Intervals for state estimation

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1 Interval analysis

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2) \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\operatorname{abs}([-7,1]) &= [0,7]
\end{aligned}$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned} [f]([x_1], [x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ &\quad + \sin [x_1] \cdot \sin [x_2] + 2. \end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

2 Set inversion

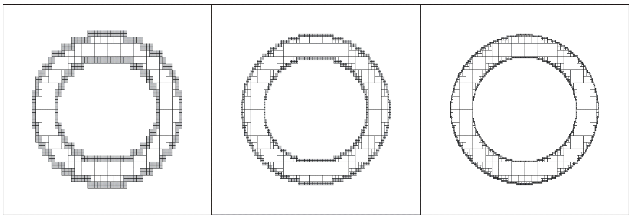
A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.

Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \in [1, 2]\}.$$



Set operations such as $\mathbb{Z} := \mathbb{X} + \mathbb{Y}$, $\mathbb{X} := \mathbf{f}^{-1}(\mathbb{Y})$, $\mathbb{Z} := \mathbb{X} \cap \mathbb{Y} \dots$ can be approximated by subpaving operations.

Let $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and let $\mathbb{Y}$ be a subset of $\mathbb{R}^m$. Set inversion is the characterization of

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

We shall use the following tests.

$$\begin{array}{lll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

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Algorithm Sivia(in: $[\mathbf{x}](0), \mathbf{f}, \mathbb{Y}$)

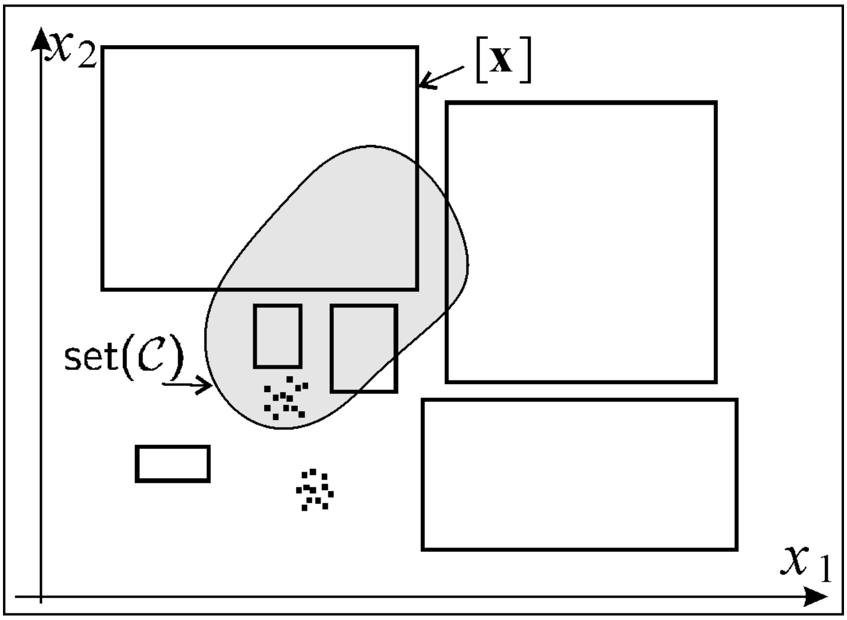
1 $\mathcal{L} := \{[\mathbf{x}](0)\}$;
2 pull $[\mathbf{x}]$ from $\mathcal{L}$;
3 if $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$, draw($[\mathbf{x}]$, 'red');
4 elseif $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$, draw($[\mathbf{x}]$, 'blue');
5 elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$, 'yellow')};
6 else bisect $[\mathbf{x}]$ and push into $\mathcal{L}$;
7 if $\mathcal{L} \neq \emptyset$, go to 2

3 Contractors

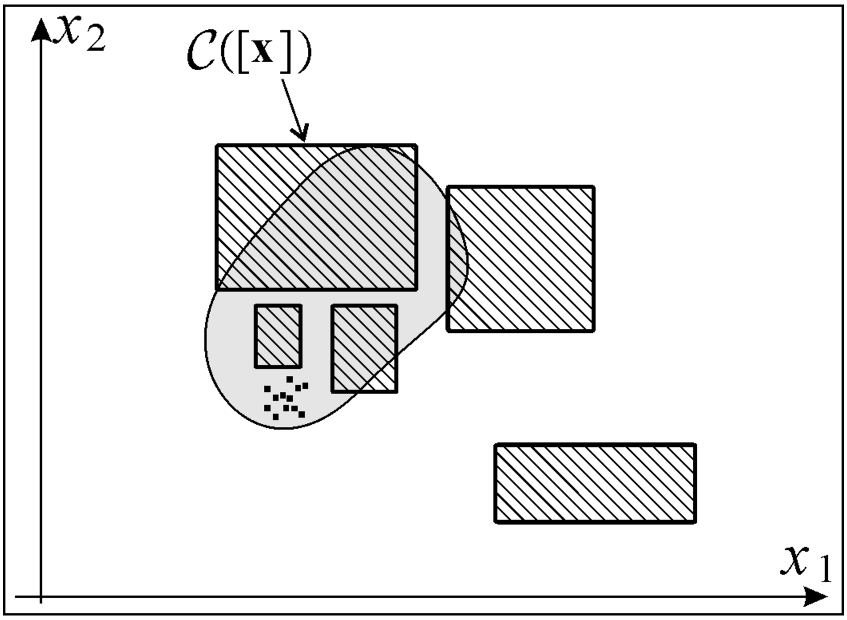
3.1 Definition

The operator $\mathcal{C}_{\mathbb{X}} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* for $\mathbb{X} \subset \mathbb{R}^n$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)}, \\ \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness)}. \end{cases}$$



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
x_1

3.2 Primitive contractors

Let x, y, z be 3 variables such that

$$\begin{aligned} x &\in [-\infty, 5], \\ y &\in [-\infty, 4], \\ z &\in [6, \infty], \\ z &= x + y. \end{aligned}$$

Which values for x, y, z are consistent.

Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

$$
\begin{aligned}
z = x + y \Rightarrow \quad z \in \quad & [6, \infty] \cap ([-\infty, 5] + [-\infty, 4]) \\
& = [6, \infty] \cap [-\infty, 9] = [6, 9]. \\
x = z - y \Rightarrow \quad x \in \quad & [-\infty, 5] \cap ([6, \infty] - [-\infty, 4]) \\
& = [-\infty, 5] \cap [2, \infty] = [2, 5]. \\
y = z - x \Rightarrow \quad y \in \quad & [-\infty, 4] \cap ([6, \infty] - [-\infty, 5]) \\
& = [-\infty, 4] \cap [1, \infty] = [1, 4].
\end{aligned}
$$

The contractor associated with $z = x + y$ is.

Algorithm pplus(inout: $[z], [x], [y]$)
1 $[z] := [z] \cap ([x] + [y])$;
2 $[x] := [x] \cap ([z] - [y])$;
3 $[y] := [y] \cap ([z] - [x])$.

The projection procedure developed for plus can be extended to other ternary constraints such as mult: $z = x \cdot y$, or equivalently

$$\text{mult} \triangleq \left\{ (x, y, z) \in \mathbb{R}^3 \mid z = x \cdot y \right\}.$$

The resulting projection procedure becomes

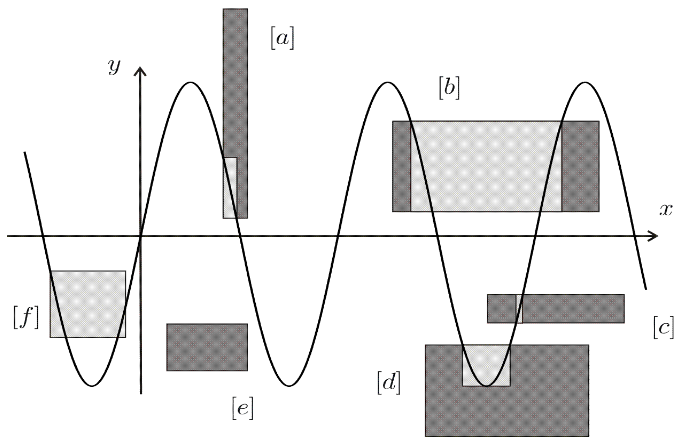
Algorithm pmult(inout: $[z], [x], [y]$)
1 $[z] := [z] \cap ([x] \cdot [y])$;
2 $[x] := [x] \cap ([z] \cdot 1/[y])$;
3 $[y] := [y] \cap ([z] \cdot 1/[x])$.

Consider the binary constraint

$$\mathsf{exp} \triangleq \{(x, y) \in \mathbb{R}^n | y = \exp(x)\}.$$

The associated contractor is

Algorithm pexp(inout: $[y], [x]$)
1 $[y] := [y] \cap \exp([x])$;
2 $[x] := [x] \cap \log([y])$.


[a]
y
[b]
x
[f]
[c]
[d]
[e]

3.3 Solvers

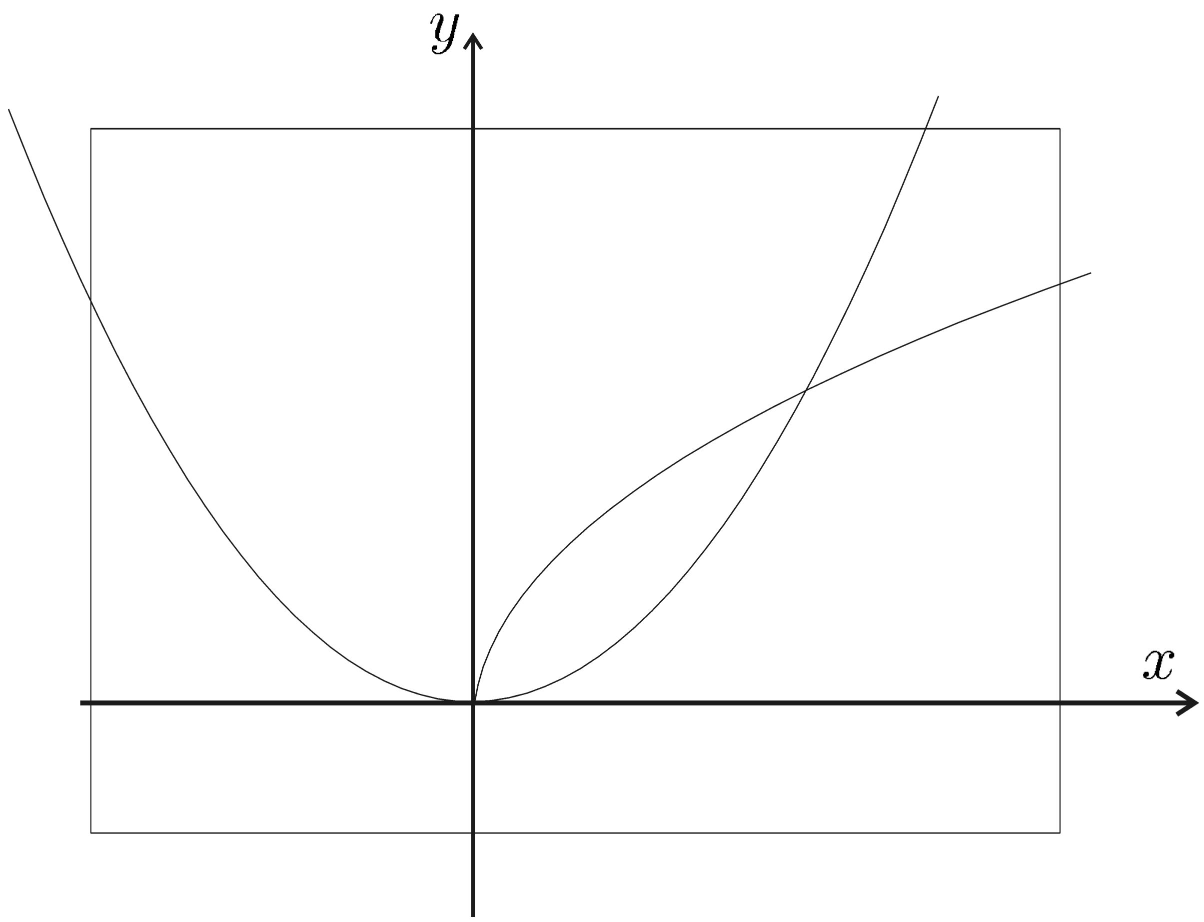
Example. Consider the system.

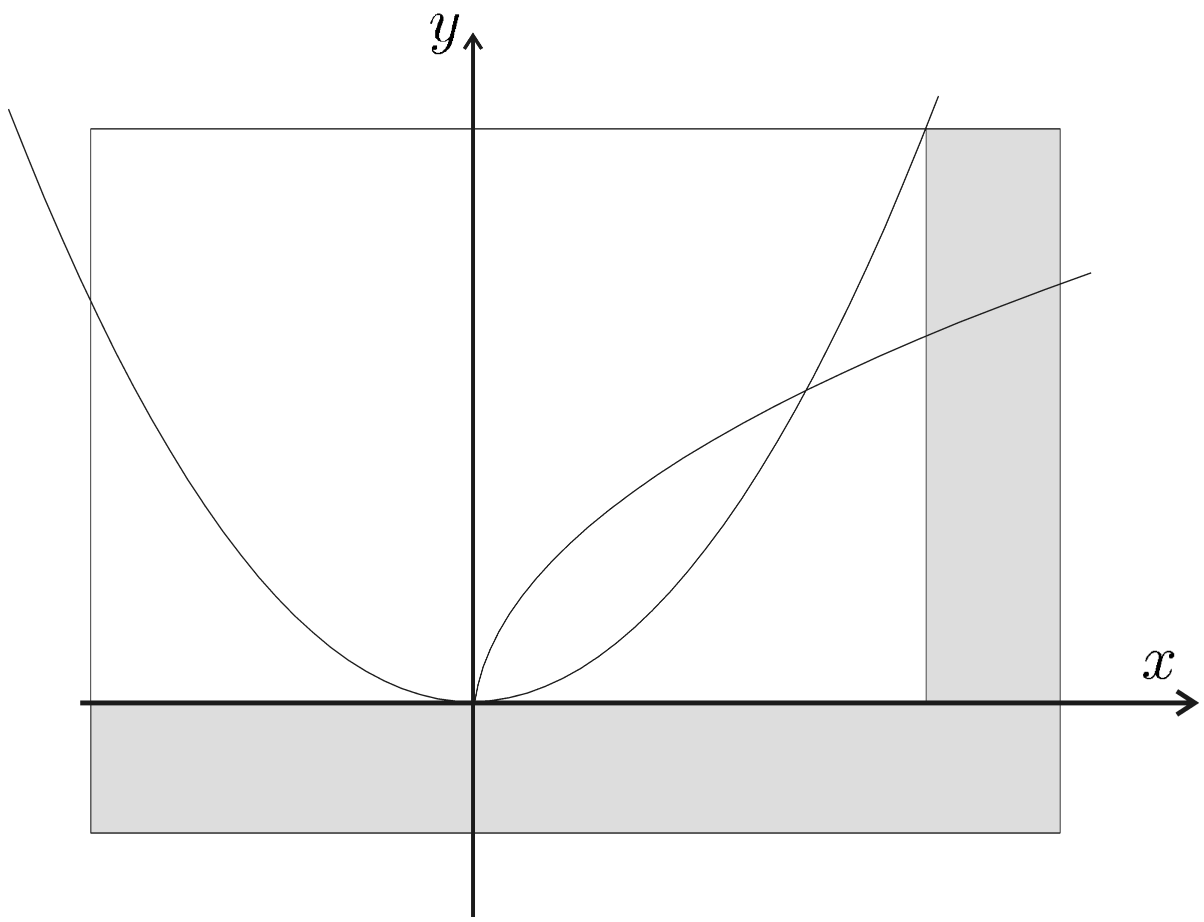
$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

We build two contractors

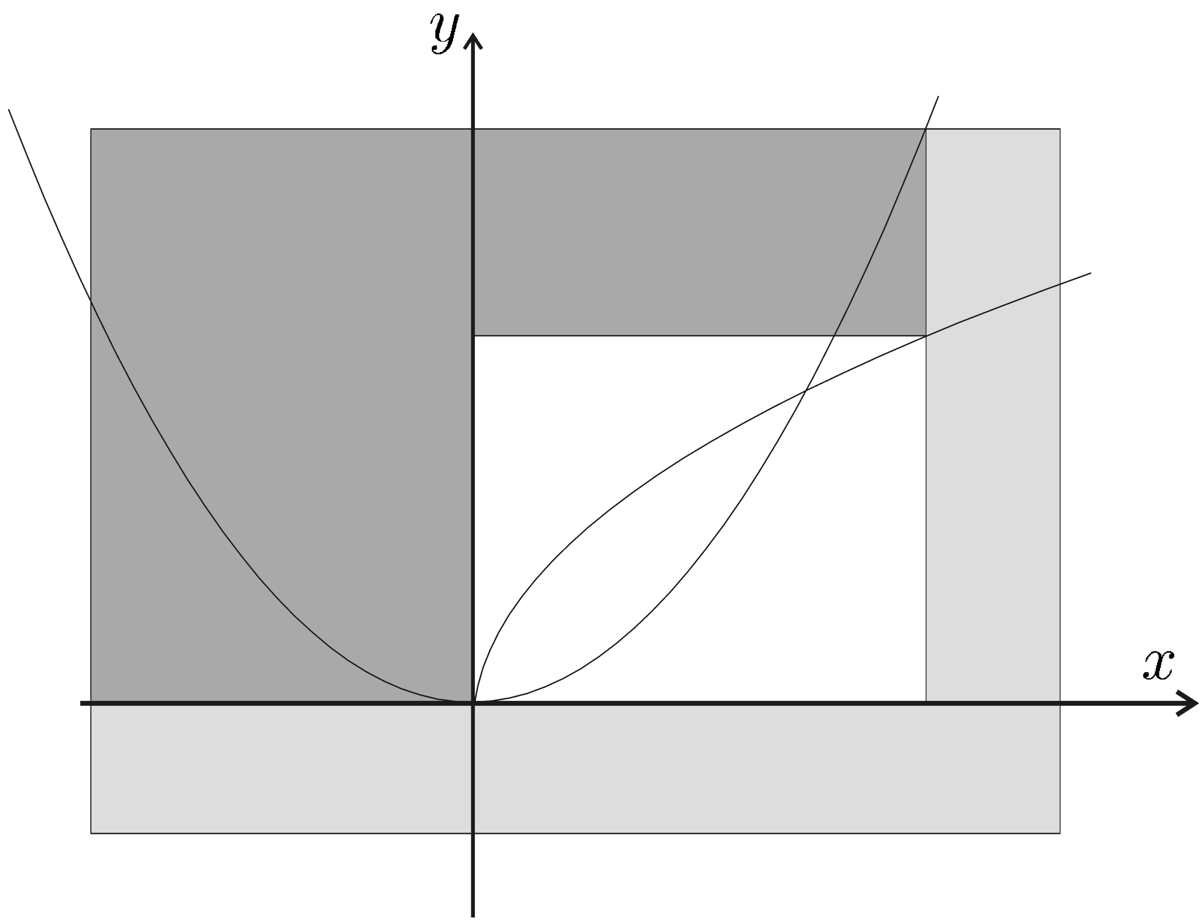
$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated to } y = x^2$$

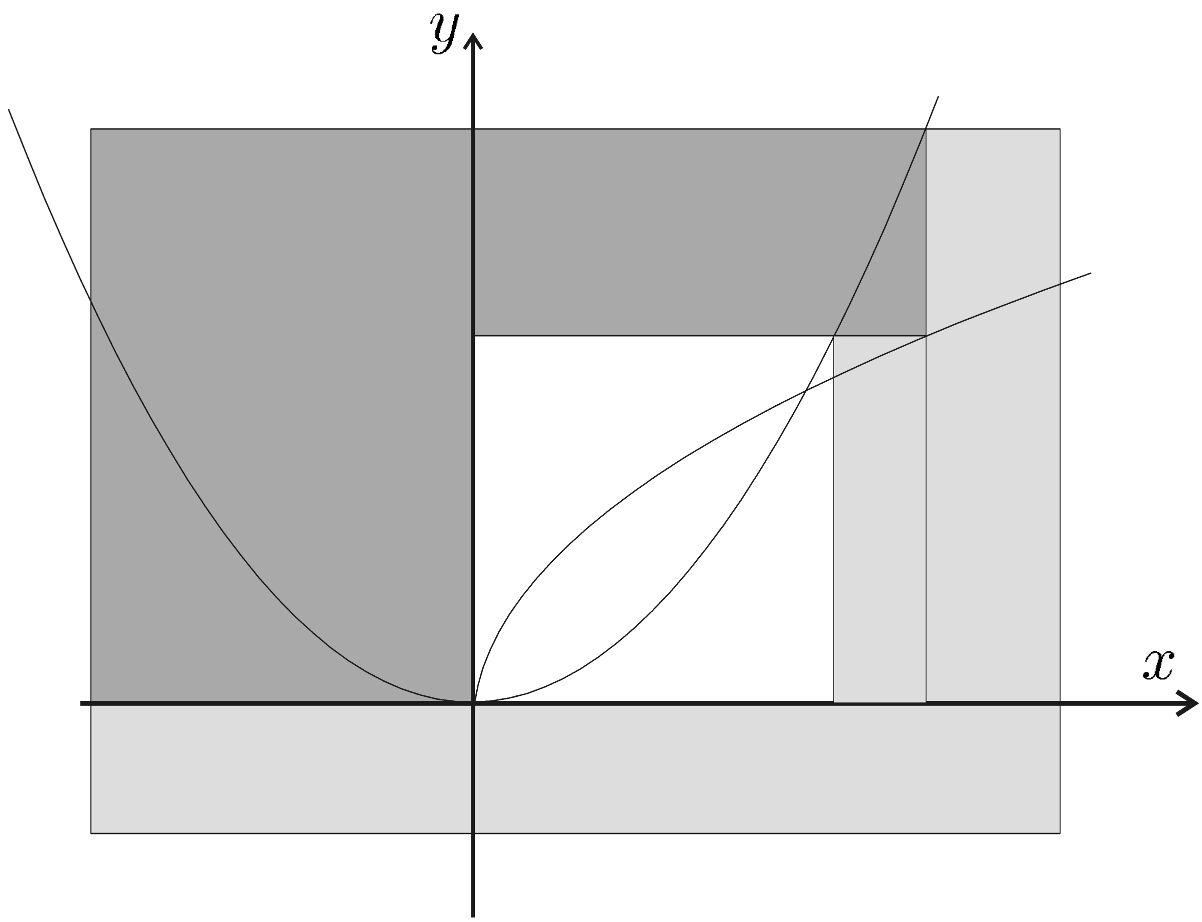
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated to } y = \sqrt{x}$$

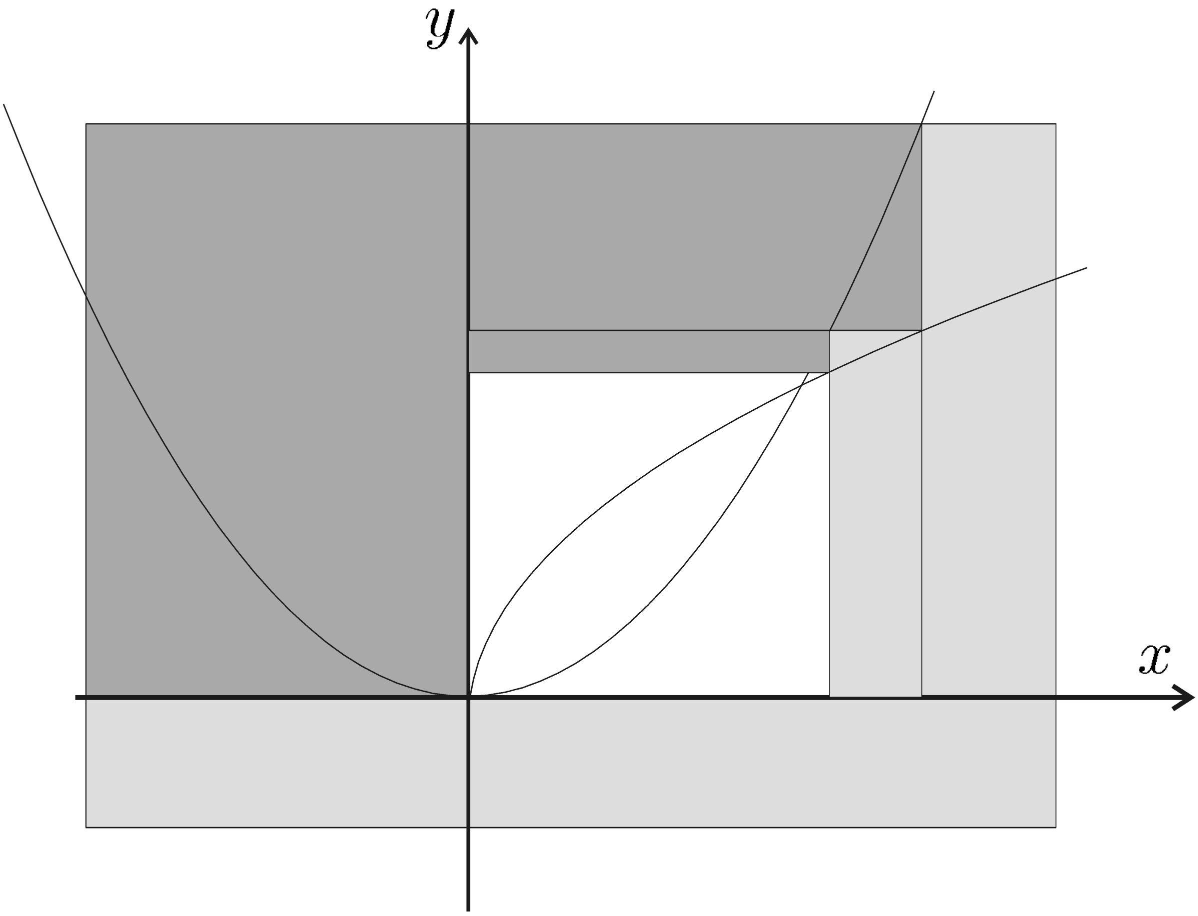

y
x

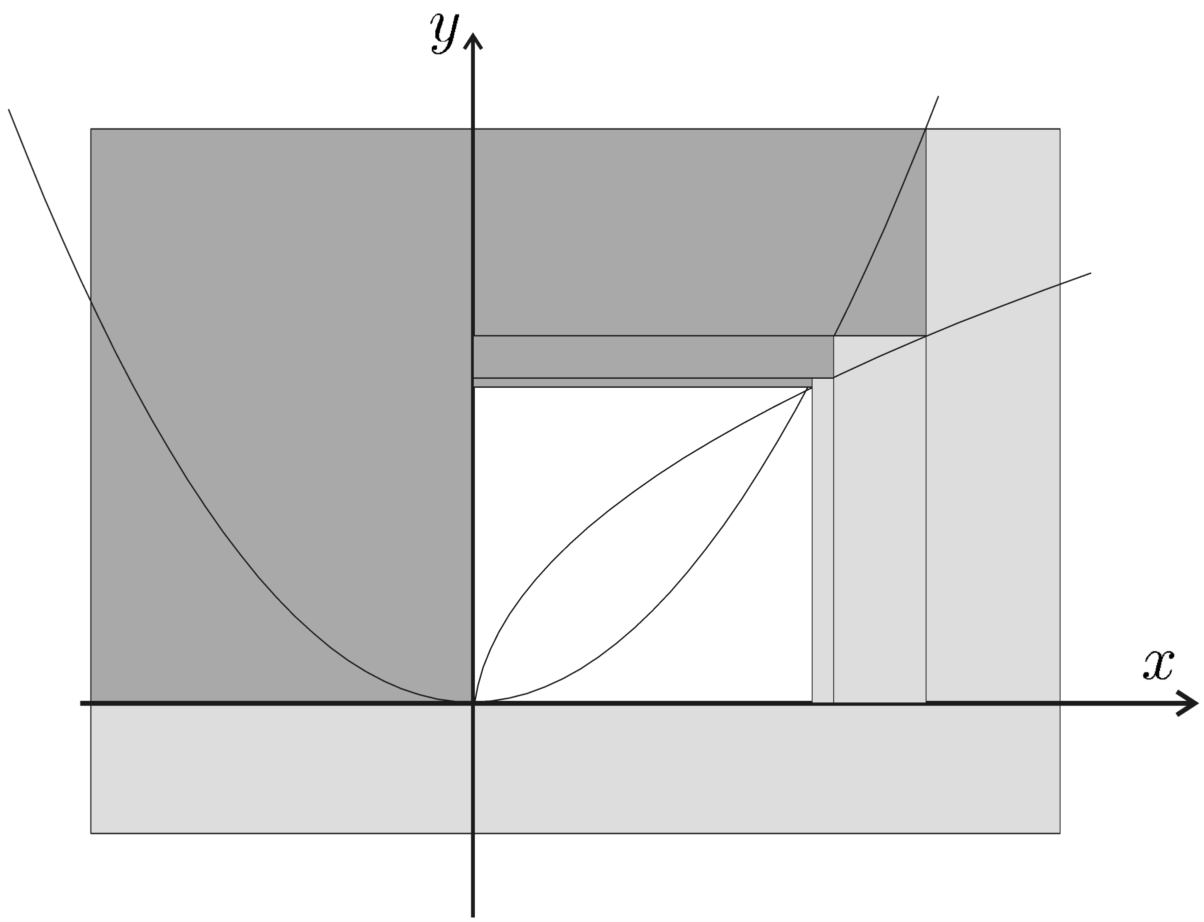


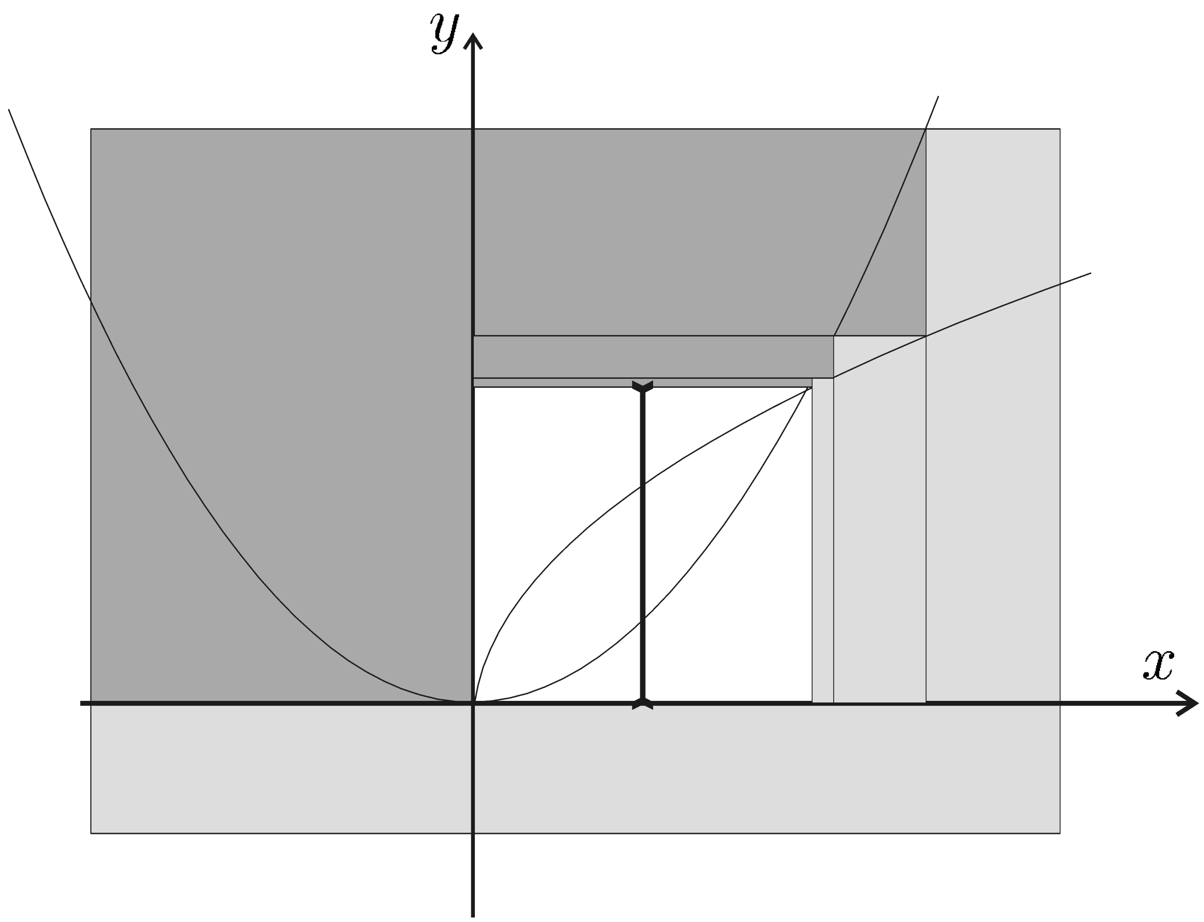
y
x

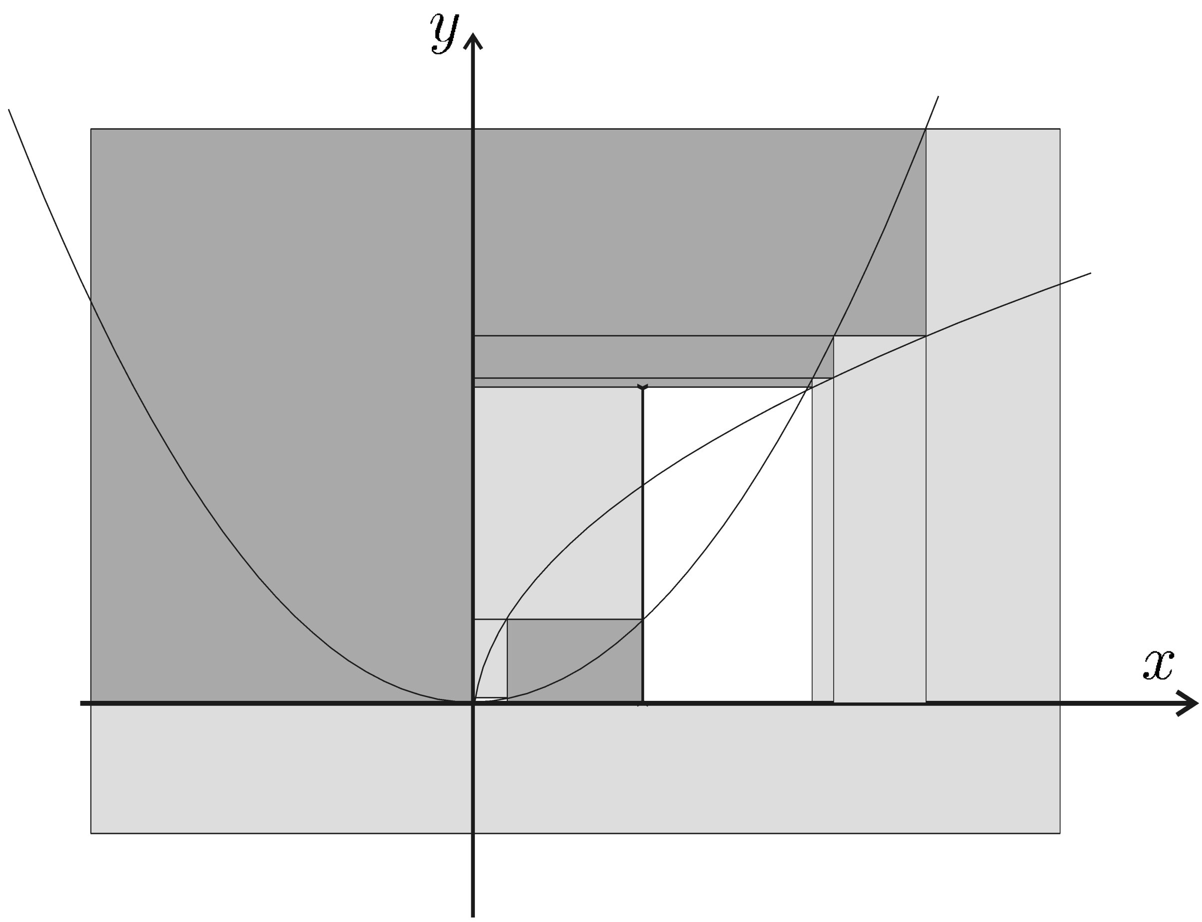

y
x

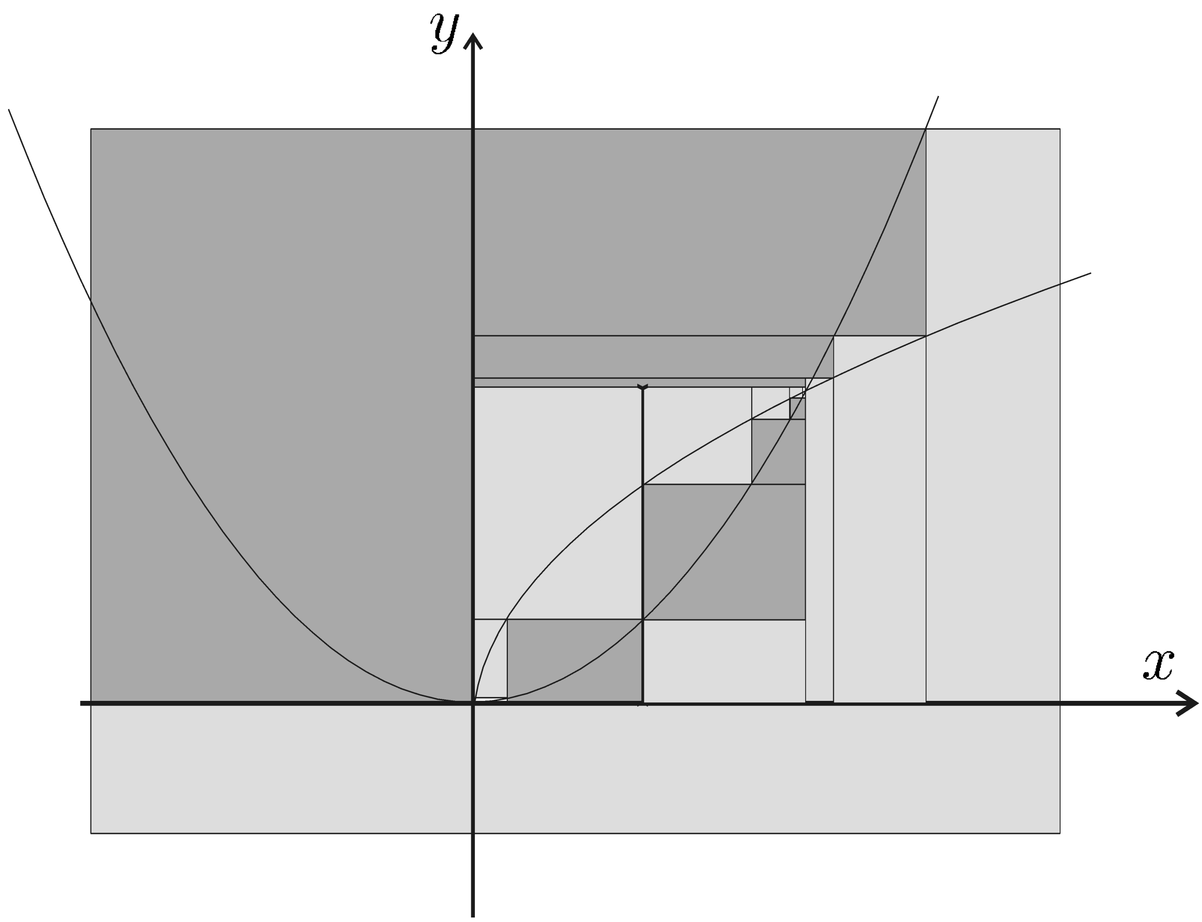

y
x


y
x


y
x


y
x


y
x



y
x

3.4 Decomposition into primitive constraints

$$x + \mathsf{sin}(xy) \leq 0,$$
$$x \in [-1, 1], y \in [-1, 1]$$

How to contract?

$$x + \mathsf{sin}(xy) \leq 0,$$
$$x \in [-1, 1], y \in [-1, 1]$$

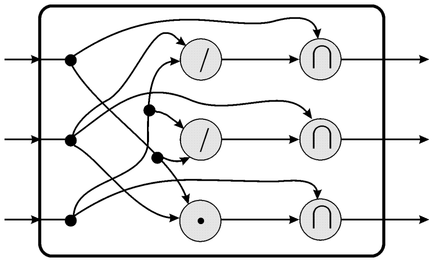
can be decomposed into

$$\begin{cases} a = xy \\ b = \mathsf{sin}(a) \\ c = x + b \end{cases}, \quad \begin{array}{ll} x \in [-1, 1] & a \in [-\infty, \infty] \\ y \in [-1, 1] & b \in [-\infty, \infty] \\ & c \in [-\infty, 0] \end{array}$$

4 Matrices-contractors-algebra

$$\begin{array}{ccc} \text{linear application} & \longrightarrow & \text{matrices} \\ \mathcal{L}: \left\{ \begin{array}{lcl} \alpha & = & 2a + 3h \\ \gamma & = & h - 5a \end{array} \right. & \longrightarrow & \mathbf{A} = \left(\begin{array}{cc} 2 & 3 \\ 1 & -5 \end{array} \right) \end{array}$$

We have a matrix algebra and Matlab.
We have: $\text{var}(\mathcal{L}) = \{a, h\}$, $\text{covar}(\mathcal{L}) = \{\alpha, \gamma\}$.
But we cannot write: $\text{var}(\mathbf{A}) = \{a, h\}$, $\text{covar}(\mathbf{A}) = \{\alpha, \gamma\}$.

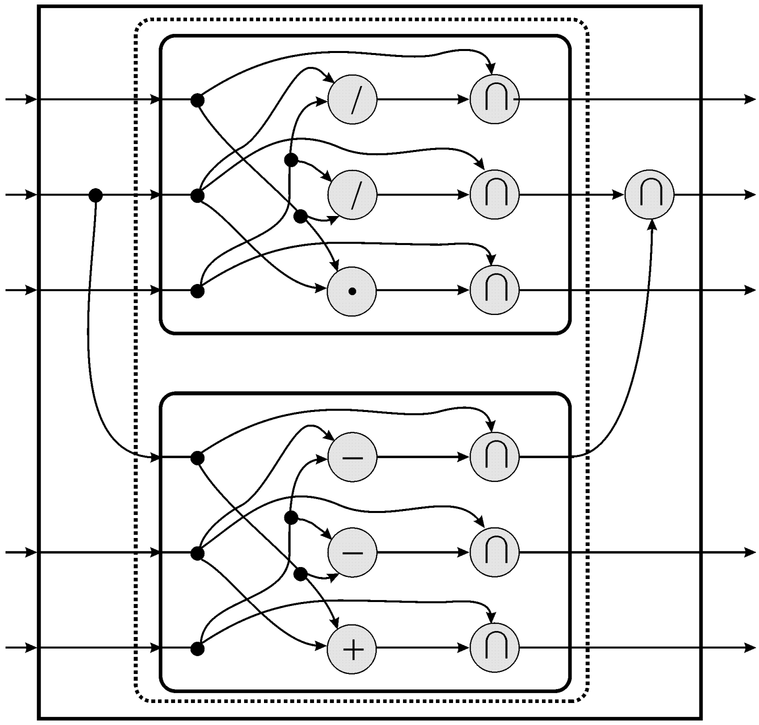
constraint	$\longrightarrow$	contractor
$a \cdot b = z$	$\longrightarrow$	

Contractor fusion

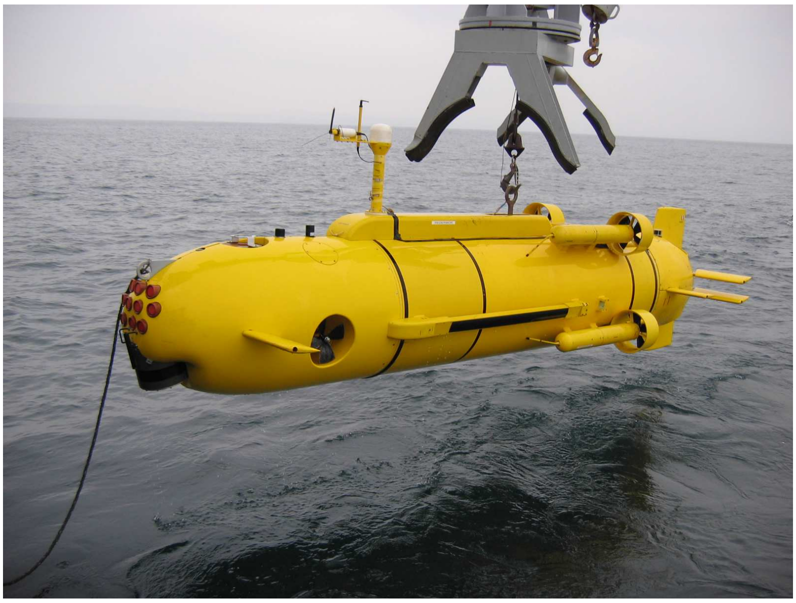
$$\begin{cases} a \cdot b = z & \rightarrow \ \mathcal{C}_1 \\ b + c = d & \rightarrow \ \mathcal{C}_2 \end{cases}$$

Since b occurs in both constraints, we fuse the two contractors as:

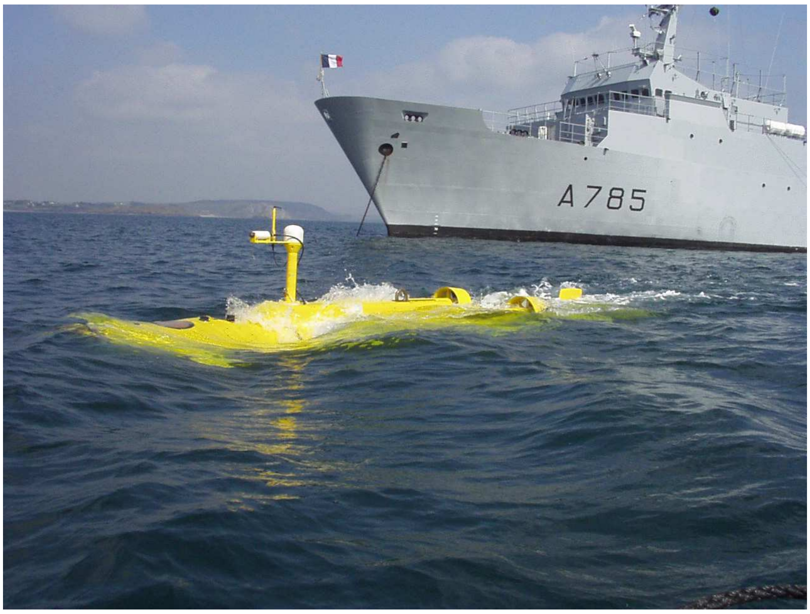
$$\begin{aligned} \mathcal{C} &= \mathcal{C}_1 \times \mathcal{C}_2 \rfloor_{(2,1)} \\ &= \mathcal{C}_1 | \mathcal{C}_2 \text{ (for short)} \end{aligned}$$



5 Underwater SLAM



The *Redermor*, GESMA



The *Redermor* at the surface

Why choosing an interval constraint approach for SLAM ?

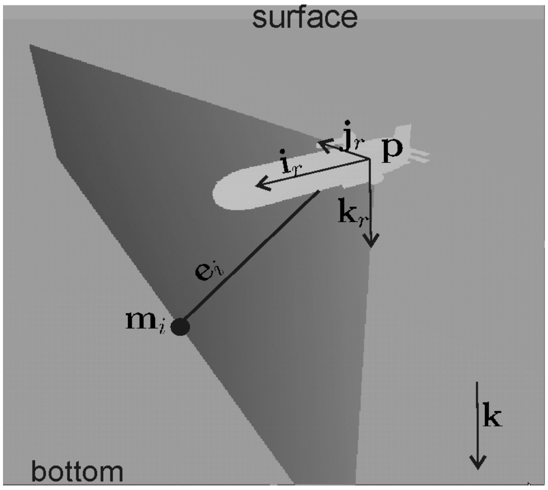
1) A reliable method is needed.
2) The model is nonlinear.
3) The pdf of the noises are unknown.
4) Reliable error bounds are provided by the sensors.
5) A huge number of redundant data are available.

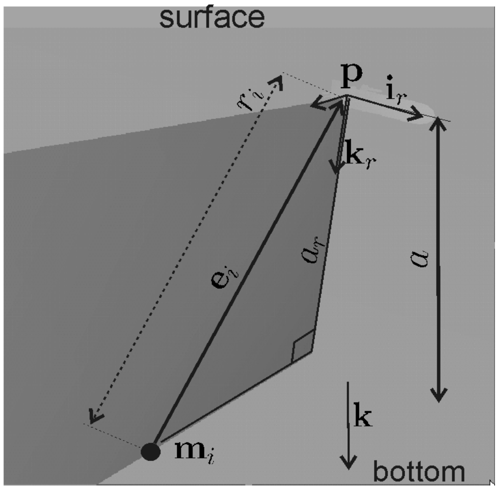
5.1 Sensors

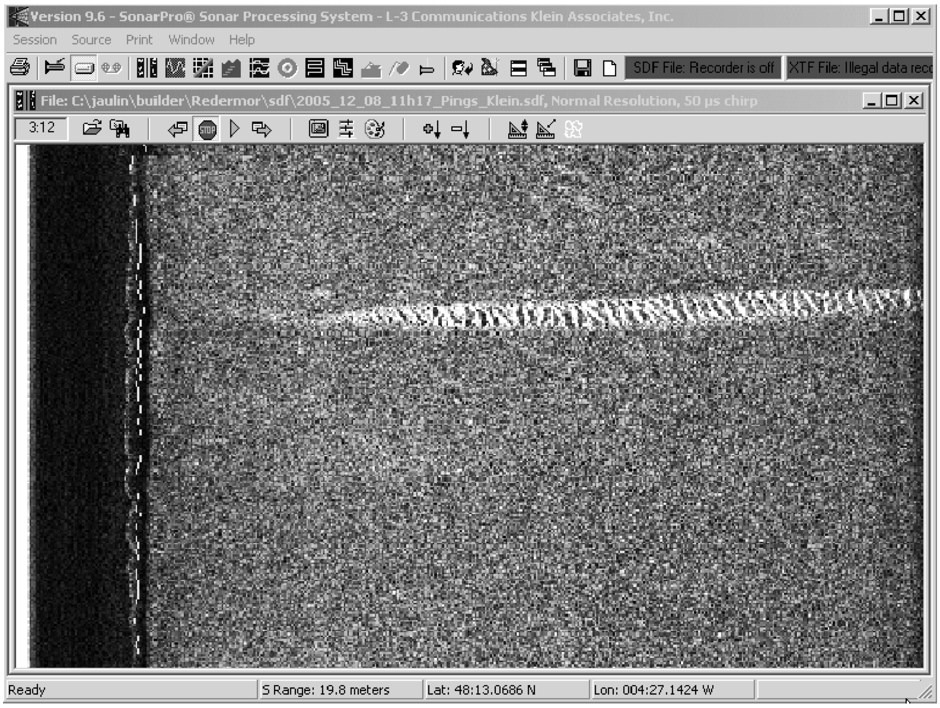
A GPS (Global positioning system) at the surface only.

$$t_0 = 6000 \text{ s}, \quad \ell^0 = (-4.4582279^\circ, 48.2129206^\circ) \pm 2.5m$$
$$t_f = 12000 \text{ s}, \quad \ell^f = (-4.4546607^\circ, 48.2191297^\circ) \pm 2.5m$$

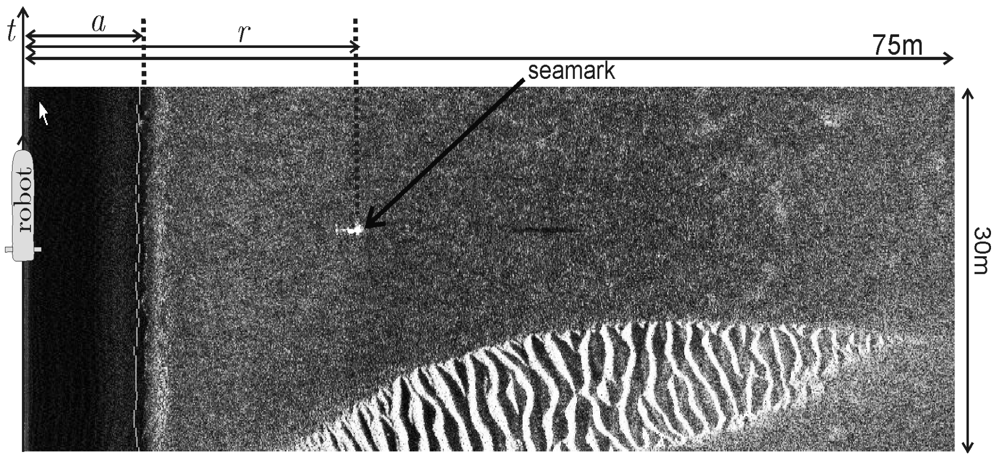
A sonar (KLEIN 5400 side scan sonar). Gives the distance r between the robot to the detected object.







Screenshot of SonarPro



Detection of a mine using SonarPro

A Loch-Doppler. Returns the speed of the robot $\mathbf{v}_r$ and the altitude a of the robot $\pm$ 10cm.

A Gyrocompass (Octans III from IXSEA). Returns the roll ϕ, the pitch θ and the head ψ.

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4}.\,[-1,1] \\ 1.75 \times 10^{-4}.\,[-1,1] \\ 5.27 \times 10^{-3}.\,[-1,1] \end{pmatrix}.$$

5.2 Data

For each time $t \in \{6000.0, 6000.1, 6000.2, \ldots, 11999.4\}$, we get intervals for

$$\phi(t), \theta(t), \psi(t), v_r^x(t), v_r^y(t), v_r^z(t), a(t).$$

Six mines have been detected by the sonar:

i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau(i)$	7054	7092	7374	7748	9038	9688	10024	10817	11172	11232	11279	11688
$\sigma(i)$	1	2	1	0	1	5	4	3	3	4	5	1
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98	37.90	36.71	37.37	31.03	33.51	15.05

5.3 Constraints satisfaction problem

$$t \in \{6000.0, 6000.1, 6000.2, \ldots, 11999.4\},$$

$$i \in \{0, 1, \ldots, 11\},$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) * \frac{\pi}{180}\right) & 0 \end{pmatrix} \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix},$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t)),$$

$$\mathbf{R}_\psi(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{R}_\theta(t) = \begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix},$$

$$\mathbf{R}_{\varphi}(t) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & -\sin\varphi(t) \\ 0 & \sin\varphi(t) & \cos\varphi(t) \end{array} \right),$$

$$\mathbf{R}(t) = \mathbf{R}_{\psi}(t).\mathbf{R}_{\theta}(t).\mathbf{R}_{\varphi}(t),$$

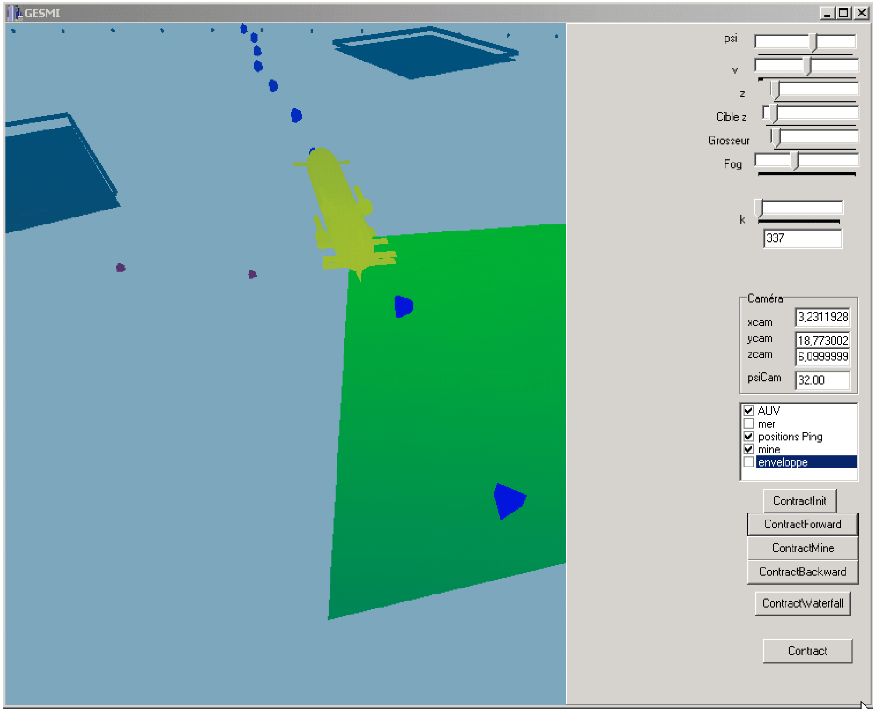
$$\dot{\mathbf{p}}(t) = \mathbf{R}(t).\mathbf{v}_r(t)$$

$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| \ = r(i),$$

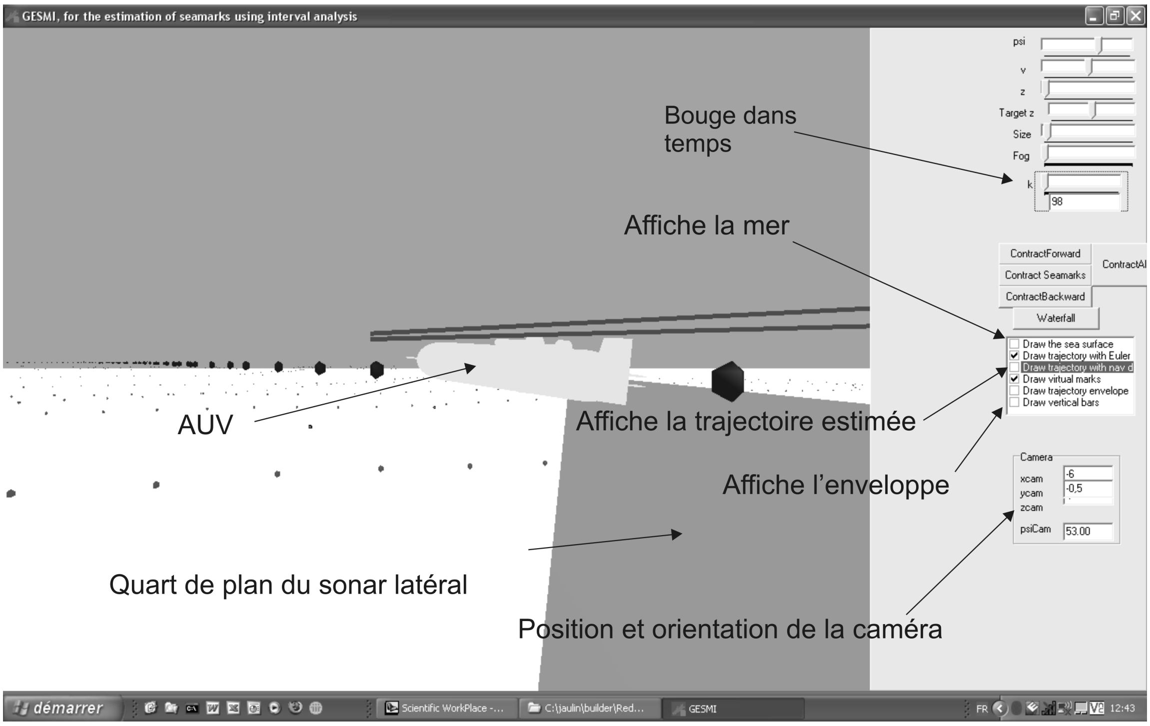
$$\mathbf{R}^{\mathsf{T}}(\tau(i))\,(\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0, \infty]^{\times 2},$$

$$m_z(\sigma(i)) - p_z(\tau(i)) - a(\tau(i)) \in [-0.5, 0.5].$$

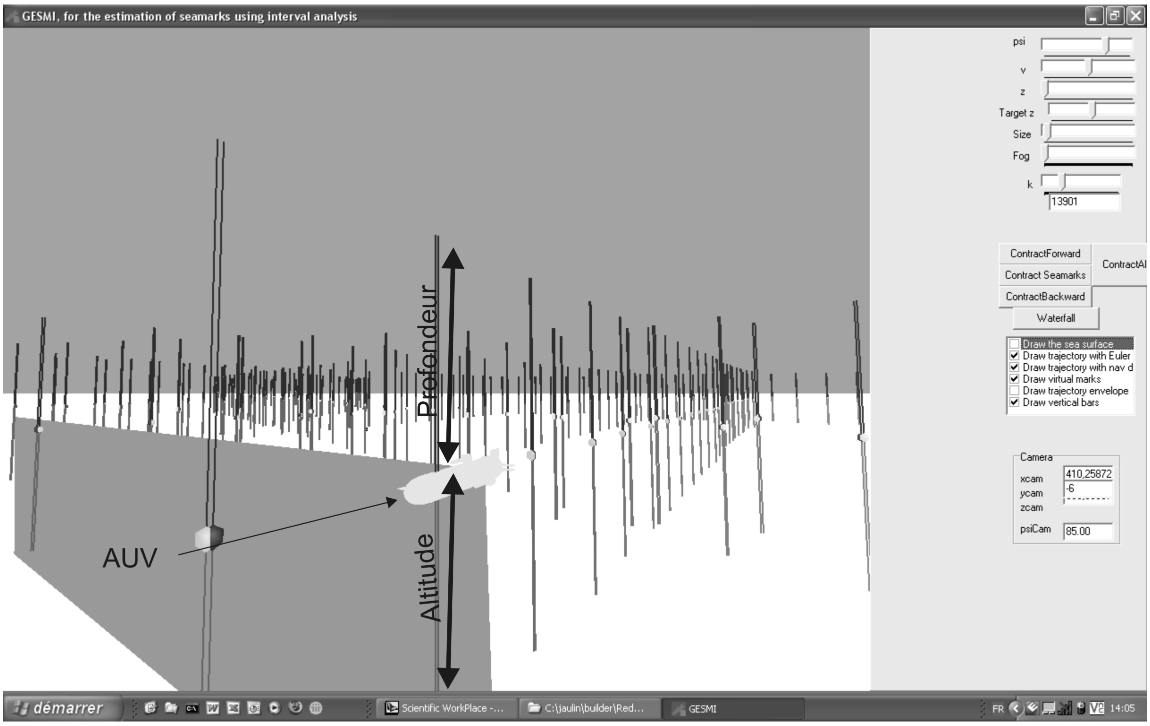
5.4 GESMI



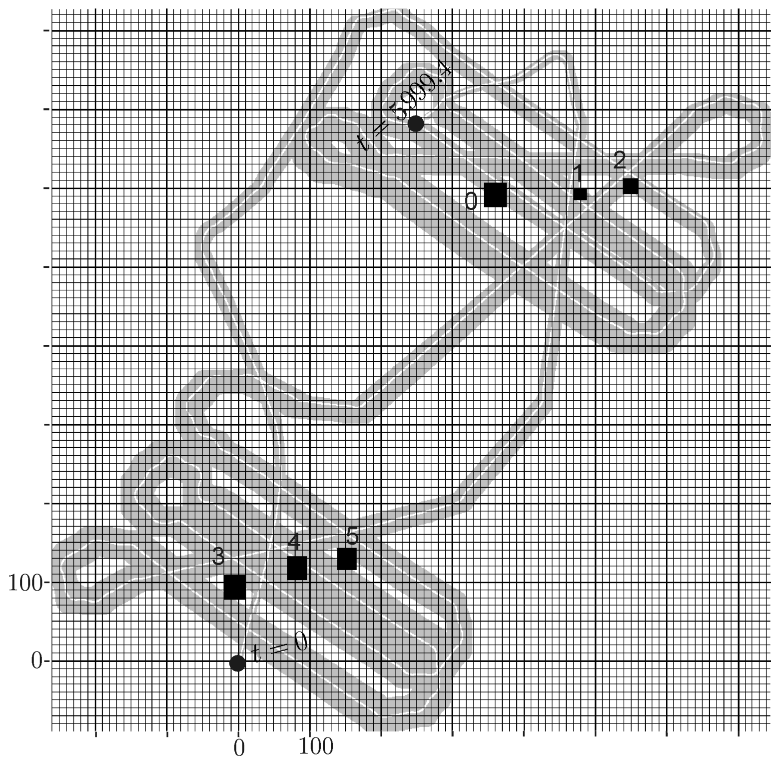
GESMI (Guaranteed Estimation of Sea Mines with Intervals)



GESMI, for the estimation of seamarks using interval analysis
psi
v
z
Target z
Size
Fog
k
98
Bouge dans temps
Affiche la mer
ContractForward
Contract Seamarks
ContractBackward
Waterfall
Draw the sea surface
Draw trajectory with Euler
Draw trajectory with nav d
Draw virtual marks
Draw trajectory envelope
Draw vertical bars
AUV
Affiche la trajectoire estimée
Affiche l'enveloppe
Camera
xcam -6
ycam -0,5
zcam
psiCam 53.00
Quart de plan du sonar latéral
Position et orientation de la caméra
démarrer
Scientific WorkPlace -...
GESMI
FR
12:43



GESMI, for the estimation of seamarks using interval analysis
psi
v
z
Target z
Size
Fog
k
13901
ContractForward
Contract Seamarks
ContractBackward
Waterfall
Draw the sea surface
Draw trajectory with Euler
Draw virtual marks
Draw trajectory envelope
Draw vertical bars
Camera
xcam
ycam
zcam
410,25872
-6
psiCam
85.00
Profondeur
AUV
Altitude
démarrer
Scientific WorkPlace -...
GESMI
FR
14:05


t = 5999.4
0
1
2
3
4
5
t = 0
100
0
0
100

6 Probabilistic-set approach

Bounded-error estimation

$$\mathbf{y} = \boldsymbol{\psi}(\mathbf{p}) + \mathbf{e},$$

where

$\mathbf{e} \in \mathbb{E} \subset \mathbb{R}^m$ is the error vector,

$\mathbf{y} \in \mathbb{R}^m$ is the collected data vector,

$\mathbf{p} \in \mathbb{R}^n$ is the parameter vector to be estimated.

Or equivalently

$$\mathbf{e} = \mathbf{y} - \boldsymbol{\psi}\left(\mathbf{p}\right) = \mathbf{f_y}\left(\mathbf{p}\right),$$

The *posterior feasible set* for the parameters is

$$\mathbb{P} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\mathbb{E}\right).$$

Probabilistic set approach. We decompose the error space into two subsets: $\mathbb{E}$ on which we bet $\mathbf{e}$ will belong and $\overline{\mathbb{E}}$. We set

$$\pi = \Pr\left(\mathbf{e} \in \mathbb{E}\right)$$

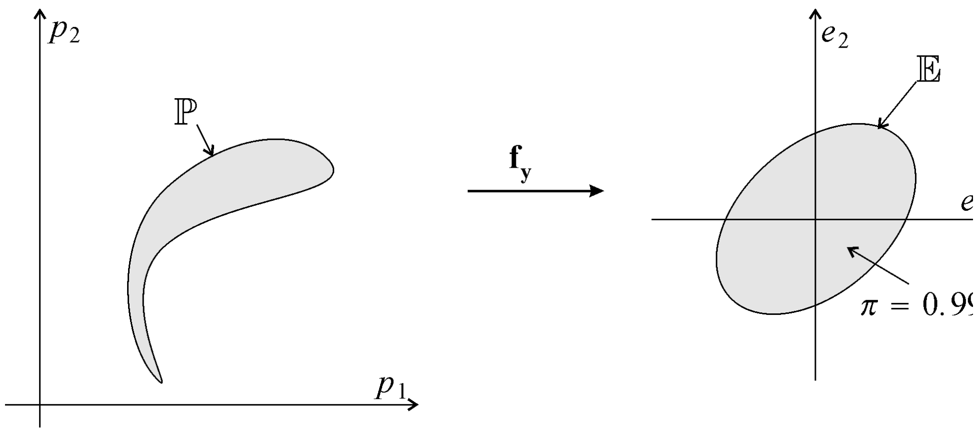
The event $\mathbf{e} \in \overline{\mathbb{E}}$ is considered as *rare*, i.e., $\pi \simeq 1$.

Once $\mathbf{y}$ is collected, we compute

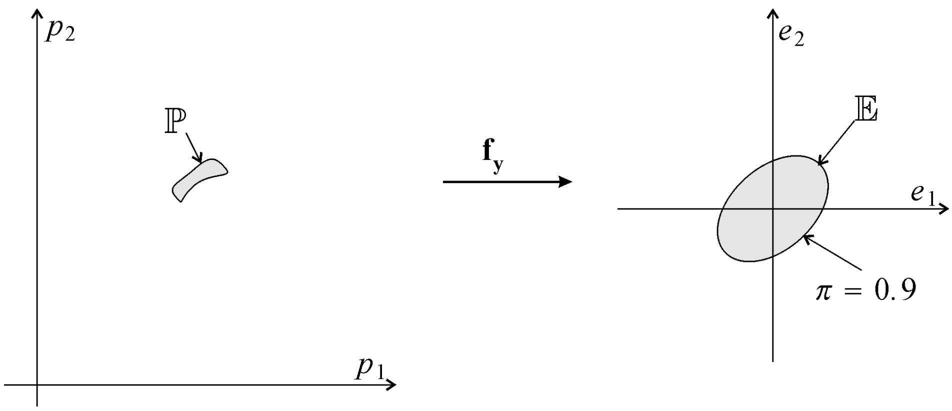
$$\mathbb{P} = \mathbf{f}_{\mathbf{y}}^{-1}(\mathbb{E}) \,.$$

If $\mathbb{P} \neq \emptyset$, we conclude that $\mathbf{p} \in \mathbb{P}$ with a prior probability of π.

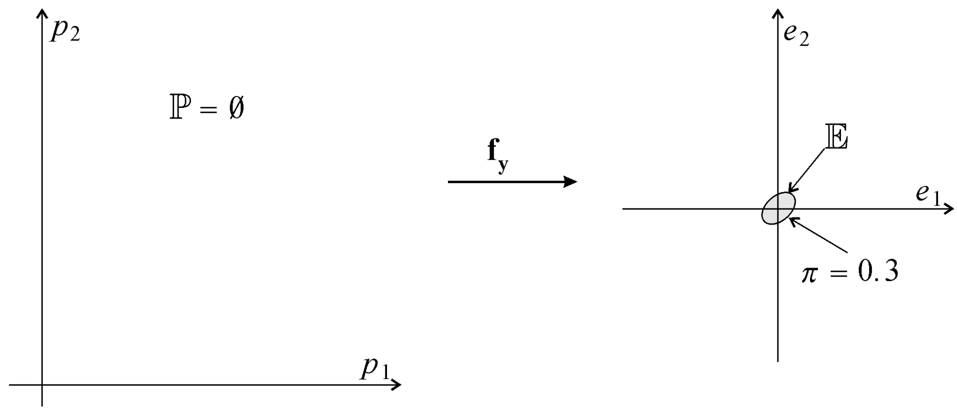
If $\mathbb{P} = \emptyset$, than we conclude the rare event $\mathbf{e} \in \overline{\mathbb{E}}$ occurred.



p_2
$\mathbb{P}$
p_1
$\mathbf{f_y}$
e_2
$\mathbb{E}$
$\pi = 0.9$



p2
ℙ
p1
fy
e2
𝔼
e1
π = 0.9



p_2
$\mathbb{P} = \emptyset$
p_1
$\mathbf{f_y}$
e_2
$\mathbb{E}$
e_1
$\pi = 0.3$

7 Robust regression

Consider the error model

$$\underbrace{\begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}}_{=\mathbf{e}} = \underbrace{\begin{pmatrix} y_1 - \psi_1(\mathbf{p}) \\ \vdots \\ y_m - \psi_m(\mathbf{p}) \end{pmatrix}}_{=\mathbf{f_y}(\mathbf{p})}$$

The data y_i is an *inlier* if $e_i \in [e_i]$ and an *outlier* otherwise. We assume that

$$\forall i,\ \mathsf{Pr}\left(e_i \in [e_i]\right) = \pi$$

and that all e_i's are independent.

Equivalently,

$$\begin{cases} y_1 - \psi_1(\mathbf{p}) \in [e_1] & \text{with a probability } \pi \\ \vdots & \vdots \\ y_m - \psi_m(\mathbf{p}) \in [e_m] & \text{with a probability } \pi \end{cases}$$

The probability of having k inliers is

$$\frac{m!}{k!\,(m-k)!}\pi^k.\,(1-\pi)^{m-k}\,.$$

The probability of having strictly more than q outliers is thus

$$\gamma\left(q,m,\pi\right)\stackrel{\text{def}}{=}\sum_{k=0}^{m-q-1}\frac{m!}{k!\left(m-k\right)!}\pi^{k}.\left(1-\pi\right)^{m-k}.$$

Denote by $\mathbb{E}^{\{q\}}$ the set of all $\mathbf{e} \in \mathbb{R}^m$ consistent with at least $m - q$ error intervals $[e_i]$.

For $m = 3$, we have

$$
\begin{aligned}
\mathbb{E}^{\{0\}} &= [e_1] \times [e_2] \times [e_3] \\
\mathbb{E}^{\{1\}} &= ([e_1] \cap [e_2]) \cup ([e_2] \cap [e_3]) \cup ([e_1] \cap [e_3]) \\
\mathbb{E}^{\{2\}} &= [e_1] \cup [e_2] \cup [e_3] \\
\mathbb{E}^{\{3\}} &= \mathbb{R}^3.
\end{aligned}
$$

Define

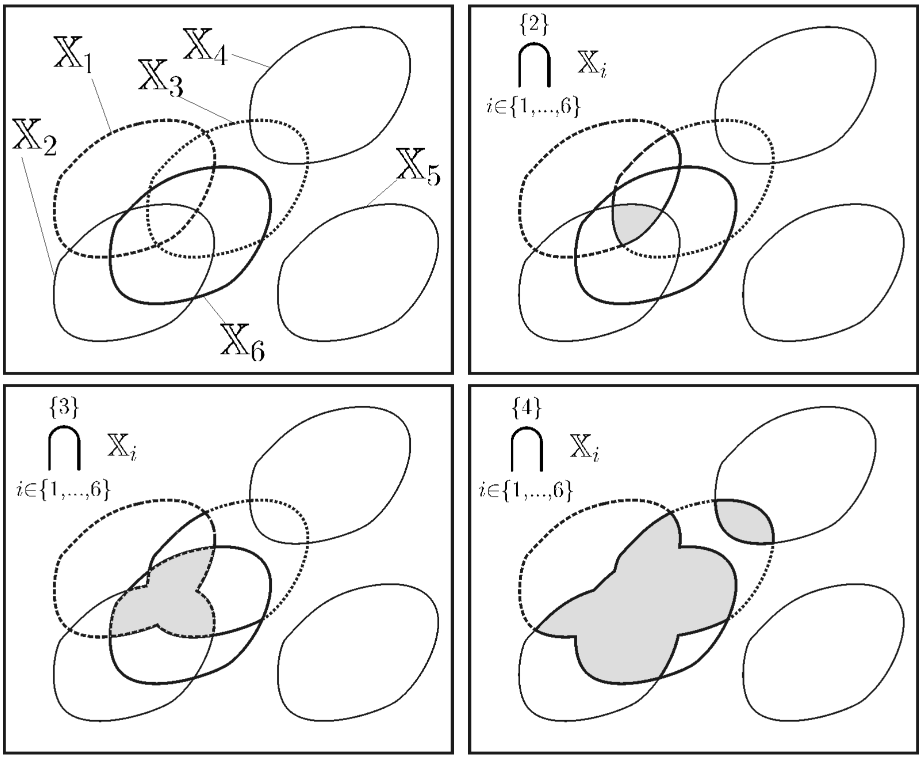
$$\mathbb{P}^{\{q\}} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\mathbb{E}^{\{q\}}\right).$$

We have

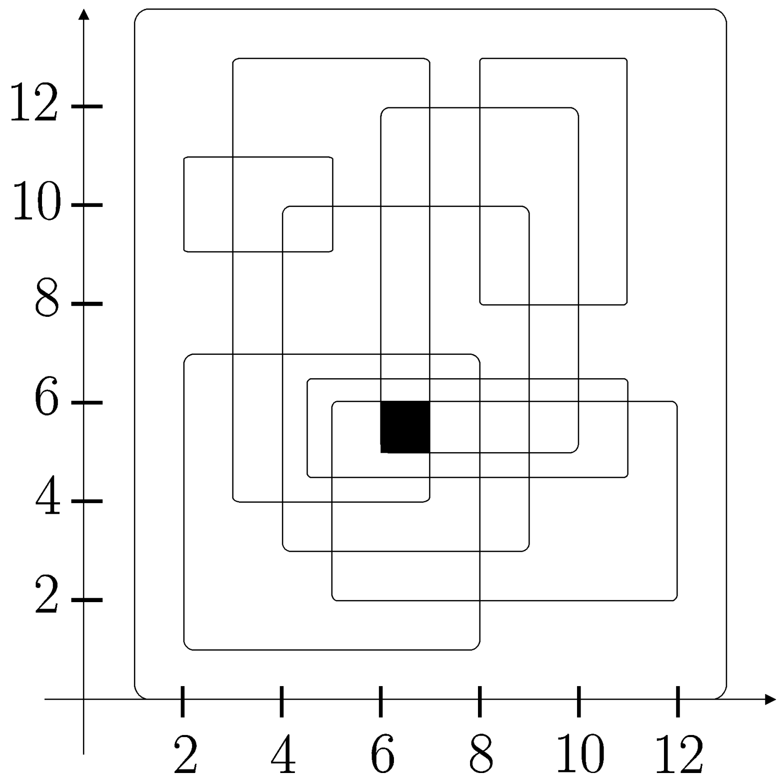
$$\begin{aligned}\text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{q\}}\right) &= \mathbf{1} - \gamma\left(q, m, \pi\right)\\ \text{prob}\left(\mathbf{p} \in \overline{\mathbb{P}^{\{q\}}}\right) &= \gamma\left(q, m, \pi\right).\end{aligned}$$

Thus $\mathbb{P}^{\{q\}}$ is the inverse of $\mathbb{E}^{\{q\}}$ and inner/outer approximations can thus be found.

8 Relaxed intersection



q-relaxed intersection



The black box is the 2-intersection of 9 boxes

$$\mathbb{P}^{\{q\}} = \mathbf{f}_{\mathbf{y}}^{-1}\left(\mathbb{E}^{\{q\}}\right) = \bigcap_{i \in \{1,\ldots,m\}}^{\{q\}} f_{y_i}^{-1}\left([e_i]\right).$$

Proposition. We have

$$\overline{\mathbb{P}^{\{q\}}} = \bigcap^{\{m-q-1\}} f_{y_i}^{-1}\left(\overline{[e_i]}\right).$$

This proposition allows to obtain an inner approximation of $\mathbb{P}^{\{q\}}$.

9 Application to localization

A robot measures distances to three beacons.

beacon	x_i	y_i	$[d_i]$
1	1	3	$[1, 2]$
2	3	1	$[2, 3]$
3	-1	-1	$[3, 4]$

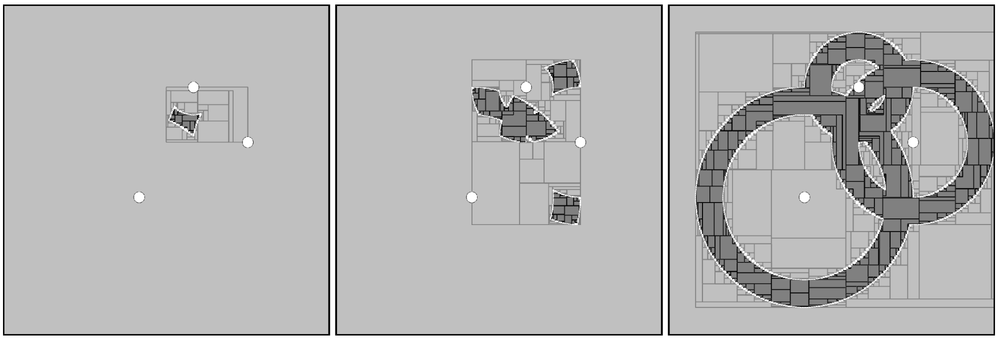
The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

The feasible sets associated to each data is

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} - d_i \in [-0.5, 0.5] \right\},$$

where $d_1 = 1.5, d_2 = 2.5, d_3 = 3.5$.

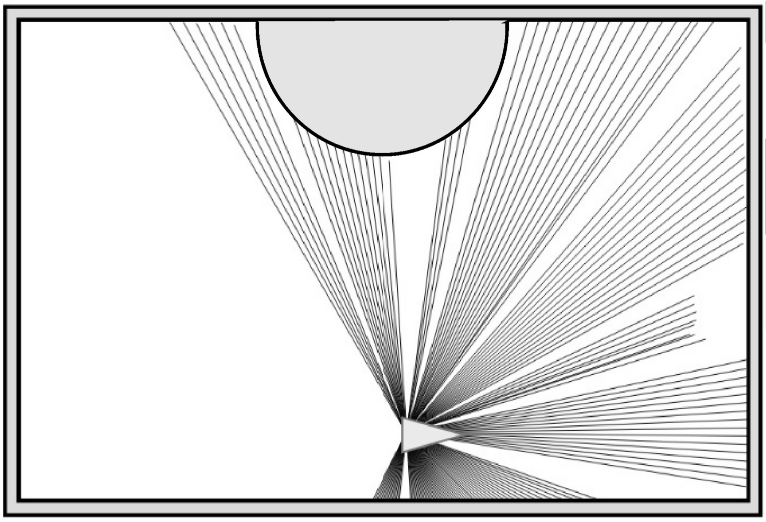
$$
\begin{aligned}
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{0\}}\right) &= \mathsf{0.729} \\
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{1\}}\right) &= \mathsf{0.972} \\
\mathsf{prob}\left(\mathbf{p} \in \mathbb{P}^{\{2\}}\right) &= \mathsf{0.999}
\end{aligned}
$$



Probabilistic sets $\mathbb{P}^{\{0\}}, \mathbb{P}^{\{1\}}, \mathbb{P}^{\{2\}}$.

10 With real data

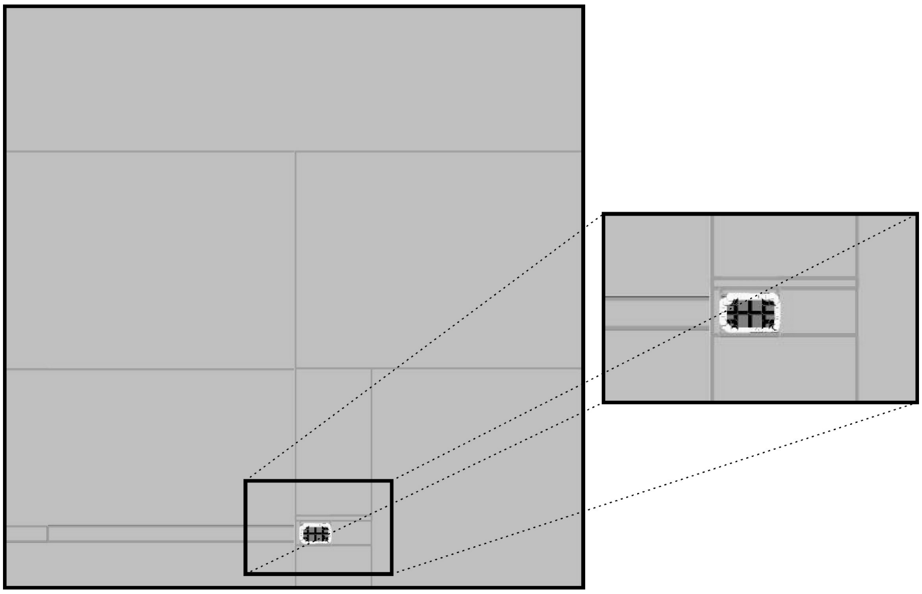
Robot equipped with a laser rangefinder and a compass.



143 distances collected by the rangefinder $\pm 10cm$

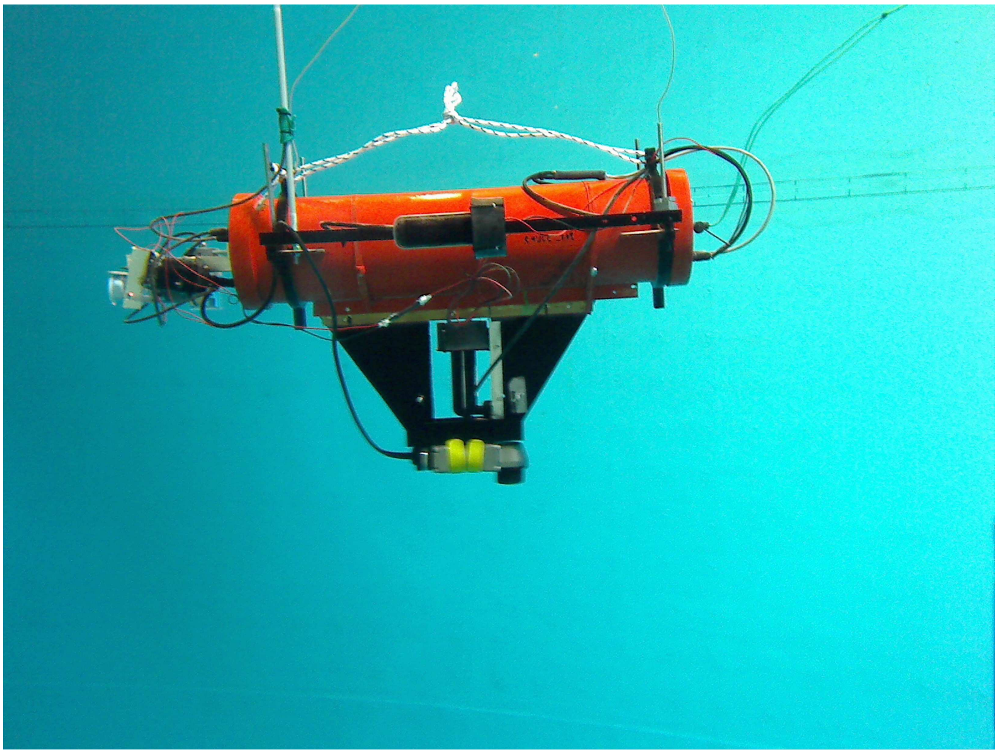
For $q = 16, m = 143, \pi = 0.95$, the probability of being wrong is

$$\alpha = \gamma\left(q, m, \pi\right) = 8.46 \times 10^{-4}.$$



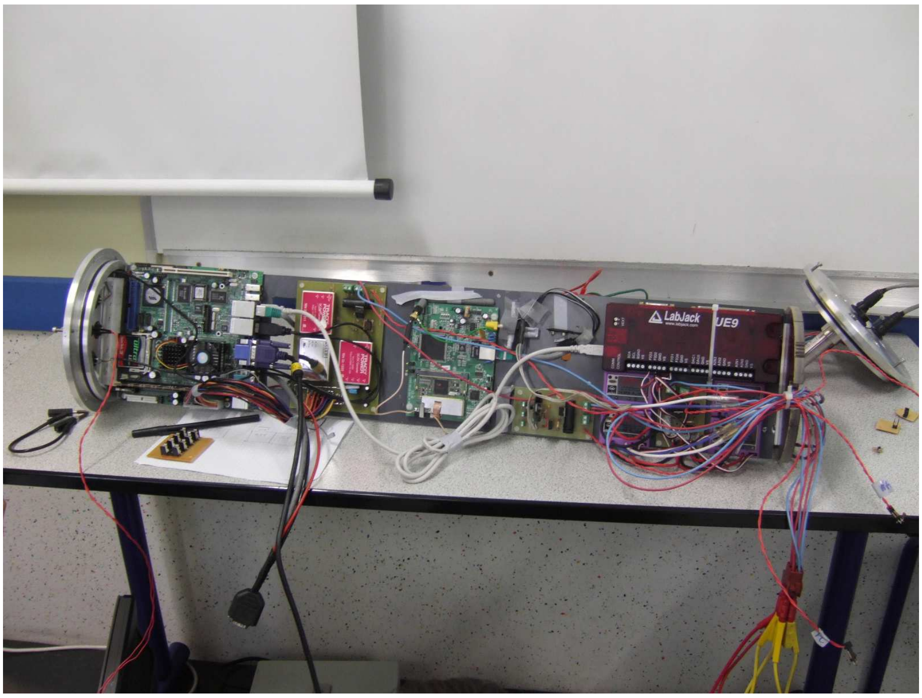
$\mathbb{P}^{\{16\}}$ contains $\mathbf{p}^*$ with a probability $1 - \alpha = 0.99915$.

11 SAUC'ISSE

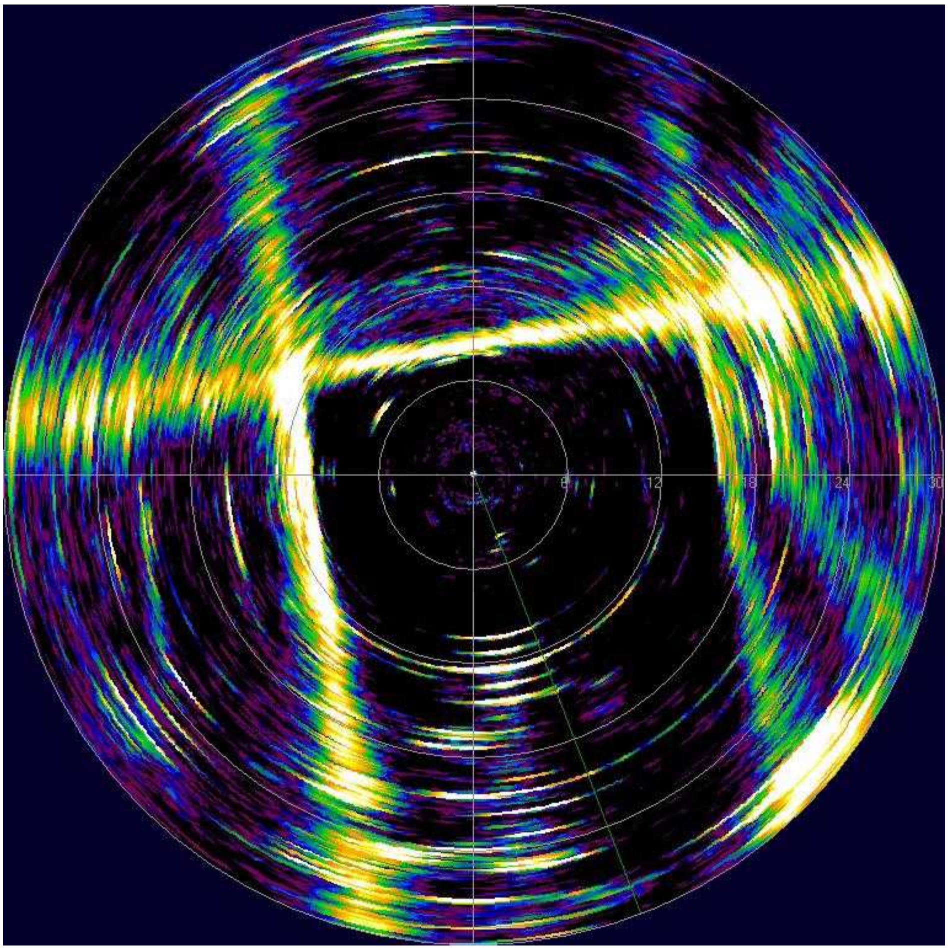


Robot SAUC'ISSE

Portsmouth, July 12-15, 2007.


LabJack
UE9


AR HAUT


6
12
18
24
30

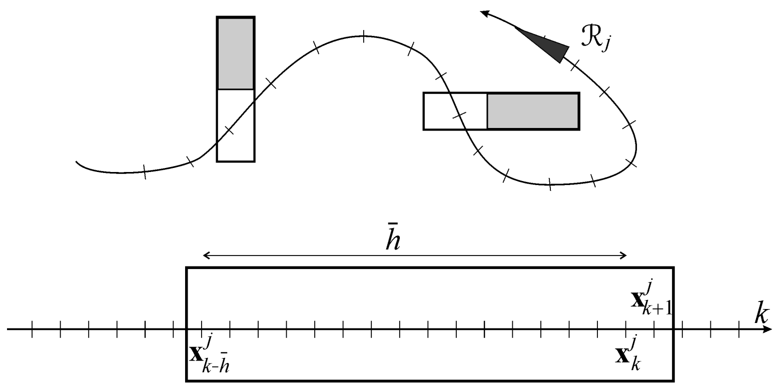
12 State estimation

$$\begin{cases} \mathbf{x}(k+1) & = & \mathbf{f}_k(\mathbf{x}(k), \mathbf{n}\,(k)) \\ \mathbf{y}(k) & = & \mathbf{g}_k(\mathbf{x}(k)), \end{cases}$$

with $\mathbf{n}\,(k) \in \mathbb{N}\,(k)$ and $\mathbf{y}\,(k) \in \mathbb{Y}\,(k)$.

Without outliers

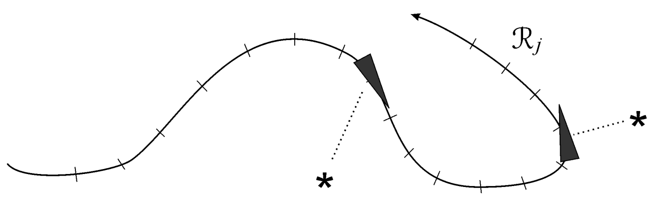
$$\mathbb{X}(k+\mathbf{1}) = \mathbf{f}_k\left(\mathbb{X}(k) \cap \mathbf{g}_k^{-1}\left(\mathbb{Y}(k)\right), \quad \mathbb{N}\left(k\right)\right).$$

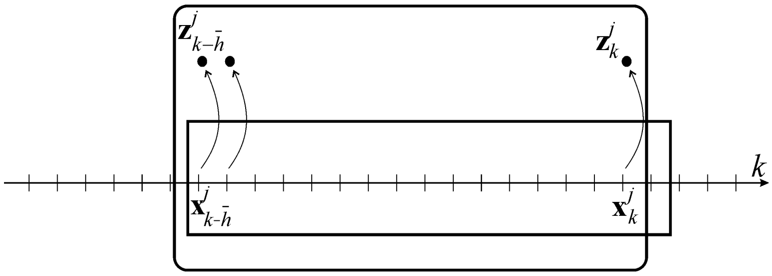


$$\mathbf{x}^j_{h+1} = \mathbf{f}(\mathbf{x}^j_h, \mathbf{u}^j_h), h \in \{k - \bar{h}, \dots, k\}$$

The observer: $C^{k,j}_{\mathbf{x}} = \bigcap_{h \in \{k-\bar{h},\dots,k\}} C^j_{\mathbf{x}(h)}$

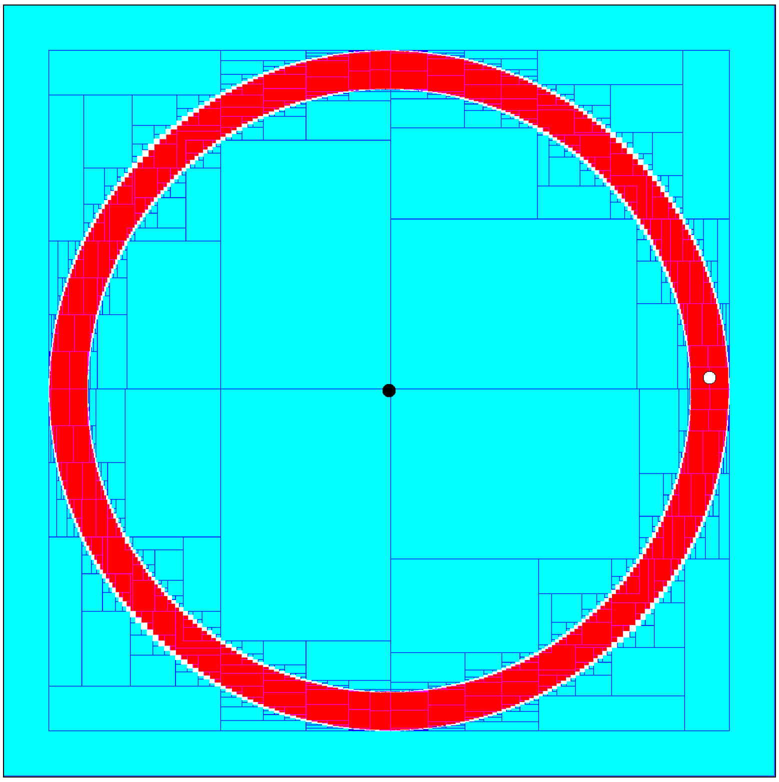
var($\mathcal{C}^{k,j}_{\mathbf{x}}$) =var($\mathcal{C}^{k,j}_{\mathbf{x}(h)}$) $= \left\{\mathbf{x}^j_{k-\bar{h}}, \dots, \mathbf{x}^j_k, \mathbf{x}^j_{k+1}\right\}$.



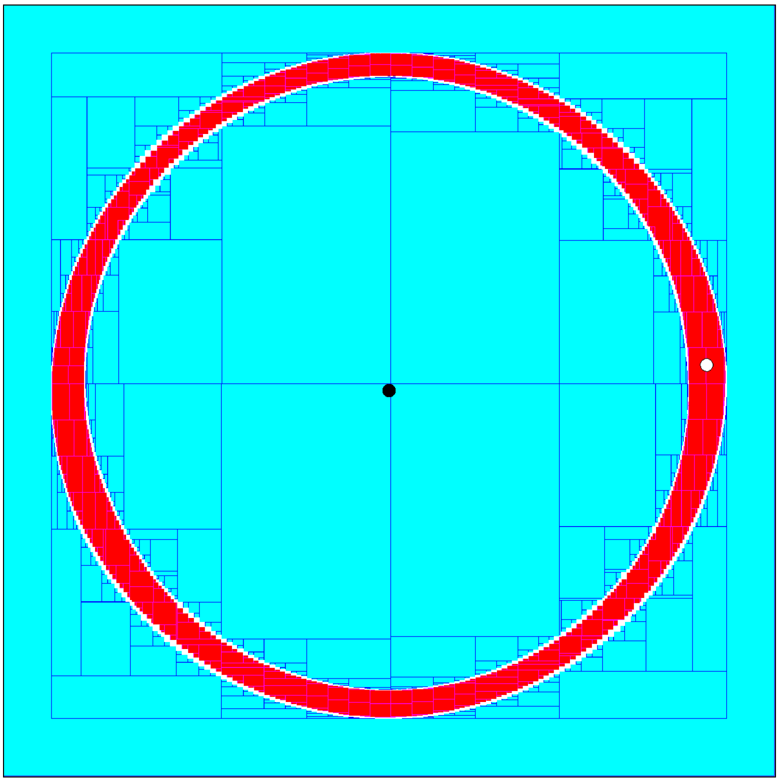


$$\mathbf{z}^j_h = \mathbf{h}(\mathbf{x}^j_h)$$

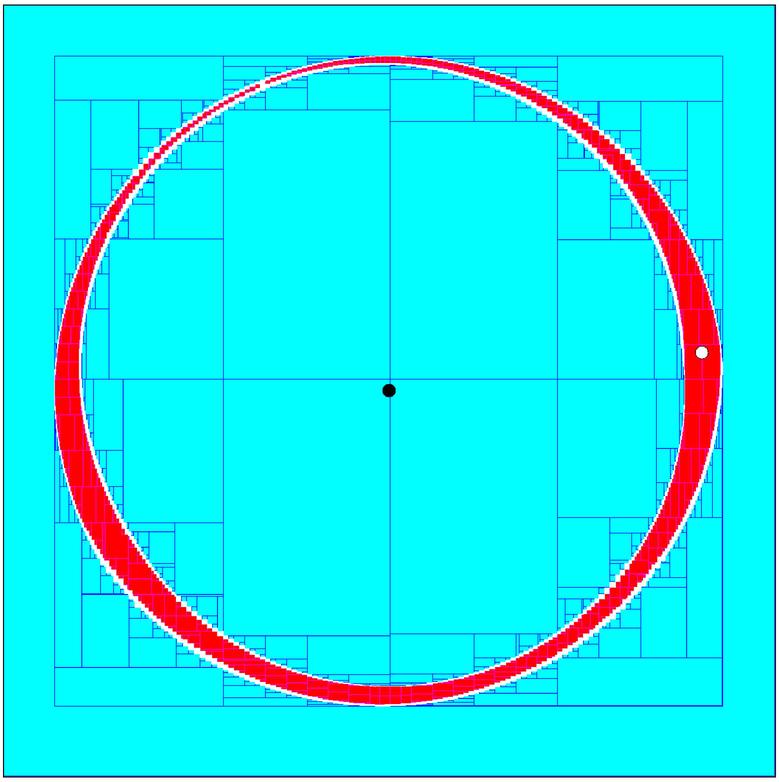
Observer RSO: $C^{k,j}_{\mathbf{x},z} = C^{k,j}_{\mathbf{x}} \cap \bigcap^{\{q_1\}}_{h \in \{k-\bar{h},\ldots,k\}} \left(C^{k,j}_{\mathbf{x}} | C^{h,j}_{z} \right).$



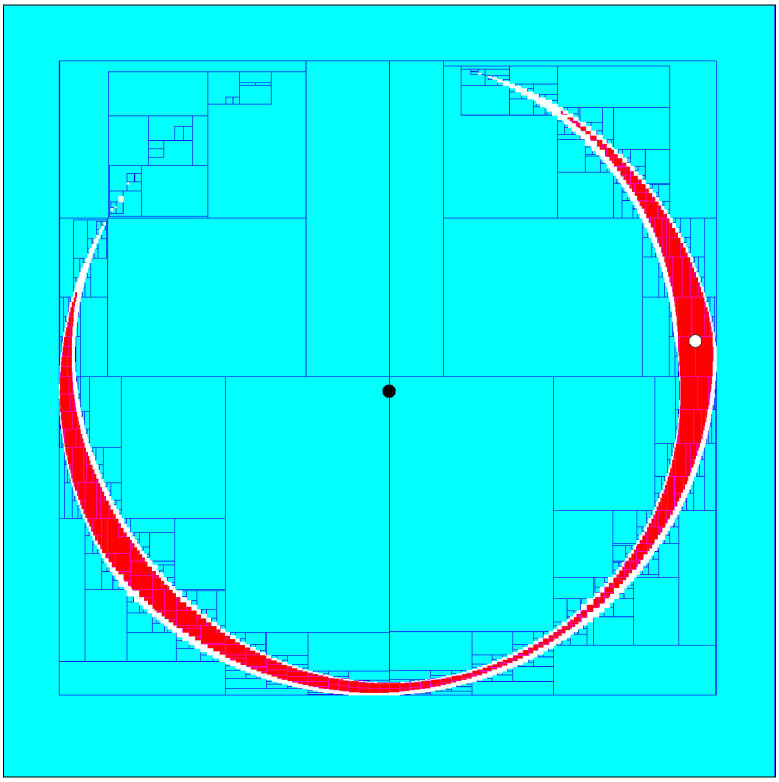
$t = 0$



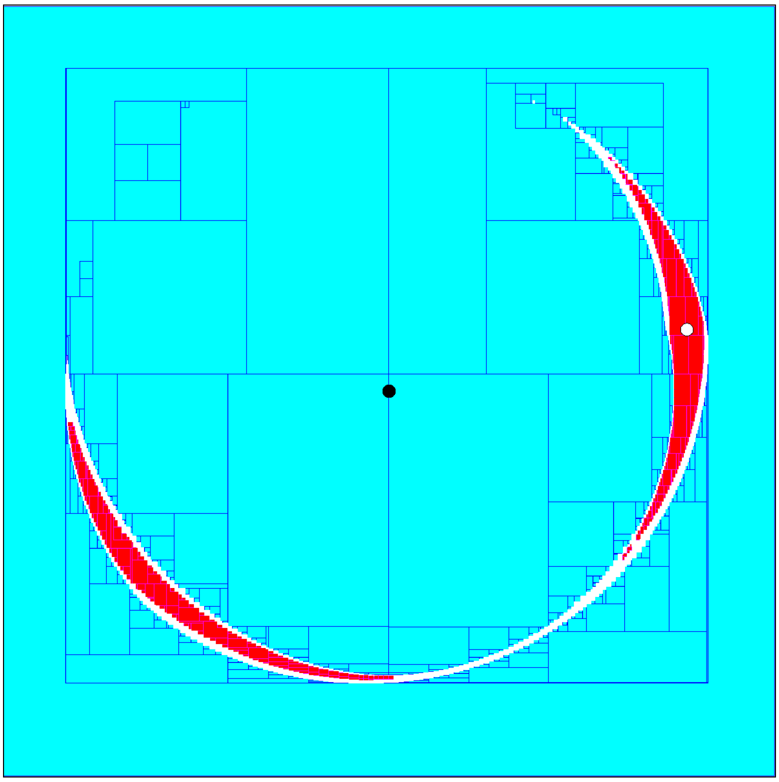
$t = 0.1$



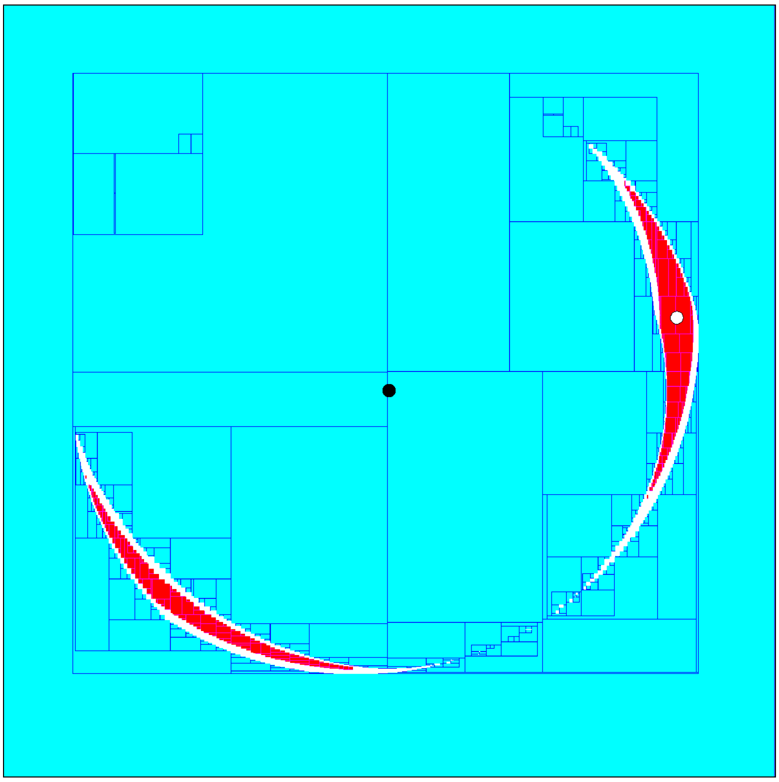
$t = 0.2$



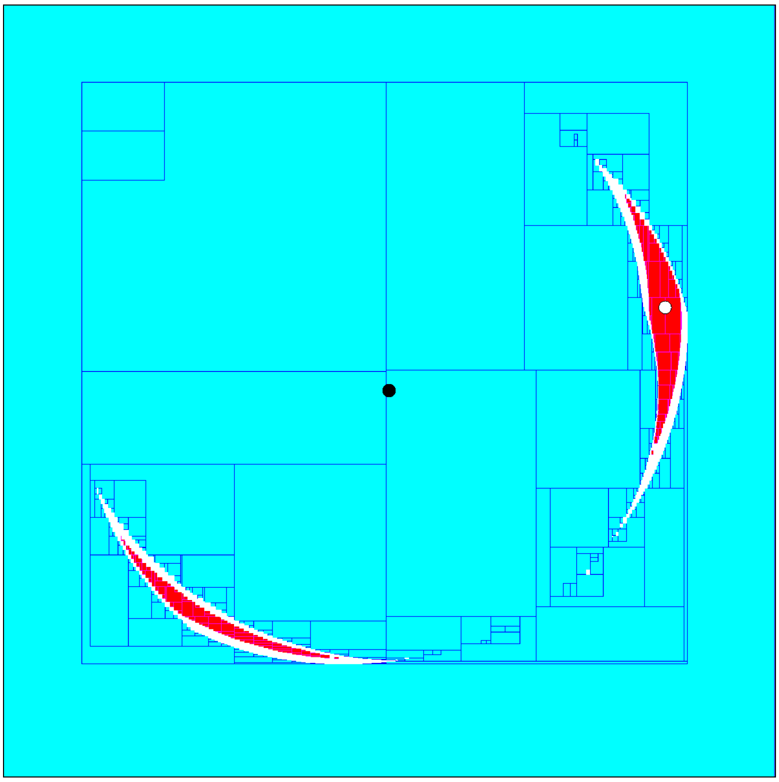
$t = 0.3$



$t = 0.4$



$t = 0.5$

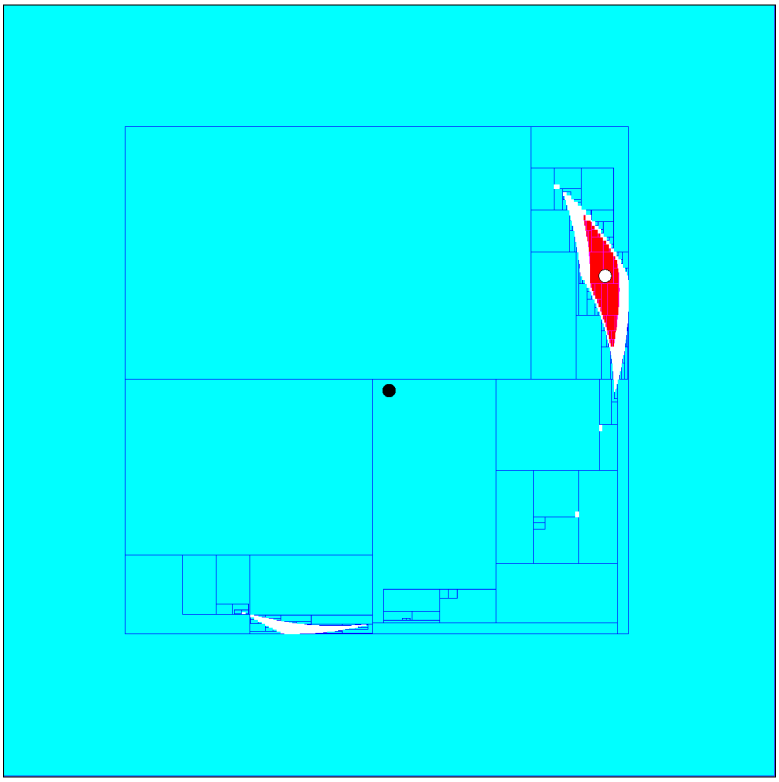


$t = 0.6$

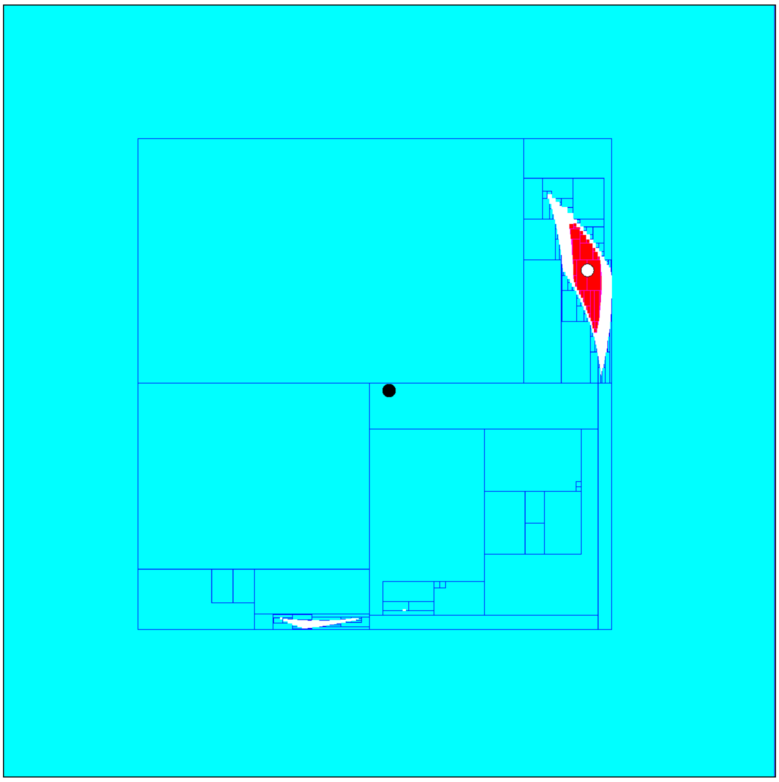
$t = 0.7$

$t = 0.8$

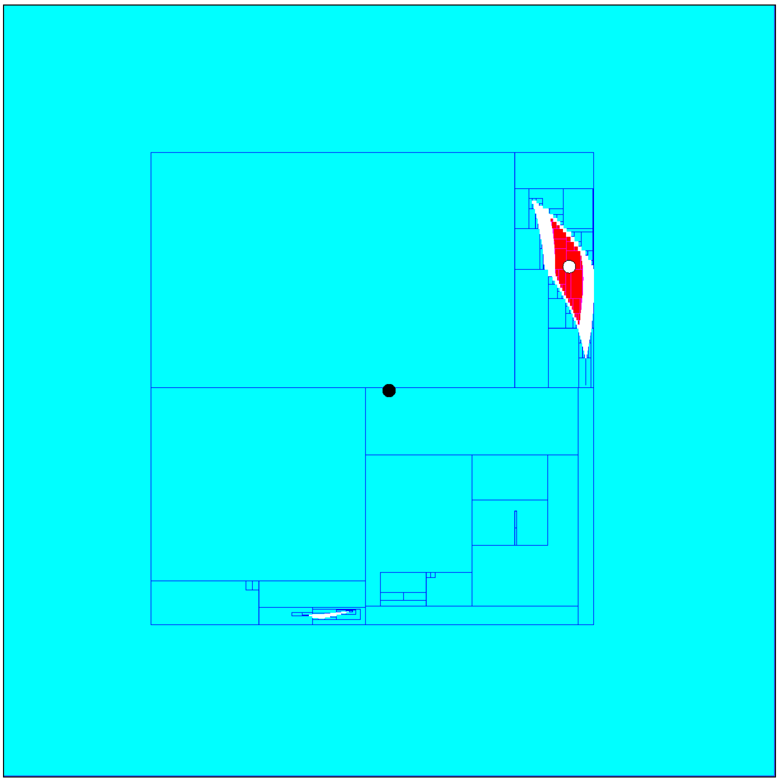
$t = 0.9$



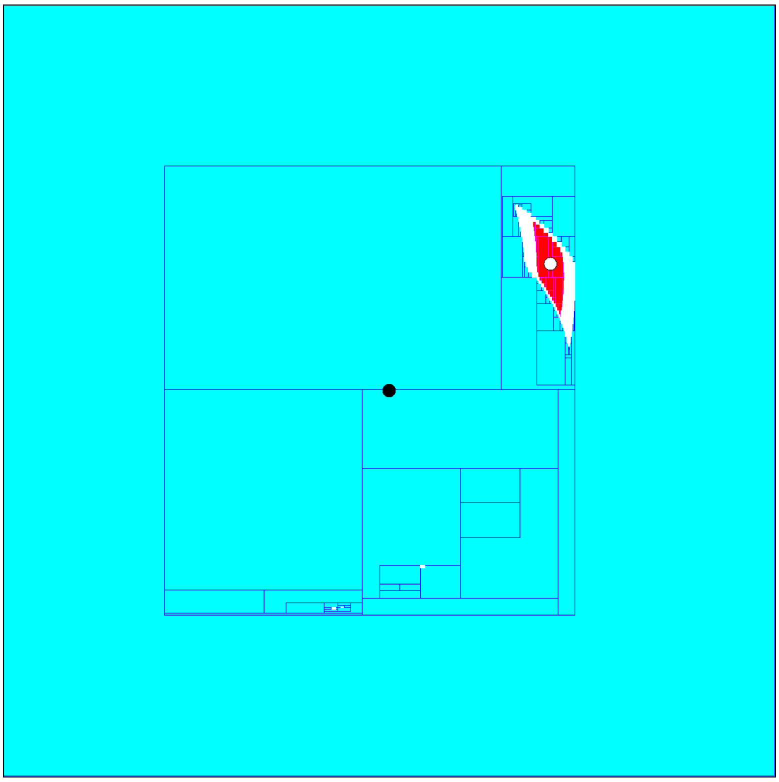
$t = 1.0$



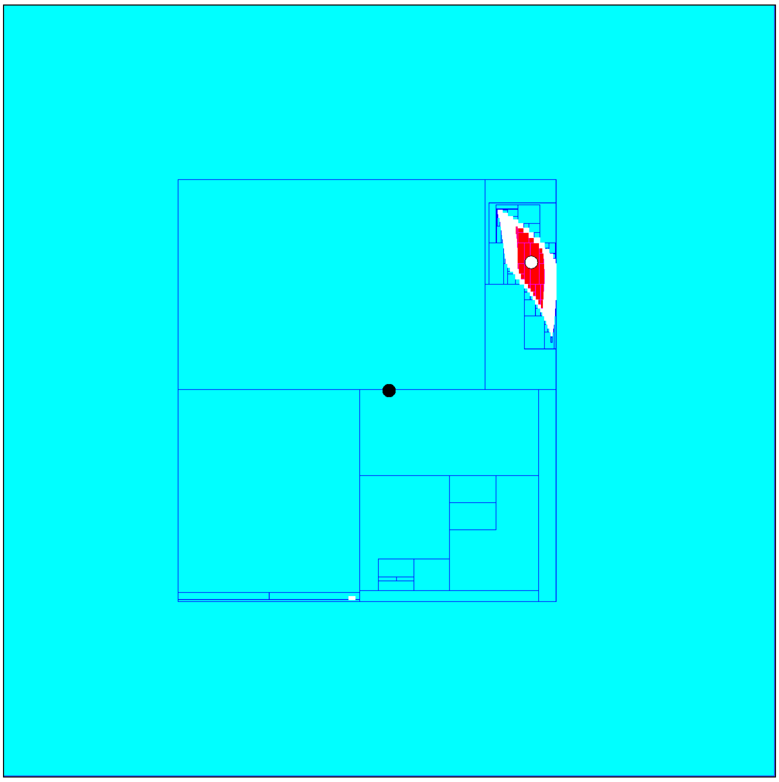
$t = 1.1$



$t = 1.2$

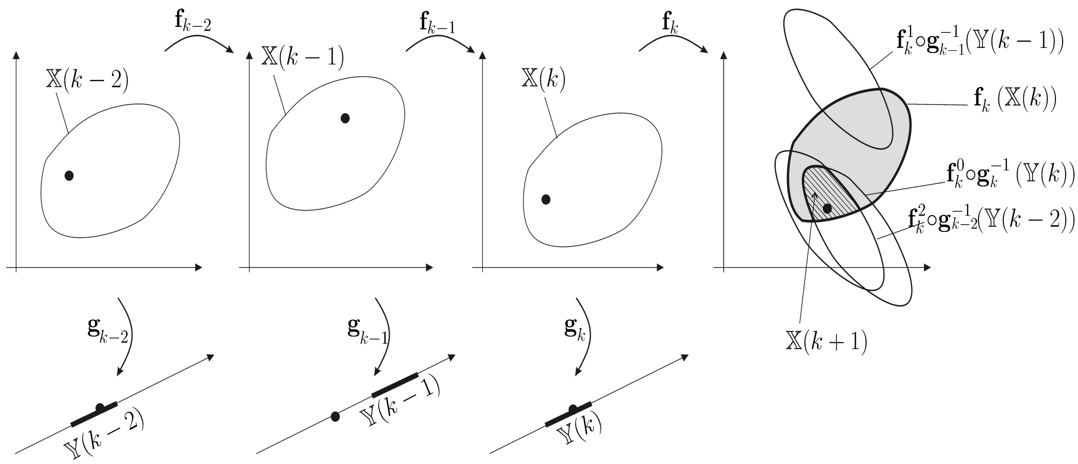


$t = 1.3$



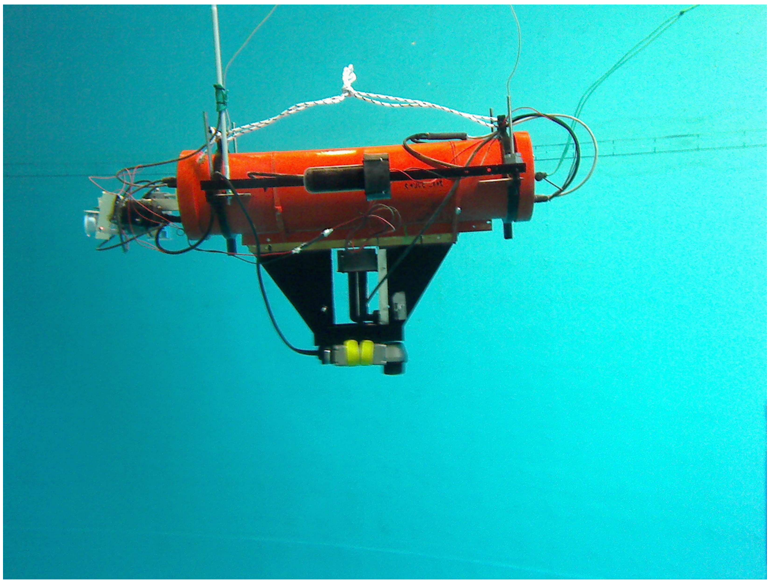
$t = 1.4$

12.1 Set interpretation



$\mathbf{f}_{k-2}$
$\mathbf{f}_{k-1}$
$\mathbf{f}_k$
$\mathbb{X}(k-2)$
$\mathbb{X}(k-1)$
$\mathbb{X}(k)$
$\mathbf{f}_k^1 \circ \mathbf{g}_{k-1}^{-1}(\mathbb{Y}(k-1))$
$\mathbf{f}_k(\mathbb{X}(k))$
$\mathbf{f}_k^0 \circ \mathbf{g}_k^{-1}(\mathbb{Y}(k))$
$\mathbf{f}_k^2 \circ \mathbf{g}_{k-2}^{-1}(\mathbb{Y}(k-2))$
$\mathbf{g}_{k-2}$
$\mathbf{g}_{k-1}$
$\mathbf{g}_k$
$\mathbb{X}(k+1)$
$\mathbb{Y}(k-2)$
$\mathbb{Y}(k-1)$
$\mathbb{Y}(k)$

12.2 Application to localization



Sauc'isse robot inside a swimming pool

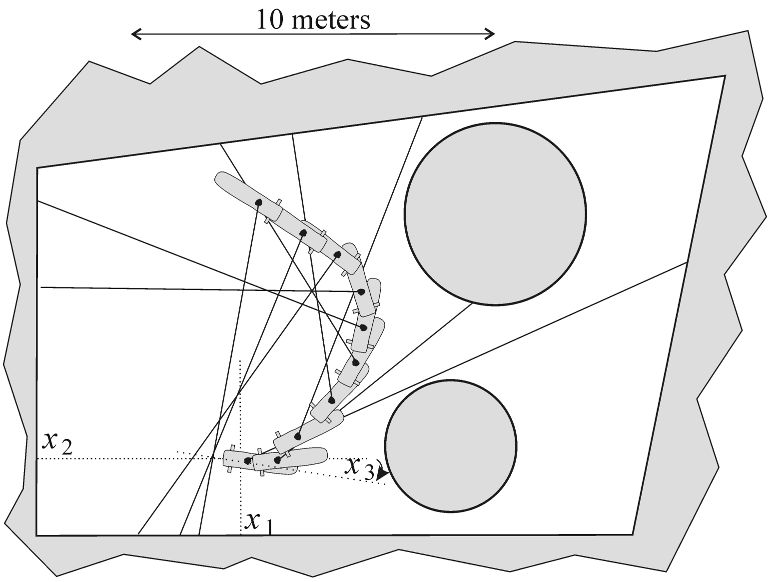
The robot evolution is described by

$$\begin{cases} \dot{x}_1 &= x_4 \cos x_3 \\ \dot{x}_2 &= x_4 \sin x_3 \\ \dot{x}_3 &= u_2 - u_1 \\ \dot{x}_4 &= u_1 + u_2 - x_4, \end{cases}$$

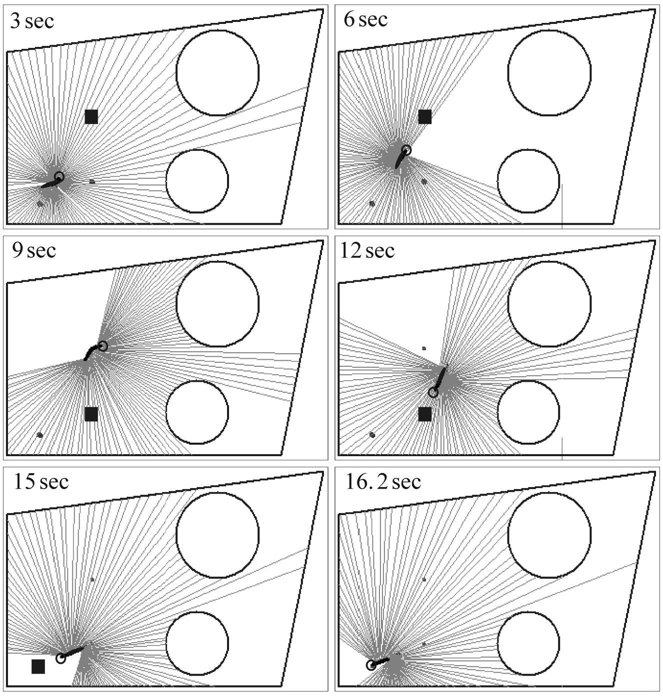
where x_1, x_2 are the coordinates of the robot center, x_3 is its orientation and x_4 is its speed. The inputs u_1 and u_2 are the accelerations provided by the propellers.

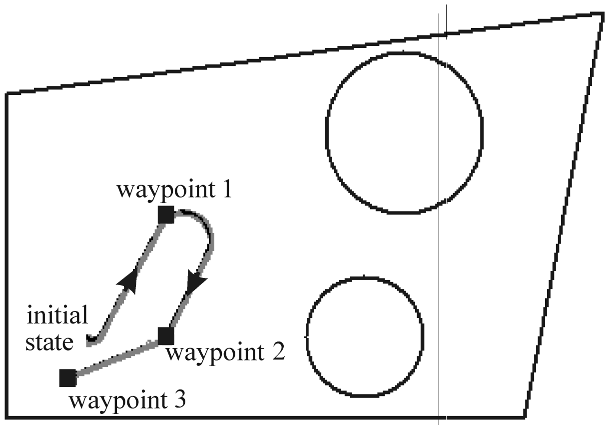
The system can be discretized by $\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k)$, where,

$$\mathbf{f}_k \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + \delta . x_4 . \cos(x_3) \\ x_2 + \delta . x_4 . \sin(x_3) \\ x_3 + \delta . (u_2(k) - u_1(k)) \\ x_4 + \delta . (u_1(k) + u_2(k) - x_4) \end{pmatrix}$$



Underwater robot moving inside a pool


3 sec
6 sec
9 sec
12 sec
15 sec
16. 2 sec


waypoint 1
initial
state
waypoint 2
waypoint 3