Leftrightarrow & \neg\exists \mathbf{a} \in \mathbb{A}, (\mathbf{a},\mathbf{p}) \in \mathbf{f}^{-1}\left(\overline{\mathbb{B}}\right) \end{array}$$

Thus

$$\mathbb{P} = \overline{proj_{\mathbf{p}}\{(\mathbb{A} \times \mathbb{R}^p) \cap \mathbf{f}^{-1}\left(\overline{\mathbb{B}}\right)\}}.$$

If $\mathscr{S}_{\mathbb{A}}, \mathscr{S}_{\mathbb{B}}$ are separators for $\mathbb{A}, \mathbb{B}$ then a separator $\mathscr{S}_{\mathbb{P}}$ for $\mathbb{P}$ is:

$$\mathscr{S}_{\mathbb{P}} = \overline{\mathit{proj}_{\mathbf{p}}\{(\mathscr{S}_{\mathbb{A}} \times \mathscr{S}_{\mathbb{R}^p}) \cap \mathbf{f}^{-1}\left(\overline{\mathscr{S}_{\mathbb{B}}}\right)\}}.$$

With a paver, we obtain an inner and outer approximation of $\mathbb{P}$.

Minkowski sum and difference

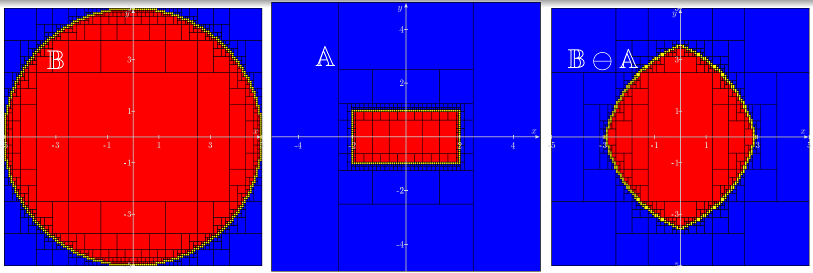
Computing Minkowski sum and difference can be seen as a registation problem.

Minkowski difference

Given two sets $\mathbb{A} \subset \mathscr{P}(\mathbb{R}^n)$, $\mathbb{B} \subset \mathscr{P}(\mathbb{R}^n)$, the Minkowski difference is defined by

$$\mathbb{B} \ominus \mathbb{A} = \{\mathbf{p} \mid \mathbb{A} + \mathbf{p} \subset \mathbb{B}\} = \overline{proj_{\mathbf{p}}\{(\mathbb{A} \times \mathbb{R}^p) \cap \mathbf{f}^{-1}\left(\overline{\mathbb{B}}\right)\}},$$

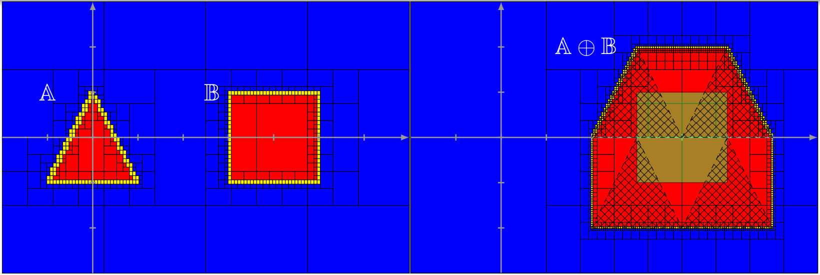
where $\mathbf{f}(\mathbf{a},\mathbf{p}) = \mathbf{a} + \mathbf{p}$.


$\mathbb{B}$
$\mathbb{A}$
$\mathbb{B} \ominus \mathbb{A}$

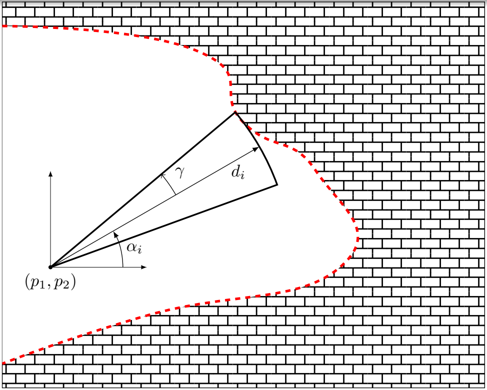
Minkowski sum

Minkowski sum, denoted by $\oplus$, is defined by:

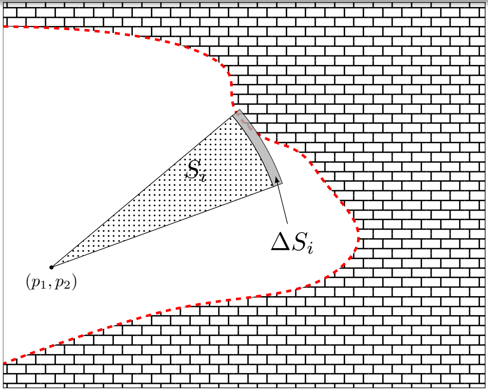
$$\mathbb{A} \oplus \mathbb{B} == \{\mathbf{a}+\mathbf{b}, \mathbf{a} \in \mathbb{A}, \mathbf{b} \in \mathbb{B}\} = \overline{\overline{\mathbb{B}} \ominus -\mathbb{A}}. \qquad (1)$$


$\mathbb{A}$
$\mathbb{B}$
$\mathbb{A} \oplus \mathbb{B}$

Sonar localization



Emission cone



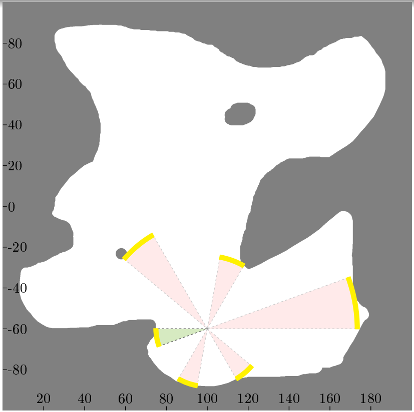
Free sector $\mathbb{S}_i$ and impact pie $\Delta\mathbb{S}_i$

The set of positions consistent with $[d_i]$ is

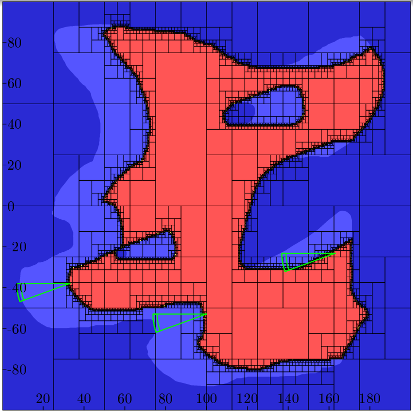
$$\begin{aligned}\mathbb{P}(i) &= \{\mathbf{p} \in \mathbb{R}^2 \mid (\mathbf{p} + \mathbb{S}_i) \subset \mathbb{M} \text{ and } (\mathbf{p} + \Delta\mathbb{S}_i) \cap \overline{\mathbb{M}} \neq \emptyset\} \\ &= (\mathbb{M} \ominus \mathbb{S}_i) \cap (\overline{\mathbb{M}} \oplus -\Delta\mathbb{S}_i).\end{aligned}$$

With several measurements $[d_i]$ the set consistent positions is

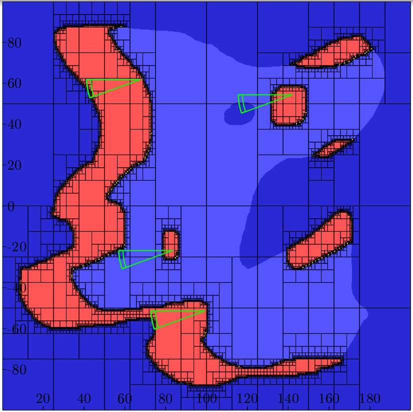
$$\mathbb{P} = \bigcap_i (\mathbb{M} \ominus \mathbb{S}_i) \cap (\overline{\mathbb{M}} \oplus -\Delta\mathbb{S}_i).$$



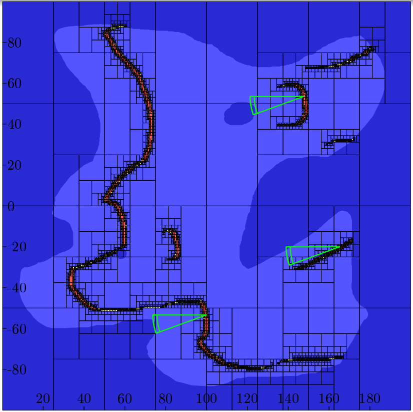
A robot which collects 6 sonar range measurements.



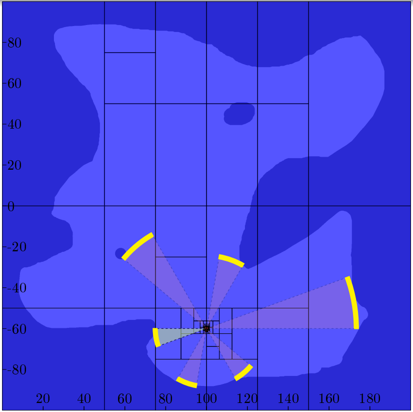
$$\mathbb{M} \ominus \mathbb{S}_1$$



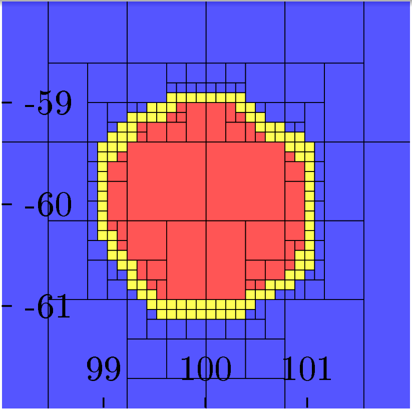
$$\mathbb{M} \cap \left(\overline{\mathbb{M}} \oplus -\Delta \mathbb{S}_1 \right)$$



$$(\mathbb{M} \ominus \mathbb{S}_1) \cap (\overline{\mathbb{M}} \oplus -\Delta\mathbb{S}_1)$$



$$\bigcap_i (\mathbb{M} \ominus \mathbb{S}_i) \cap (\overline{\mathbb{M}} \oplus -\Delta \mathbb{S}_i)$$



$$\bigcap_i (\mathbb{M} \ominus \mathbb{S}_i) \cap (\overline{\mathbb{M}} \oplus -\Delta\mathbb{S}_i)$$