Nonlinear state estimation with delays

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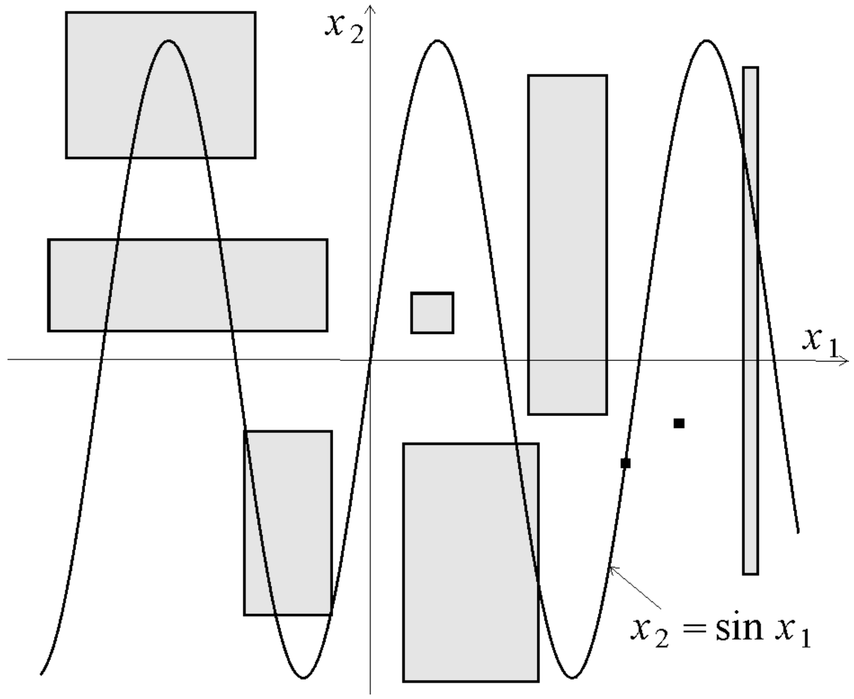
1 Contractors

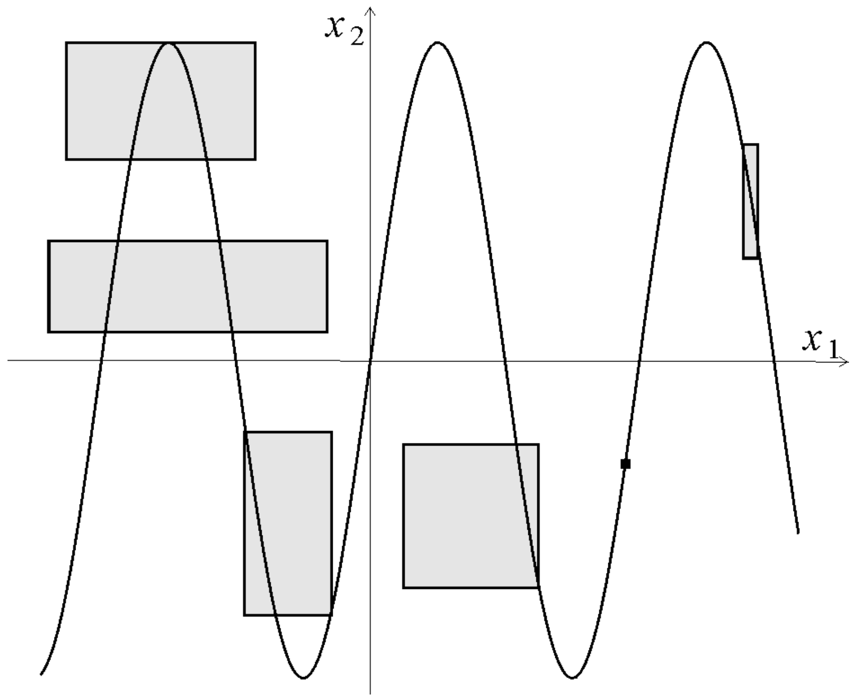
The operator $\mathcal{C} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

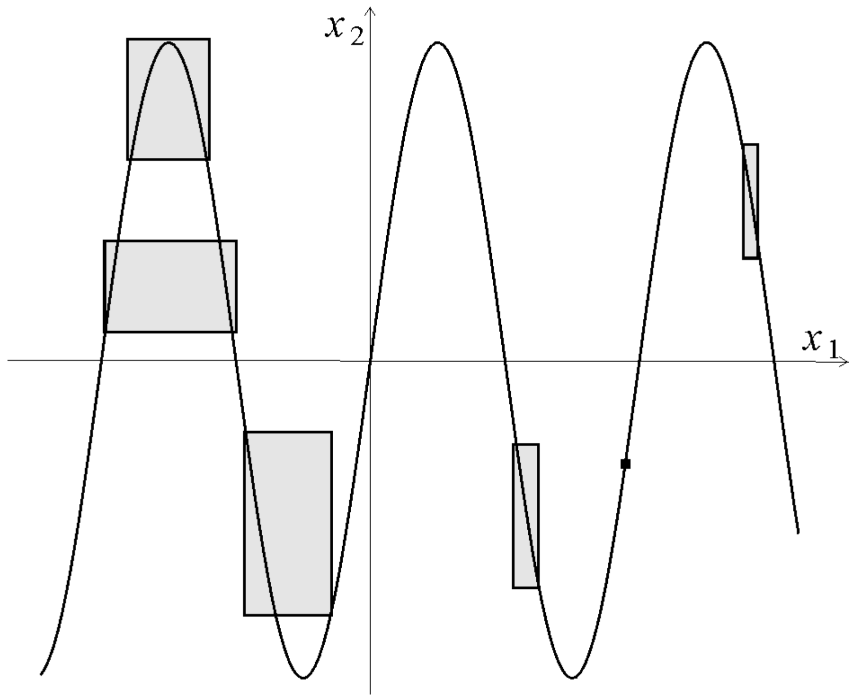
Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$


x_2
x_1
$x_2 = \sin x_1$



Forward contraction



Backward contraction

Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

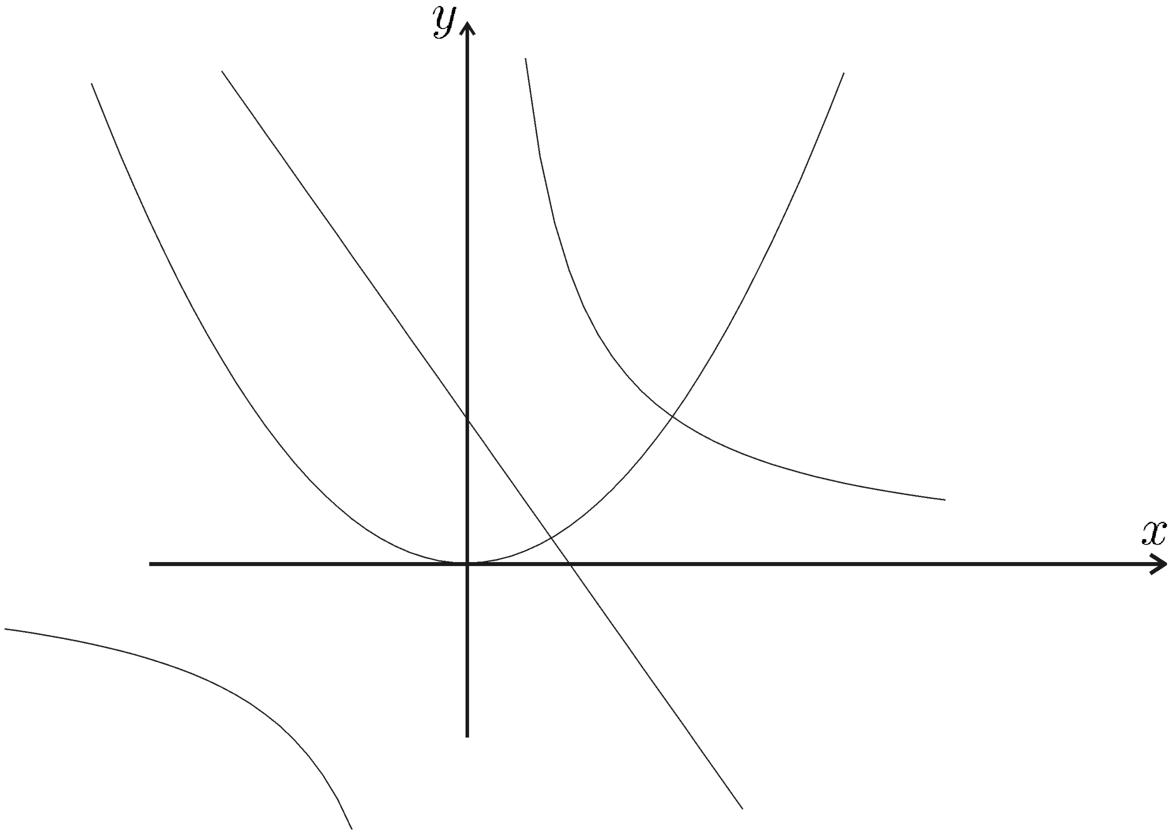
$$
\begin{array}{llll}
x_3 = x_1 + x_2 \Rightarrow & x_3 \in & [x_3] \cap ([x_1] + [x_2]) & \text{// forward} \\
x_1 = x_3 - x_2 \Rightarrow & x_1 \in & [x_1] \cap ([x_3] - [x_2]) & \text{// backward} \\
x_2 = x_3 - x_1 \Rightarrow & x_2 \in & [x_2] \cap ([x_3] - [x_1]) & \text{// backward}
\end{array}
$$

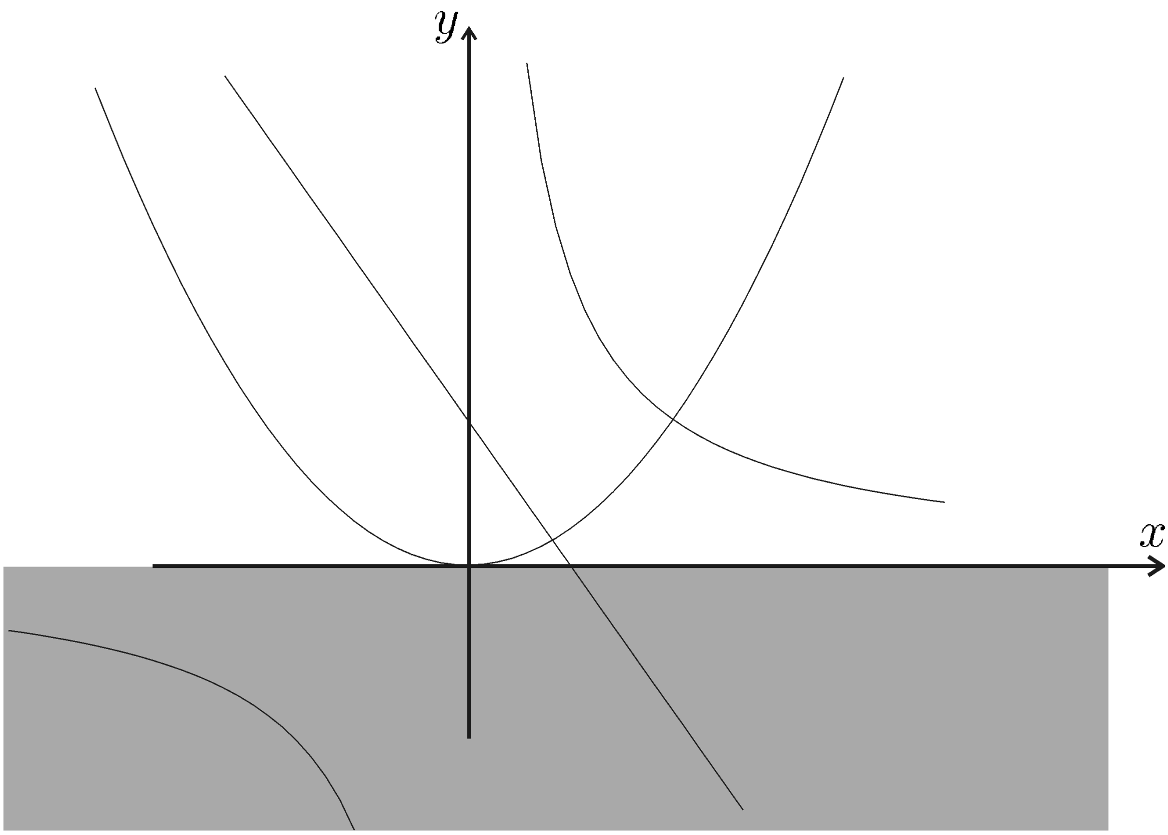
The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

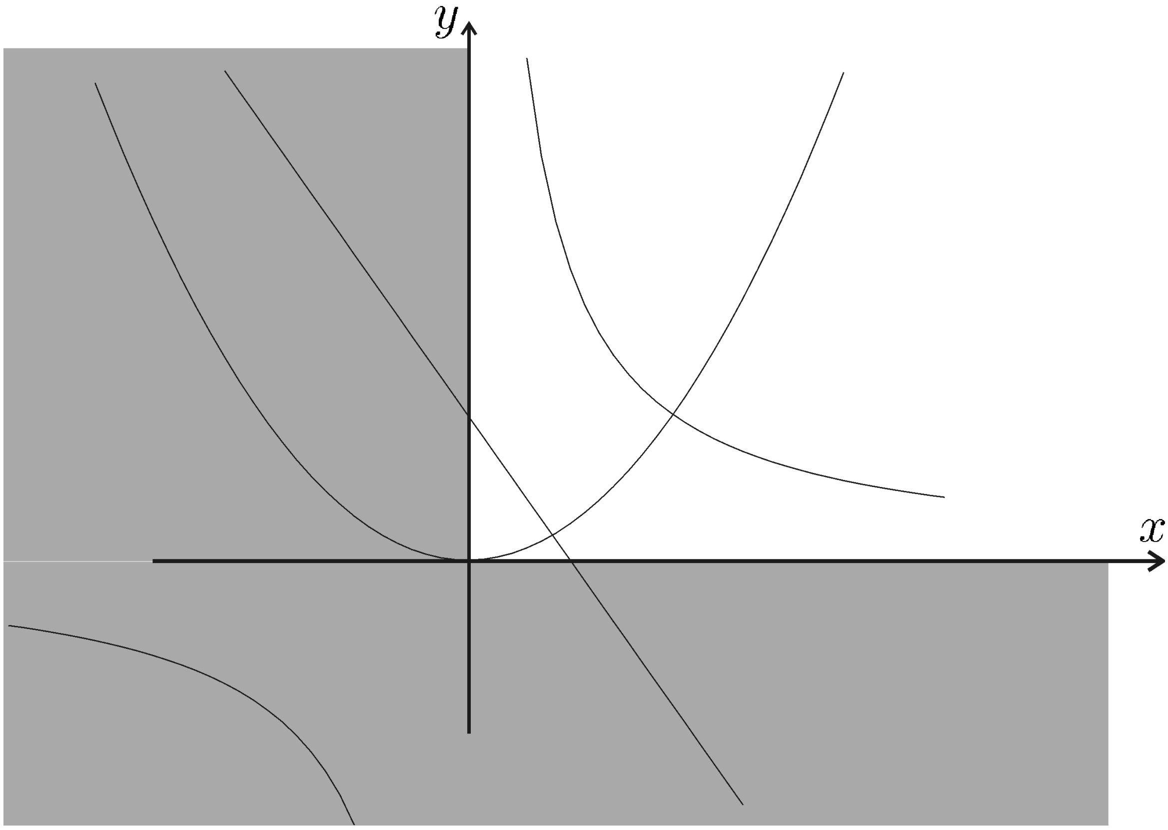
Consider the following problem

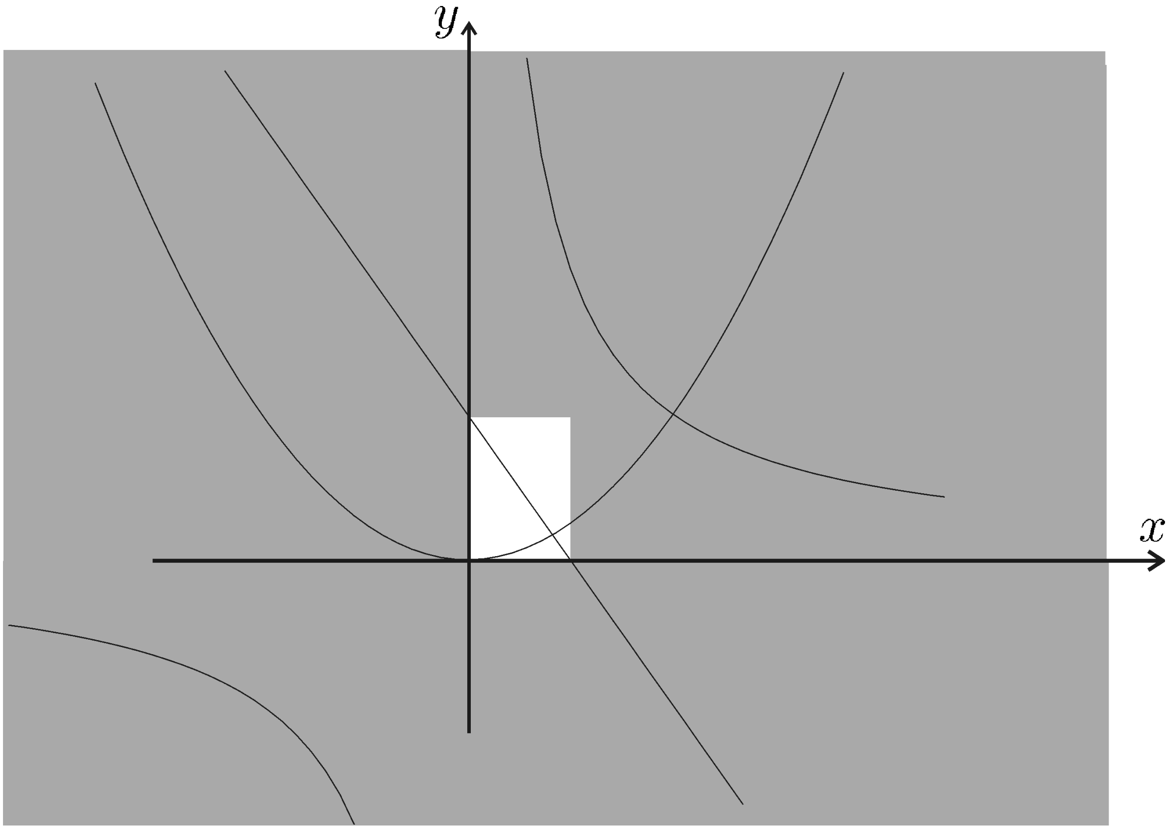
$$\begin{cases} (C_1): & y = x^2 \\ (C_2): & xy = 1 \\ (C_3): & y = -2x + 1 \end{cases}$$

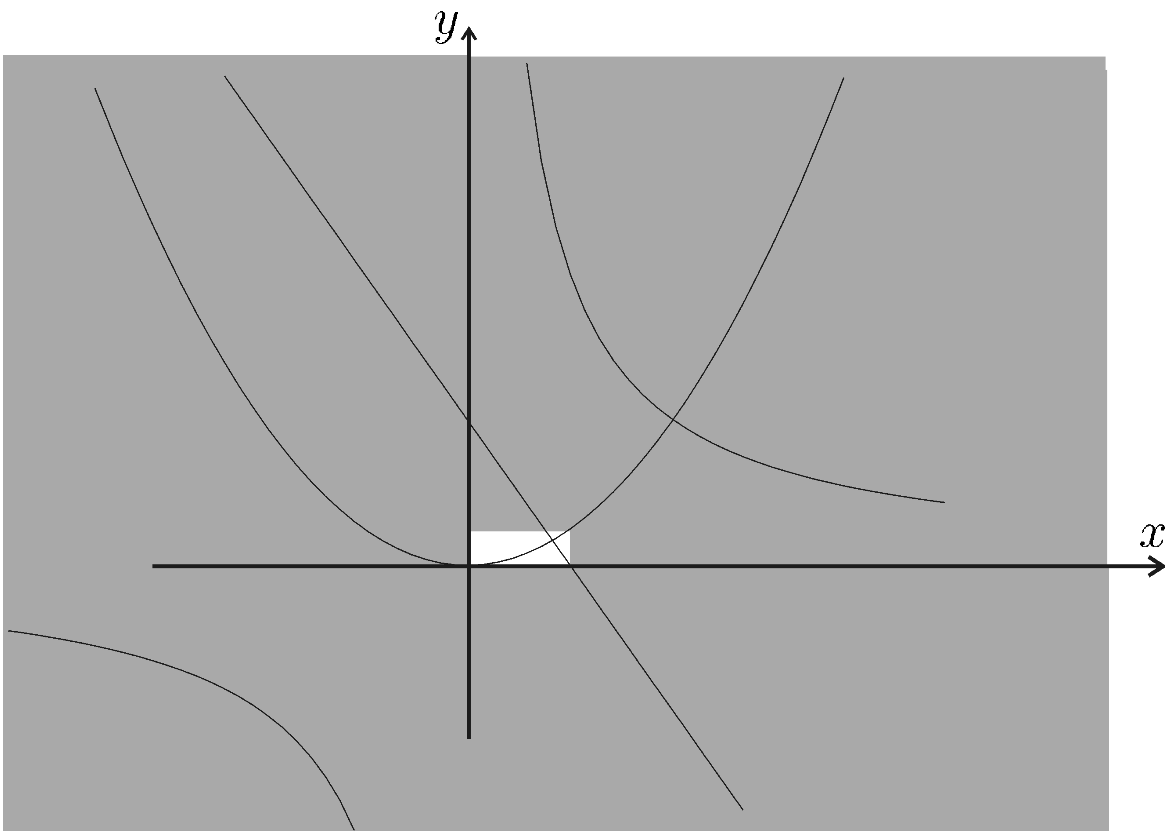

y
x

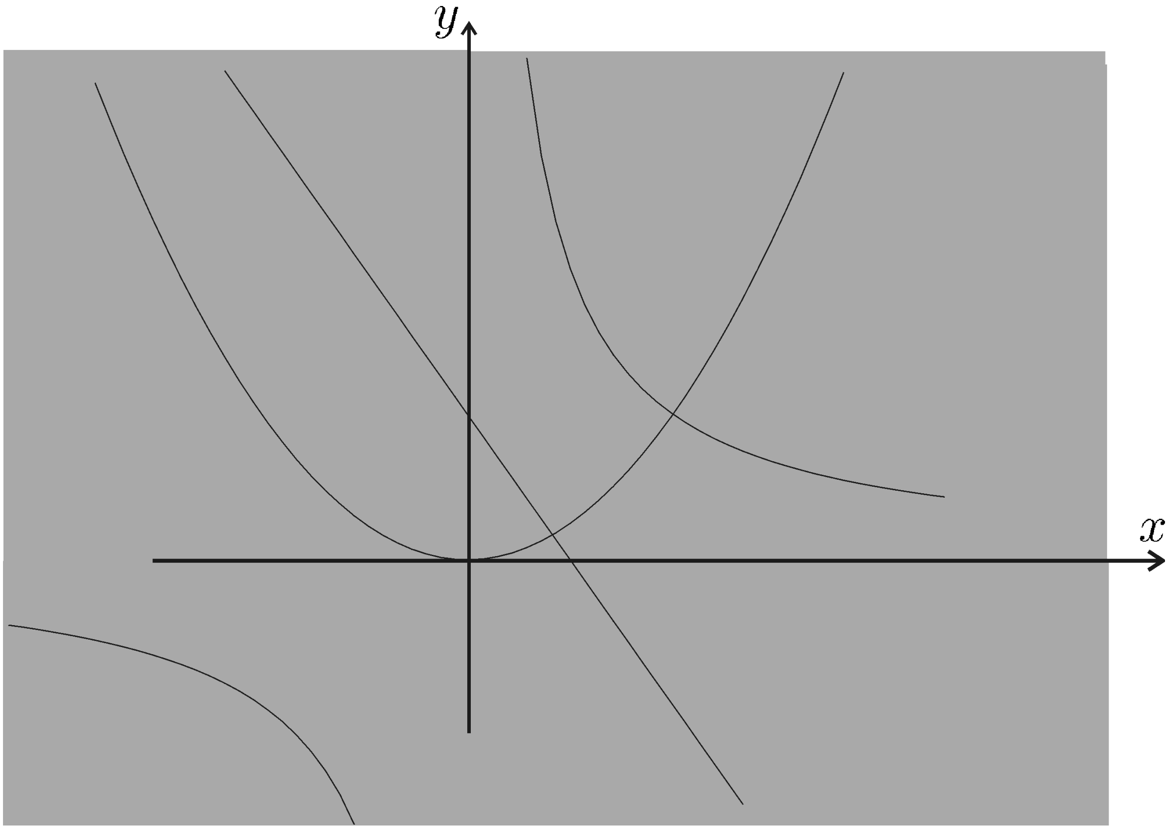


y
x


y
x


y
x


y
x


y
x

$$
\begin{array}{lll}
(C_1) \Rightarrow & y \in & [-\infty, \infty]^2 \;=\; [0, \infty] \\
(C_2) \Rightarrow & x \in & 1/[0, \infty] \;= [0, \infty] \\
(C_3) \Rightarrow & y \in & [0, \infty] \;\cap\; ((-2)\,.[0, \infty] + 1) \\
 & & = [0, \infty] \;\cap\; ([-\infty, 1]) = [0, 1] \\
 & x \in & [0, \infty] \;\cap\; (-[0, 1]/2 + 1/2) = [0, \frac{1}{2}] \\
(C_1) \Rightarrow & y \in & [0, 1] \cap [0, 1/2]^2 = [0, 1/4] \\
(C_2) \Rightarrow & x \in & [0, 1/2] \;\cap\; 1/[0, 1/4] = \emptyset \\
 & y \in & [0, 1/4] \cap 1/\emptyset = \emptyset
\end{array}
$$

2 Interval trajectories

A trajectory is a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$. For instance

$$\mathbf{f}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

is a trajectory.

Order relation

$$\mathbf{f} \leq \mathbf{g} \Leftrightarrow \forall t, \forall i, f_i\left(t\right) \leq g_i\left(t\right)$$

We have

$$\begin{aligned} \mathbf{h} &= \mathbf{f} \quad \wedge \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i\left(t\right) = \mathsf{min}\left(f_i\left(t\right), g_i\left(t\right)\right), \\ \mathbf{h} &= \mathbf{f} \quad \vee \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i\left(t\right) = \mathsf{max}\left(f_i\left(t\right), g_i\left(t\right)\right). \end{aligned}$$

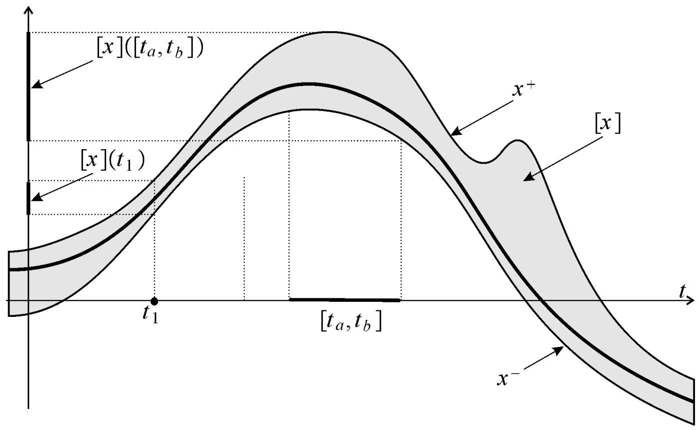


Figure 1: The tube $[x](t)$ encloses the trajectory $x(t)$

The set of all trajectories is a lattice and interval of trajectories (or tubes) can thus be defined.

Example.

$$[\mathrm{f}](t)=\left(\begin{array}{c}\cos t+\left[0,t^2\right]\\ \sin t+[-1,1]\end{array}\right)$$

is an interval trajectory (or tube).

3 Tube arithmetics

If $[x]$ and $[y]$ are two scalar tubes, we have for all t

$$[z] = [x] + [y] \Rightarrow [z](t) = [x](t) + [y](t) \qquad \text{(sum)}$$
$$[z] = \text{shift}_a([x]) \Rightarrow [z](t) = [x](t+a) \qquad \text{(shift)}$$
$$[z] = [x] \circ [y] \Rightarrow [z](t) = [x]([y](t)) \qquad \text{(compositc}$$
$$[z] = \int [x] \Rightarrow [z](t) = \left[\int_0^t x^-(\tau)\,d\tau, \int_0^t x^+(\tau)\,d\tau\right] \qquad \text{(integral)}$$

4 Tube contractors

Tube arithmetic allows us to build contractors.

Consider for instance the differential constraint

$$\begin{array}{rcl} \dot{x}(t) & = & x(t+1) \cdot u(t), \\ x(t) & \in & [x](t), \dot{x}(t) \in [\dot{x}](t), u(t) \in [u](t) \end{array}$$

We decompose as follows

$$\left\{ \begin{array}{rcl} x(t) & = & x_0 + \int_0^t y(\tau) d\tau \\ y(t) & = & a(t) \cdot u(t). \\ a(t) & = & x(t+1) \end{array} \right.$$

Possible contractors are

$$\begin{cases} [x](t) &= [x](t) \cap \left(x_0 + \int_0^t [y](\tau)\, d\tau \right) \\ [y](t) &= [y](t) \cap [a](t) \cdot [u](t) \\ [u](t) &= [u](t) \cap \frac{[y](t)}{[a](t)} \\ [a](t) &= [a](t) \cap \frac{[y](t)}{[u](t)} \\ [a](t) &= [a](t) \cap [x](t+1) \\ [x](t) &= [x](t) \cap [a](t-1) \end{cases}$$

Example. Consider $x(t) \in [x](t)$ with the constraint

$$\forall t,\ x(t) = x(t+1)$$

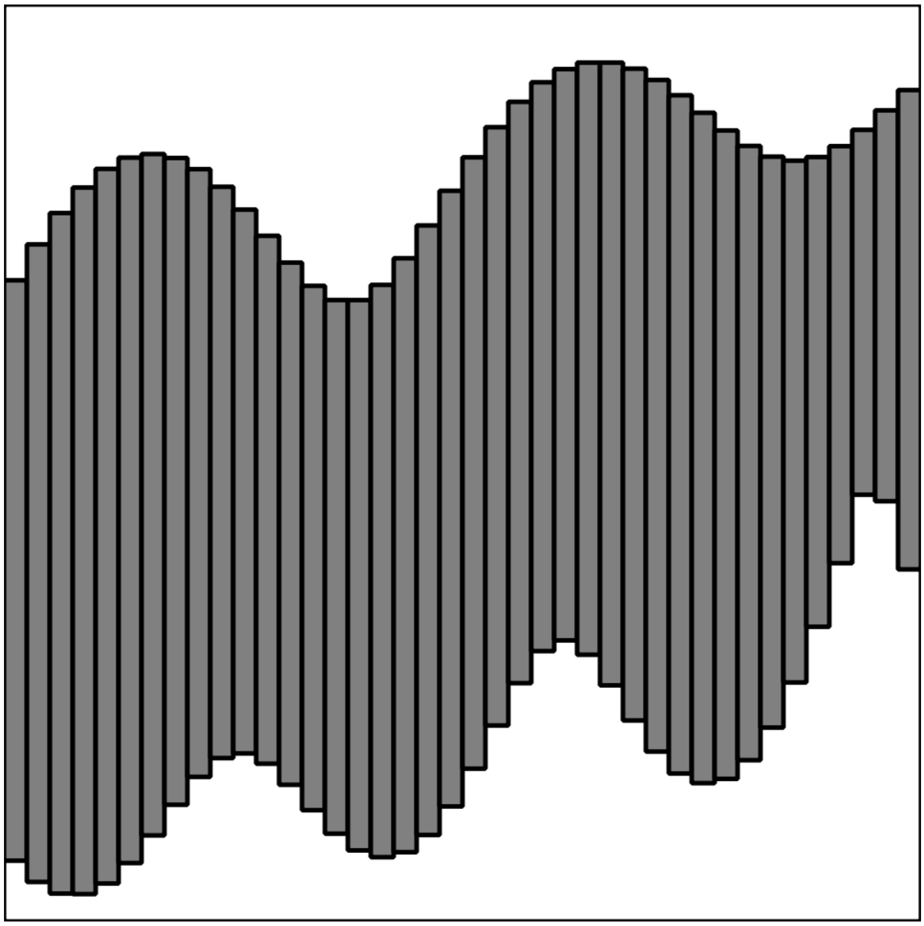
Contract the tube $[x](t)$.

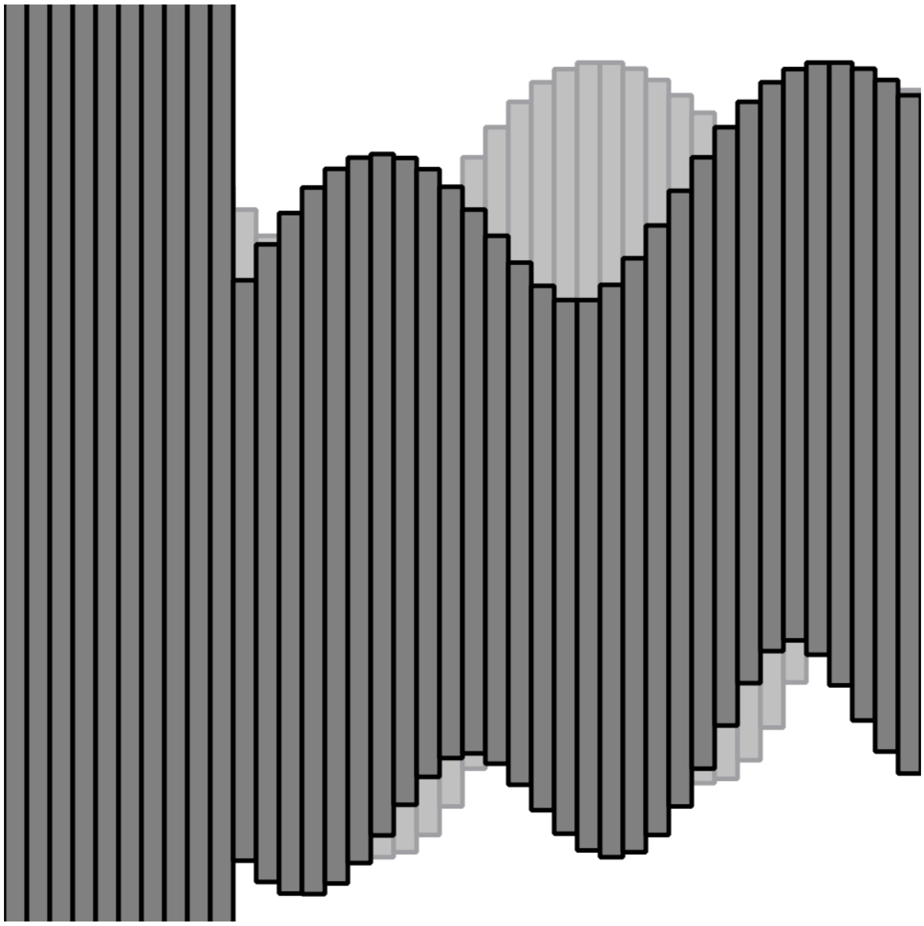
We first decompose into primitive trajectory constraints

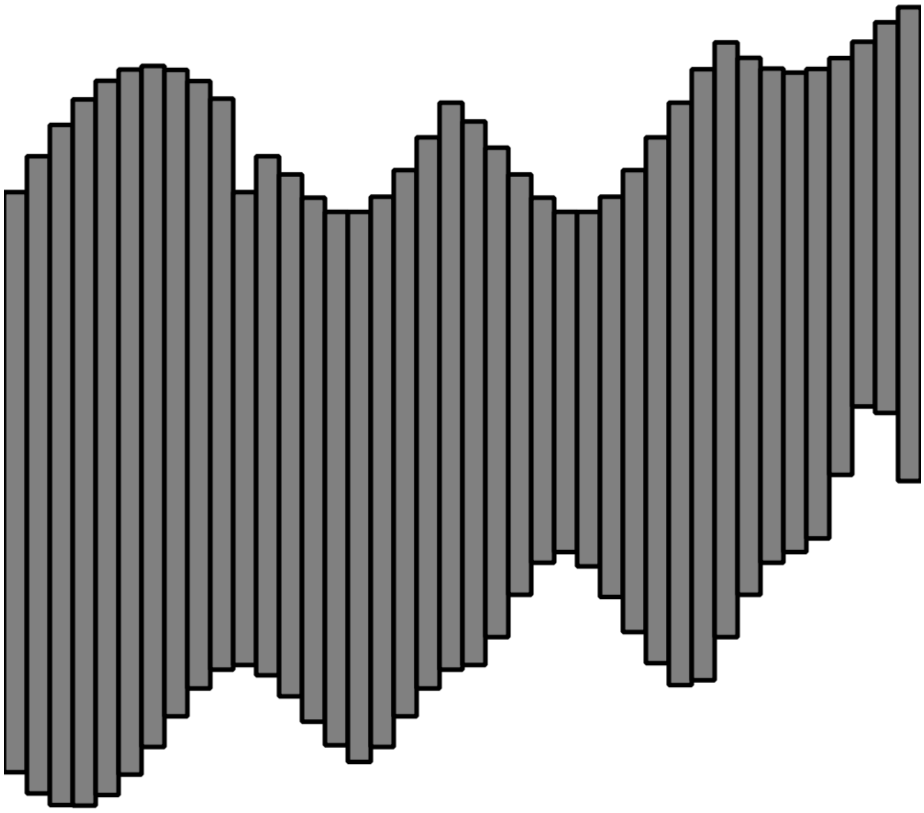
$$\begin{aligned} x(t) &= a(t+1) \\ x(t) &= a(t). \end{aligned}$$

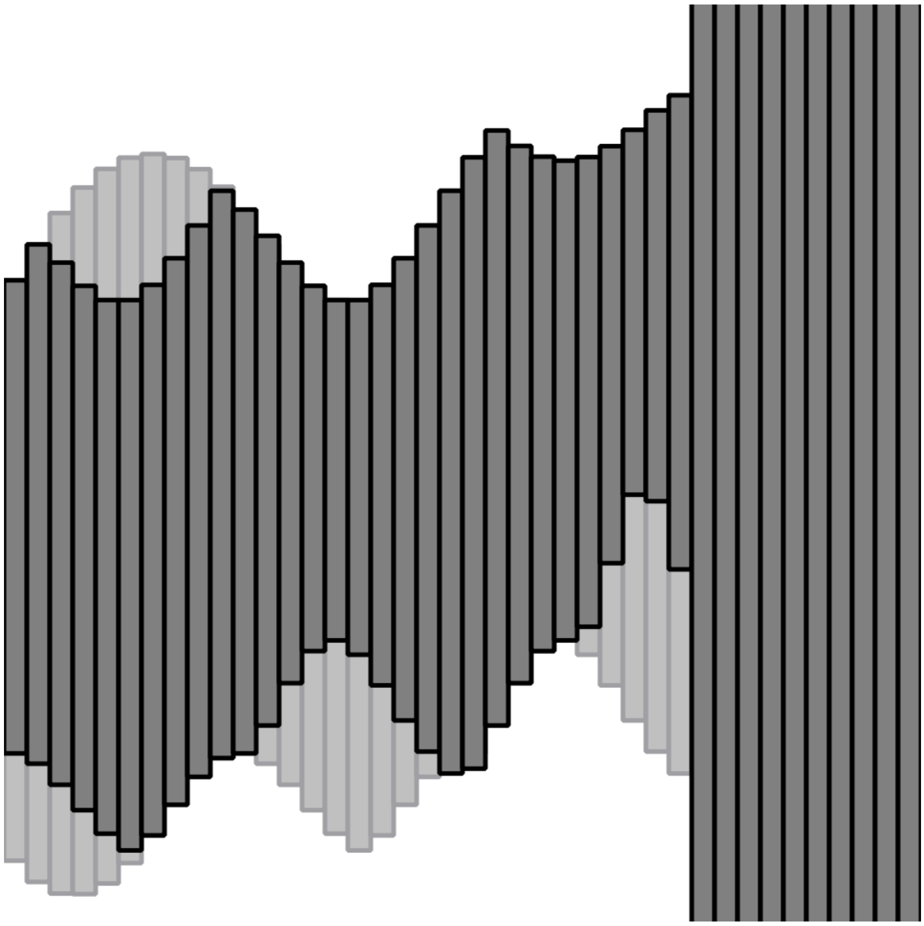
Contractors

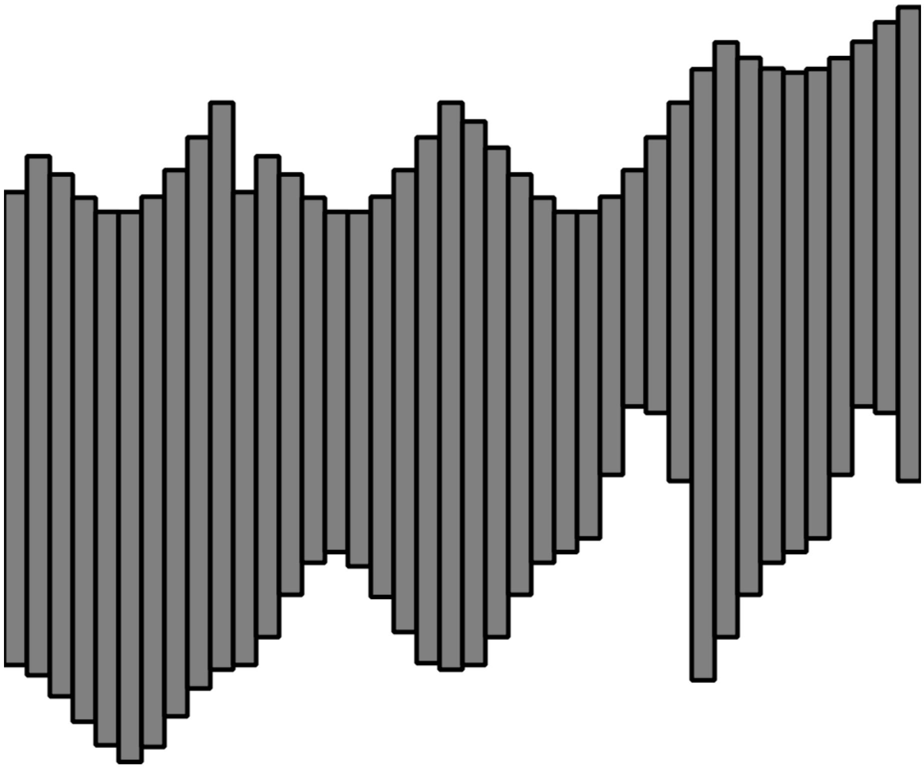
$$
\begin{aligned}
[x](t) &:= [x](t) \cap [a](t+1) \\
[a](t) &:= [a](t) \cap [x](t-1) \\
[x](t) &:= [x](t) \cap [a](t) \\
[a](t) &:= [a](t) \cap [x](t)
\end{aligned}
$$

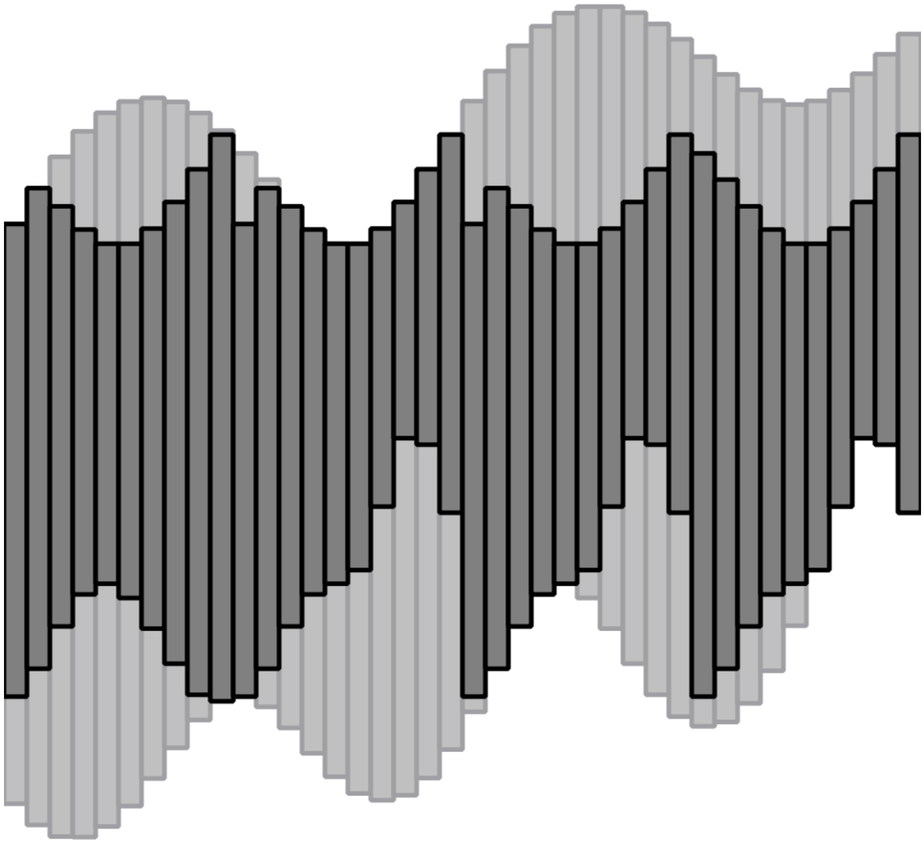


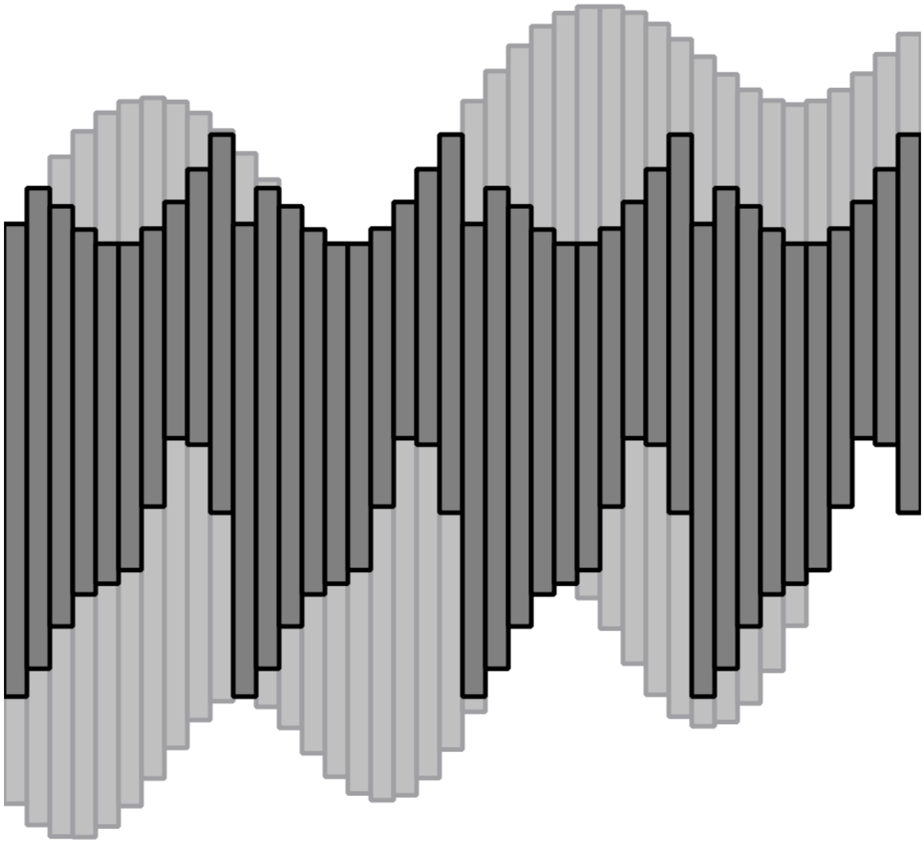












5 State estimation with intertemporal measurements

Classical state estimation

$$\left\{ \begin{array}{lcll} \dot{\mathbf{x}}\left(t\right) & = & \mathbf{f}\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) & t\in\mathbb{R} \\ \mathbf{y}\left(t_i\right) & = & \mathbf{g}\left(\mathbf{x}\left(t_i\right)\right) & t_i\in\mathbb{T}\subset\mathbb{R}. \end{array} \right.$$

With intertemporal measurements

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{y}(t_1, t_2) & = & \mathbf{g}(\mathbf{x}(t_1), \mathbf{x}(t_2)) & (t_1, t_2) \in \mathbb{T} \subset \mathbb{R} \times \mathbb{R}. \end{cases}$$

6 Mass spring problem

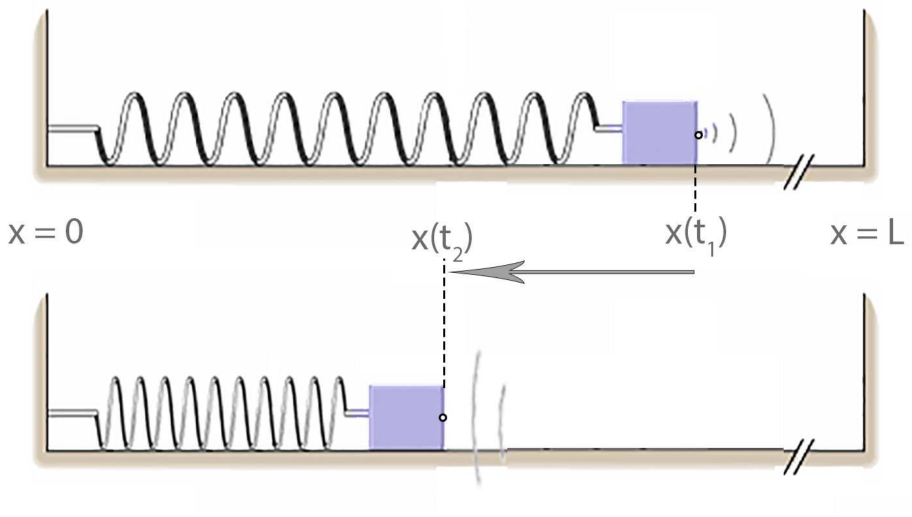
The mass spring satisfies the differential equation

$$\ddot{x} + \dot{x} + x - x^3 = 0$$

i.e.

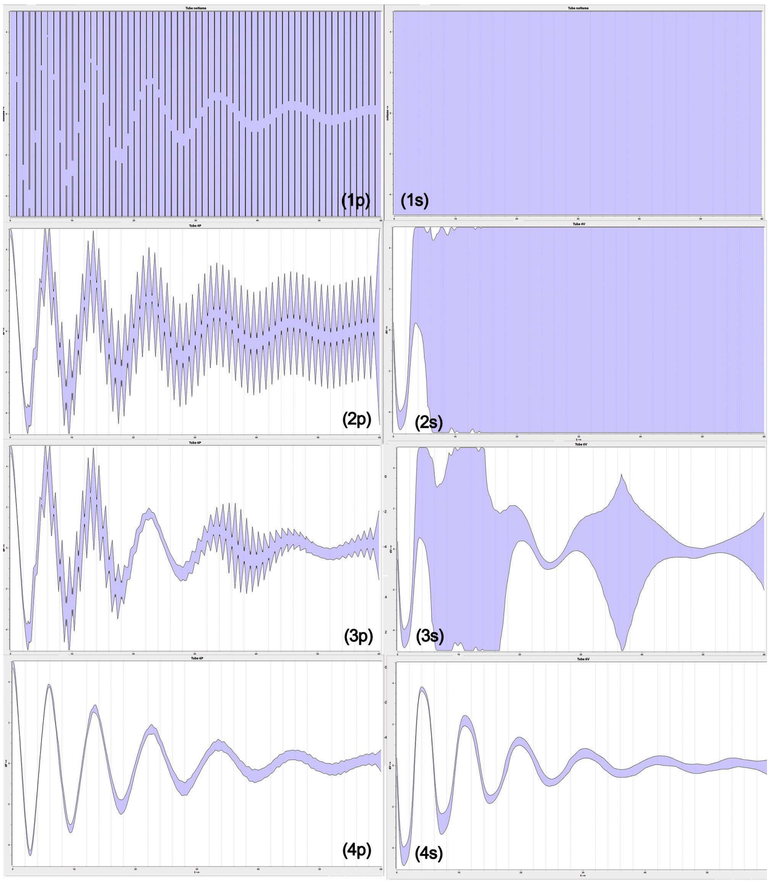
$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_2 - x_1 + x_1^3 \end{cases}$$

The initial state is unknown.

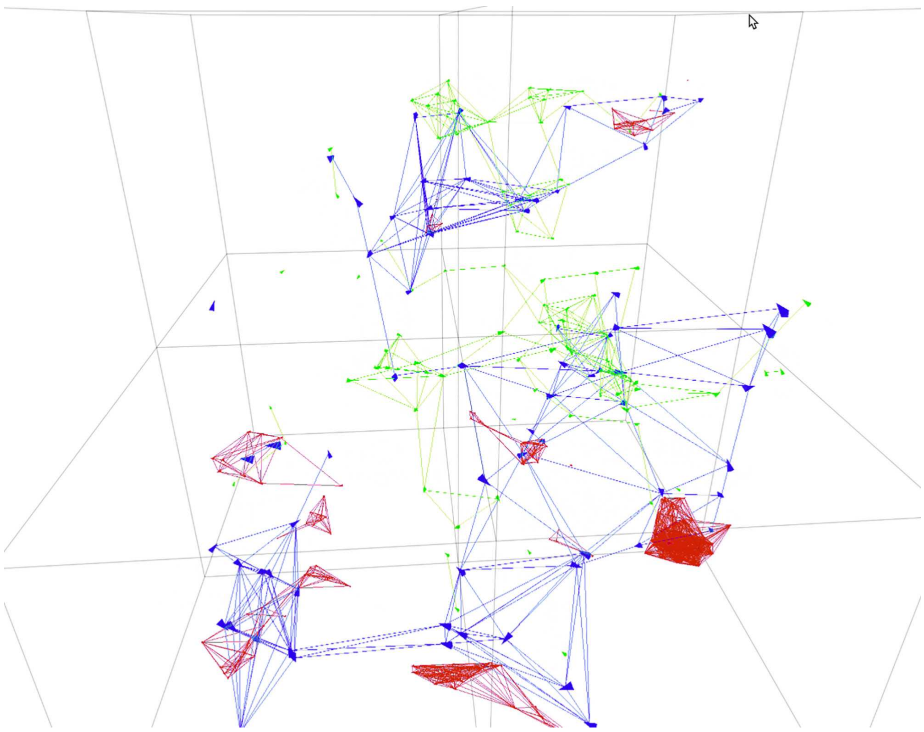


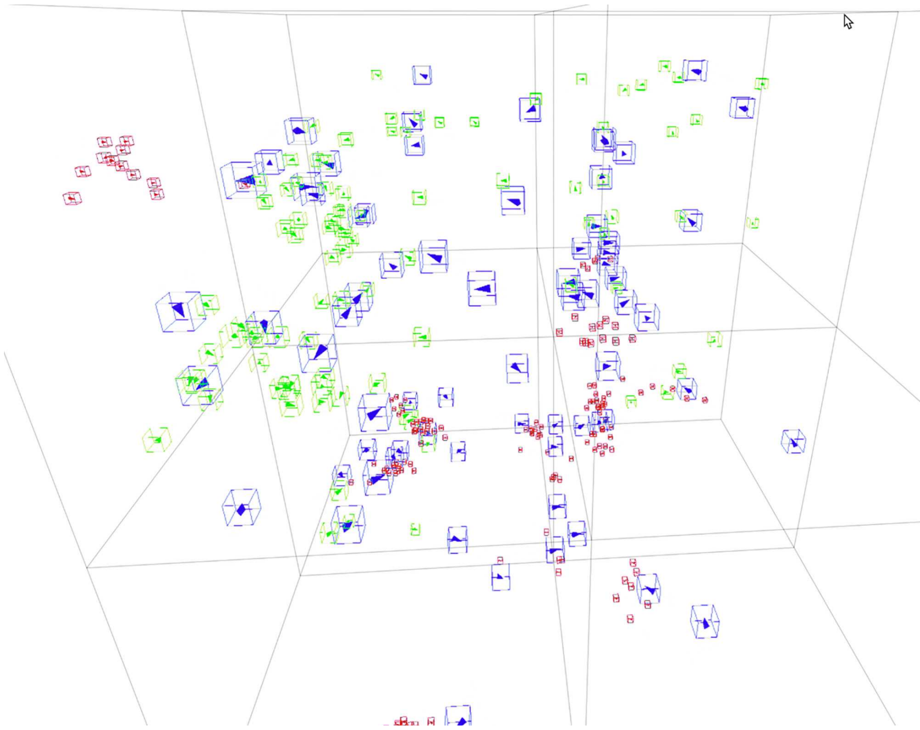
x = 0
x(t_2)
x(t_1)
x = L

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - x_1 + x_1^3 \\ L - x_1(t_1) + L - x_1(t_2) = c(t_2 - t_1). \end{cases}$$


(1p)
(1s)
(2p)
(2s)
(3p)
(3s)
(4p)
(4s)

7 Swarm localization





References

A. Bethencourt and L. Jaulin (2013). Cooperative localization of underwater robots with unsynchronized clocks, *Journal of Behavioral Robotics*, Volume 4, Issue 4, pp 233-244, pdf.

A. Bethencourt and L. Jaulin (2014). Solving non-linear constraint satisfaction problems involving time-dependant functions. *Mathematics in Computer Science*.