Solving set-valued constraint satisfaction problems

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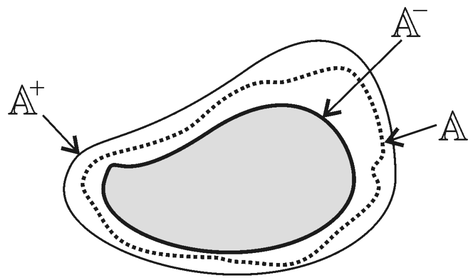
1 Set intervals

Given two sets $\mathbb{A}^-$ and $\mathbb{A}^+$ of $\mathbb{R}^n$, the pair $[\mathbb{A}] = \left[\mathbb{A}^-, \mathbb{A}^+\right]$ which encloses all sets $\mathbb{A}$ such that

$$\mathbb{A}^- \subset \mathbb{A} \subset \mathbb{A}^+$$

is a *set interval*.



The set interval $[\emptyset, \emptyset]$ is a singleton : $\emptyset \in [\emptyset, \emptyset]$.
The set interval $[\emptyset, \mathbb{R}^n]$ encloses all sets of $\mathbb{R}^n$.
The empty set interval is denoted by $[\mathbb{R}^n, \emptyset]$.

2 Arithmetic

2.1 Specific set interval operations

Set intervals are sets of sets, the intersection, the union, the inclusion can thus be defined.

Intersection.

$$\begin{array}{rcl}[\mathbb{A}] \sqcap [\mathbb{B}] & = & \{\mathbb{X}, \mathbb{X} \in [\mathbb{A}] \text{ and } \mathbb{X} \in [\mathbb{B}]\} \\ & = & \left[\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{A}^+ \cap \mathbb{B}^+\right].\end{array}$$

Proof.

$$\left\{\begin{array}{l}\mathbb{X} \in [\mathbb{A}] \\ \mathbb{X} \in [\mathbb{B}]\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\mathbb{A}^- \subset \mathbb{X} \subset \mathbb{A}^+ \\ \mathbb{B}^- \subset \mathbb{X} \subset \mathbb{B}^+\end{array}\right. \Leftrightarrow$$

$$\mathbb{A}^- \cup \mathbb{B}^- \subset \mathbb{X} \subset \mathbb{A}^+ \cap \mathbb{B}^+ \Leftrightarrow \mathbb{X} \in \left[\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{A}^+ \cap \mathbb{B}^+\right].$$

Inclusion.

$$[\mathbb{A}] \sqsubset [\mathbb{B}] \quad \Leftrightarrow \quad [\mathbb{A}] \sqcap [\mathbb{B}] = [\mathbb{B}] \, .$$

Set interval envelope.

$$\square\left\{\mathbb{A}_i, i\in\mathbb{I}\right\} = \left[\bigcap_{i\in\mathbb{I}}\mathbb{A}_i, \bigcup_{i\in\mathbb{I}}\mathbb{A}_i\right].$$

For instance,

$$\square\left\{[1,4],[3,7],[2,6]\right\} = [[3,4],[1,7]].$$

Union. We have

$$\begin{aligned} [\mathbb{A}] \sqcup [\mathbb{B}] &= \square \{\mathbb{X}, \mathbb{X} \in [\mathbb{A}] \text{ or } \mathbb{X} \in [\mathbb{B}]\} \\ &= \left[\mathbb{A}^- \cap \mathbb{B}^-, \mathbb{A}^+ \cup \mathbb{B}^+\right]. \end{aligned}$$

2.2 Set extension

All operations existing sets such as $\cap, \cup$, reciprocal image, direct image, ... can be extended to set intervals.

If $\diamond \in \{\cap, \cup, \times, \backslash, \ldots\}$,

$$[\mathbb{A}] \diamond [\mathbb{B}] = \square \left\{\mathbb{C}, \mathbb{A} \in [\mathbb{A}], \mathbb{B} \in [\mathbb{B}], \mathbb{C} = \mathbb{A} \diamond \mathbb{B}\right\}.$$

We have

$$\begin{array}{llcl}
\text{(i)} & \left[\mathbb{A}^-, \mathbb{A}^+\right] \cap \left[\mathbb{B}^-, \mathbb{B}^+\right] & = & \left[\mathbb{A}^- \cap \mathbb{B}^-, \mathbb{A}^+ \cap \mathbb{B}^+\right] \\
\text{(ii)} & \left[\mathbb{A}^-, \mathbb{A}^+\right] \cup \left[\mathbb{B}^-, \mathbb{B}^+\right] & = & \left[\mathbb{A}^- \cup \mathbb{B}^-, \mathbb{A}^+ \cup \mathbb{B}^+\right] \\
\text{(iii)} & \left[\mathbb{A}^-, \mathbb{A}^+\right] \times \left[\mathbb{B}^-, \mathbb{B}^+\right] & = & \left[\mathbb{A}^- \times \mathbb{B}^-, \mathbb{A}^+ \times \mathbb{B}^+\right] \\
\text{(iv)} & \left[\mathbb{A}^-, \mathbb{A}^+\right] \backslash \left[\mathbb{B}^-, \mathbb{B}^+\right] & = & \left[\mathbb{A}^- \backslash \mathbb{B}^+, \mathbb{A}^+ \backslash \mathbb{B}^-\right].
\end{array}$$

Extension of functions. A set-valued function f can be extended to set intervals as follows

$$f\left(\left[\mathbb{A}^{-},\mathbb{A}^{+}\right]\right)=\square\left\{f\left(\mathbb{A}\right),\mathbb{A}\in\left[\mathbb{A}^{-},\mathbb{A}^{+}\right]\right\}.$$

When f is inclusion monotonic, we have

$$f\left(\left[\mathbb{A}^{-},\mathbb{A}^{+}\right]\right)=\left[f\left(\mathbb{A}^{-}\right),f\left(\mathbb{A}^{+}\right)\right].$$

3 Natural set interval extension

Example. The natural set interval extension associated with the set expression

$$f(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3) = \mathbb{X}_1 \cup (\mathbb{X}_2 \cap g(\mathbb{X}_3))$$

is

$$[f]([\mathbb{X}_1], [\mathbb{X}_2], [\mathbb{X}_3]) = [\mathbb{X}_1] \cup ([\mathbb{X}_2] \cap g([\mathbb{X}_3])).$$

Theorem 1. If $\mathbb{X}_1 \in [\mathbb{X}_1], \ldots, \mathbb{X}_n \in [\mathbb{X}_n]$ then

$$f(\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n) \in [f]([\mathbb{X}_1], [\mathbb{X}_2], \ldots, [\mathbb{X}_n]).$$

Moreover, if in the expression of f, all $\mathbb{X}_i$ occur only once, the set interval evaluation is minimal.

Dependency problem. For instance,

$$\left[\mathbb{A}^{-},\mathbb{A}^{+}\right]\setminus\left[\mathbb{A}^{-},\mathbb{A}^{+}\right]=\left[\mathbb{A}^{-}\backslash\mathbb{A}^{+},\mathbb{A}^{+}\backslash\mathbb{A}^{-}\right]=\left[\emptyset,\mathbb{A}^{+}\backslash\mathbb{A}^{-}\right]$$

Of course, we have the inclusion property

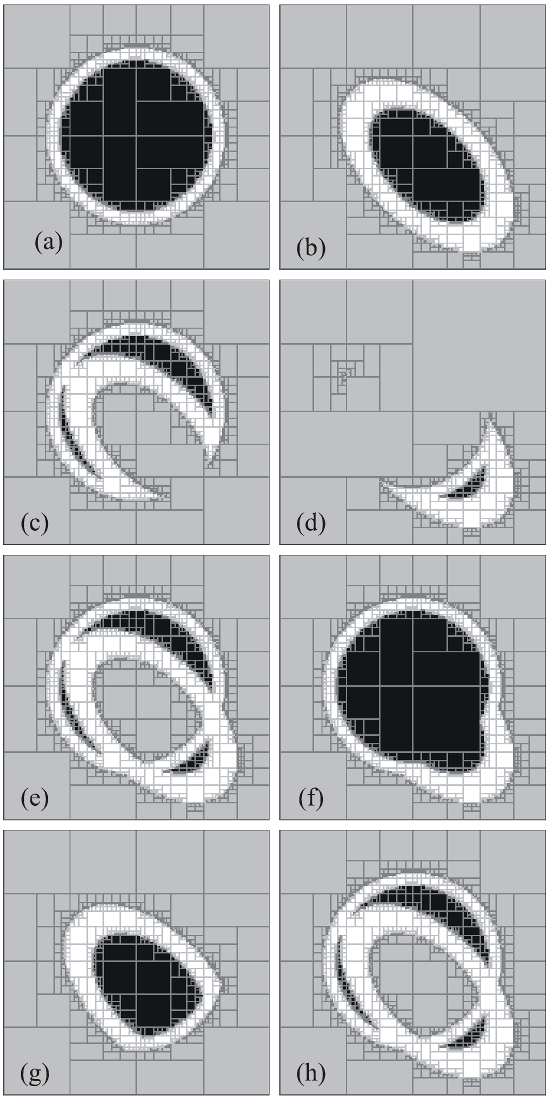
$$\left\{\mathbb{A}\backslash\mathbb{A},\mathbb{A}\in\left[\mathbb{A}^{-},\mathbb{A}^{+}\right]\right\}=[\emptyset,\emptyset]\sqsubset\left[\emptyset,\mathbb{A}^{+}\backslash\mathbb{A}^{-}\right].$$

Example. Consider two equivalent expressions of the exclusive union

$$\begin{aligned} f(\mathbb{A},\mathbb{B}) &= (\mathbb{A}\backslash\mathbb{B})\cup(\mathbb{B}\backslash\mathbb{A}) \\ g(\mathbb{A},\mathbb{B}) &= (\mathbb{A}\cup\mathbb{B})\backslash(\mathbb{A}\cap\mathbb{B}). \end{aligned}$$

The two natural set interval extensions are given by

$$\begin{aligned} [f]([\mathbb{A}],[\mathbb{B}]) &= ([\mathbb{A}]\backslash[\mathbb{B}])\cup([\mathbb{B}]\backslash[\mathbb{A}]) \\ [g]([\mathbb{A}],[\mathbb{B}]) &= ([\mathbb{A}]\cup[\mathbb{B}])\backslash([\mathbb{A}]\cap[\mathbb{B}]). \end{aligned}$$



(a) $\mathbb{A} \in \left[\mathbb{A}^-, \mathbb{A}^+\right]$

(b) $\mathbb{B} \in \left[\mathbb{B}^-, \mathbb{B}^+\right]$

(c) $[\mathbb{A}] \setminus [\mathbb{B}]$

(d) $[\mathbb{B}] \setminus [\mathbb{A}]$

(e) $[\mathbb{A}] \setminus [\mathbb{B}] \cup [\mathbb{B}] \setminus [\mathbb{A}]$

(f) $[\mathbb{A}] \cup [\mathbb{B}]$

(g) $[\mathbb{A}] \cap [\mathbb{B}]$

(h) $([\mathbb{A}] \cup [\mathbb{B}]) \setminus ([\mathbb{A}] \cap [\mathbb{B}])$

4 Contractors

Consider the SVCSP

$$\left\{ \begin{array}{c} \mathbb{A} \subset \mathbb{B} \\ \mathbb{A} \in [\mathbb{A}]\,, \mathbb{B} \in [\mathbb{B}]\,. \end{array} \right.$$

The optimal contractor is

$$\left\{ \begin{array}{ll} \text{(i)} & [\mathbb{A}] := [\mathbb{A}] \sqcap ([\mathbb{A}] \cap [\mathbb{B}]) \\ \text{(ii)} & [\mathbb{B}] := [\mathbb{B}] \sqcap ([\mathbb{A}] \cup [\mathbb{B}]) \end{array} \right.$$

Proof.

$$\mathbb{A} \subset \mathbb{B} \quad \Leftrightarrow \quad \mathbb{A} = \mathbb{A} \cap \mathbb{B} \quad \Leftrightarrow \quad \mathbb{B} = \mathbb{A} \cup \mathbb{B}.$$

Consider the SVCSP

$$\begin{cases} \mathbb{A} \cap \mathbb{B} = \emptyset \\ \mathbb{A} \in [\mathbb{A}], \mathbb{B} \in [\mathbb{B}], \end{cases}$$

The optimal contractor is

$$\begin{cases} \text{(i)} & [\mathbb{A}] := [\mathbb{A}] \sqcap ([\emptyset, \mathbb{R}^n] \setminus [\mathbb{B}]) \\ \text{(ii)} & [\mathbb{B}] := [\mathbb{B}] \sqcap ([\emptyset, \mathbb{R}^n] \setminus [\mathbb{A}]). \end{cases}$$

Proof.

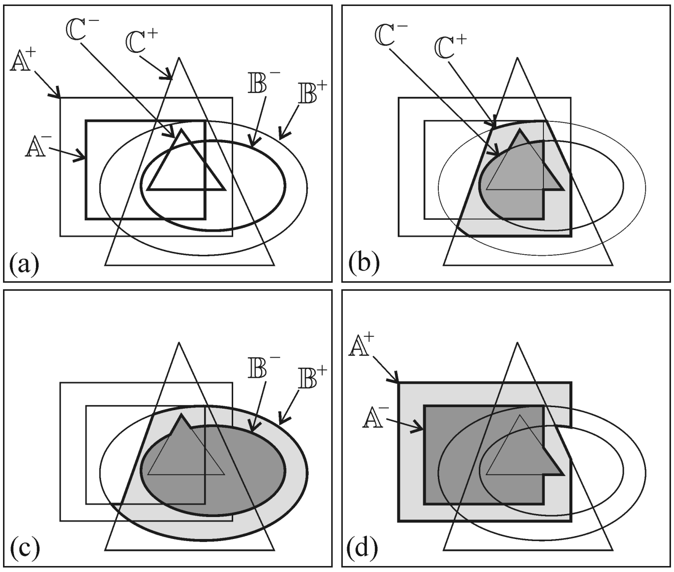
$$\begin{aligned} \mathbb{A} \cap \mathbb{B} = \emptyset \quad & \Leftrightarrow \quad \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n] \text{ such that } \mathbb{A} = \mathbb{Z} \setminus \mathbb{B} \\ & \Leftrightarrow \quad \exists \mathbb{Z} \in [\emptyset, \mathbb{R}^n] \text{ such that } \mathbb{B} = \mathbb{Z} \setminus \mathbb{A}. \end{aligned}$$

Consider the SVCSP

$$\begin{cases} \mathbb{A} \cap \mathbb{B} = \mathbb{C} \\ \mathbb{A} \in [\mathbb{A}], \mathbb{B} \in [\mathbb{B}], \mathbb{C} \in [\mathbb{C}]. \end{cases} \quad (1)$$

The optimal contractor is

$$\begin{cases} \text{(i)} & [\mathbb{C}] := [\mathbb{C}] \sqcap ([\mathbb{A}] \cap [\mathbb{B}]) \\ \text{(ii)} & [\mathbb{A}] := [\mathbb{A}] \sqcap ([\mathbb{C}] \cup ([\emptyset, \mathbb{R}^n] \setminus ([\mathbb{B}] \setminus [\mathbb{C}]))) \\ \text{(iii)} & [\mathbb{B}] := [\mathbb{B}] \sqcap ([\mathbb{C}] \cup ([\emptyset, \mathbb{R}^n] \setminus ([\mathbb{A}] \setminus [\mathbb{C}]))). \end{cases}$$



(a)
(b)
(c)
(d)

Consider the SVCSP

$$\left\{ \begin{array}{c} f(\mathbb{A}) = \mathbb{B} \\ \mathbb{A} \in [\mathbb{A}]\,, \mathbb{B} \in [\mathbb{B}] \end{array} \right.$$

where f is bijective. The optimal contractor is

$$\left\{ \begin{array}{ll} \text{(i)} & [\mathbb{B}] := [\mathbb{B}] \sqcap f\left([\mathbb{A}]\right) \\ \text{(ii)} & [\mathbb{A}] := [\mathbb{A}] \sqcap f^{-1}\left([\mathbb{B}]\right). \end{array} \right.$$

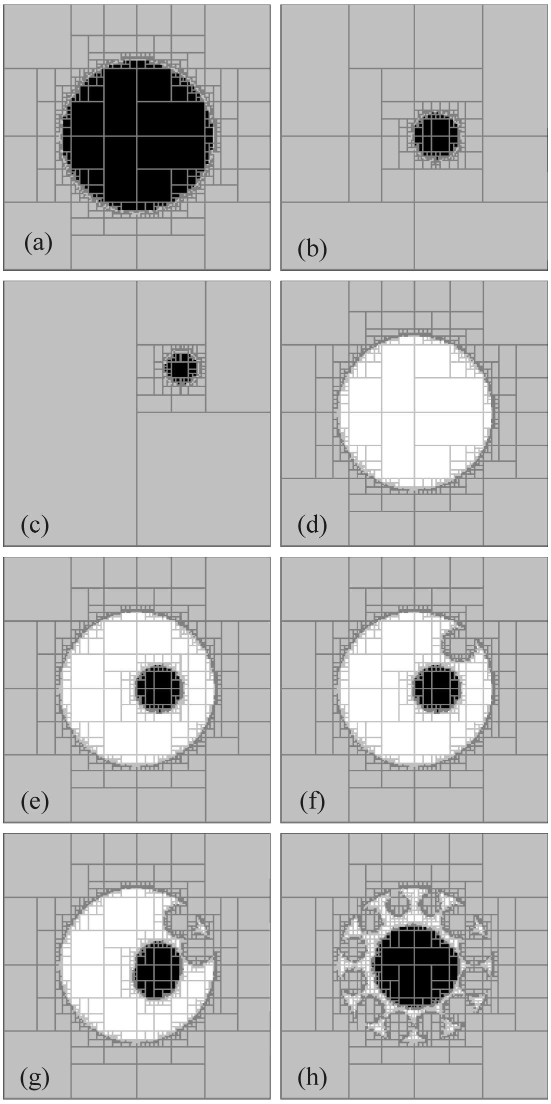
5 Application

Consider the following SVCSP

$$\begin{cases} \text{(i)} & \mathbb{X} \subset \mathbb{A} \\ \text{(ii)} & \mathbb{B} \subset \mathbb{X} \\ \text{(iii)} & \mathbb{X} \cap \mathbb{C} = \emptyset \\ \text{(iv)} & f(\mathbb{X}) = \mathbb{X}, \end{cases}$$

where $\mathbb{X}$ is an unknown subset of $\mathbb{R}^2$, f is a rotation with an angle of $-\frac{\pi}{6}$, and

$$\begin{cases} \mathbb{A} &= \left\{(x_1, x_2), x_1^2 + x_2^2 \leq 3\right\} \\ \mathbb{B} &= \left\{(x_1, x_2), (x_1 - 0.5)^2 + x_2^2 \leq 0.3\right\} \\ \mathbb{C} &= \left\{(x_1, x_2), (x_1 - 1)^2 + (x_2 - 1)^2 \leq 0.15\right\} \end{cases}$$



(a) $[\mathbb{A}]$

(b) $[\mathbb{B}]$

(c) $[\mathbb{C}]$

(d) $\mathbb{X} \subset \mathbb{A}$

(e) $\mathbb{B} \subset \mathbb{X}$

(f) $\mathbb{X} \cap \mathbb{C} = \emptyset$

(g) $f(\mathbb{X}) = \mathbb{X}$

(h) $(f(\mathbb{X}) = \mathbb{X})^{\infty}$