Guaranteed numerical methods to secure a zone with autonomous robots

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1. Interval analysis

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic

If $\diamond \in \{+, -, \cdot, /, \max, \min\}$

$$[x] \diamond [y] \quad = \quad [\ \{x \diamond y \mid x \in [x], y \in [y]\}\ ].$$

where $[\mathbb{A}]$ is the smallest interval which encloses $\mathbb{A} \subset \mathbb{R}$.

Exercise.

$$
\begin{aligned}
[-1,3]+[2,5] &= [?,?] \\
[-1,3]\cdot[2,5] &= [?,?] \\
[-2,6]/[2,5] &= [?,?]
\end{aligned}
$$

Solution.

$$\begin{aligned} [-1,3]+[2,5] &= [1,8] \\ [-1,3].[2,5] &= [-5,15] \\ [-2,6]/[2,5] &= [-1,3] \end{aligned}$$

Exercise. Compute

$$[-2,2]/[-1,1] \quad = [?,?]$$

Solution.

$$[-2,2]/[-1,1] \quad = [-\infty,\infty]$$

If $f \in \{\cos, \sin, \text{sqr}, \text{sqrt}, \log, \exp, \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

Exercise.

$$
\begin{aligned}
\sin([0,\pi]) &= ? \\
\mathsf{sqr}([-1,3]) &= [-1,3]^2 = ? \\
\mathsf{abs}([-7,1]) &= ? \\
\mathsf{sqrt}([-10,4]) &= \sqrt{[-10,4]} = ? \\
\log([-2,-1]) &= ?.
\end{aligned}
$$

Solution.

$$\begin{aligned}
\sin([0,\pi]) &= [0,1] \\
\mathsf{sqr}([-1,3]) &= [-1,3]^2 = [0,9] \\
\mathsf{abs}([-7,1]) &= [0,7] \\
\mathsf{sqrt}([-10,4]) &= \sqrt{[-10,4]} = [0,2] \\
\log([-2,-1]) &= \emptyset.
\end{aligned}$$

Inclusion functions

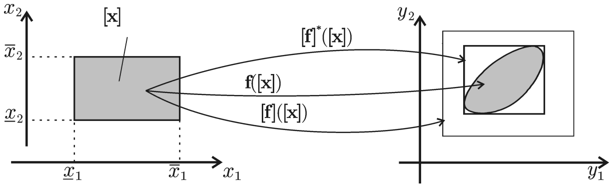
A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

$[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ is an *inclusion function* for $\mathbf{f}$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \ \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{x}]).$$



Inclusion functions $[\mathbf{f}]$ and $[\mathbf{f}]^*$; here, $[\mathbf{f}]^*$ is minimal.

Exercise. The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4.$$

For $[x] = [-3, 4]$, compute $[f]([x])$ and $f([x])$.

Solution. If $[x] = [-3, 4]$, we have

$$\begin{aligned} [f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\ &= [0,16] + [-6,8] + 4 \\ &= [-2,28]. \end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

A minimal inclusion function for

$$\mathbf{f}: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (x_1, x_2) & \mapsto & \left(x_1 x_2, x_1^2, x_1 - x_2\right). \end{array}$$

is

$$[\mathbf{f}]: \begin{array}{ccc} \mathbb{IR}^2 & \to & \mathbb{IR}^3 \\ ([x_1],[x_2]) & \to & \left([x_1]\cdot[x_2],[x_1]^2,[x_1]-[x_2]\right). \end{array}$$

If $\mathbf{f}$ is given by

Algorithm $\mathbf{f}(\text{in} : \mathbf{x} = (x_1, x_2, x_3), \text{ out} : \mathbf{y} = (y_1, y_2))$
$z := x_1$
for $k := 0$ to 100
$\quad z := x_2(z + k \cdot x_3)$
next
$y_1 := z$
$y_2 := \sin(z x_1)$

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in : $[\mathbf{x}] = ([x_1],[x_2],[x_3])$, out : $[\mathbf{y}] = ([y_1],[y_2])$)
$[z] := [x_1]$ for $k := 0$ to 100 $\quad\quad [z] := [x_2] \cdot ([z] + k \cdot [x_3])$ next $[y_1] := [z]$ $[y_2] := \sin([z] \cdot [x_1])$

Is $[\mathbf{f}]$ convergent? thin? monotonic?

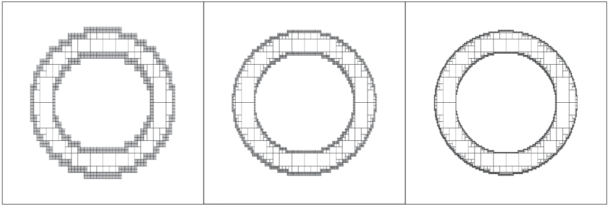
Set inversion

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$.
Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \in [1,2]\}.$$



Let $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbb{Y}$ be a subset of $\mathbb{R}^m$. Set inversion is the characterization of

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

We shall use the following tests:

$$\begin{array}{lllll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

Localization

A robot measures distances to three beacons.

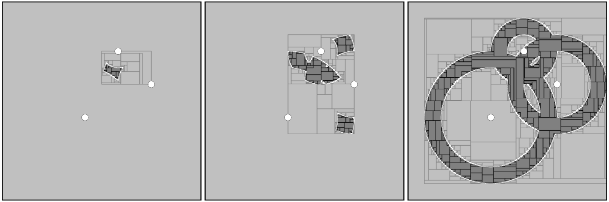
i	x_i	y_i	$[d_i]$
1	1	3	$[1,2]$
2	3	1	$[2,3]$
3	-1	-1	$[3,4]$

The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

Define

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} \in [d_i] \right\}.$$

$$\begin{aligned}
\text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{0\}}\right) &= 0.729 \\
\text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{1\}}\right) &= 0.972 \\
\text{prob}\left(\mathbf{p} \in \mathbb{P}^{\{2\}}\right) &= 0.999
\end{aligned}$$



2. Secure a zone

INFO OBS. Un sous-marin nucléaire russe repéré dans le Golfe de Gascogne

Le navire a été repéré en janvier. Ce serait la première fois depuis la fin de la Guerre Froide qu'un tel sous-marin, doté de missiles nucléaires, se serait aventuré dans cette zone au large des côtes françaises.

Bay of Biscay 220 000 km^2

An intruder

https://youtu.be/hNqIShmMQjA

- Several robots $\mathscr{R}_1, \ldots, \mathscr{R}_n$ at positions $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are moving in the ocean.
- If the intruder is in the visibility zone of one robot, it is detected.

Complementary approach

- We assume that a virtual intruder exists inside $\mathbb{G}$.
- We localize it with a set-membership observer inside $\mathbb{X}(t)$.
- The secure zone corresponds to the complementary of $\mathbb{X}(t)$.

Assumptions

- The intruder satisfies

$$\dot{\mathbf{x}} \in \mathbb{F}(\mathbf{x}(t)).$$

- Each robot $\mathscr{R}_i$ has the visibility zone $\mathbb{V}(\mathbf{a}_i)$

For instance

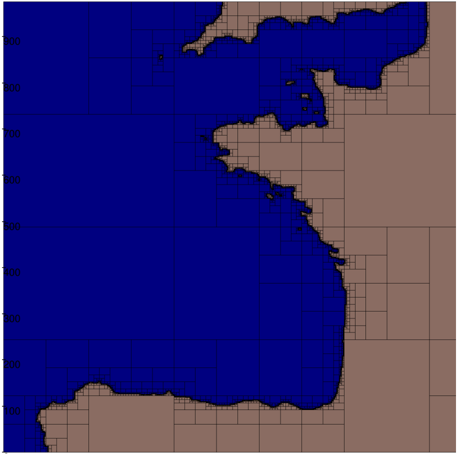
$$\mathbb{V}(\mathbf{a}_i) = \{\mathbf{x} | \text{ such that } \|\mathbf{a}_i - \mathbf{x}\| \leq 100\}$$

Theorem. An (undetected) intruder has a state vector $\mathbf{x}(t)$ inside the set

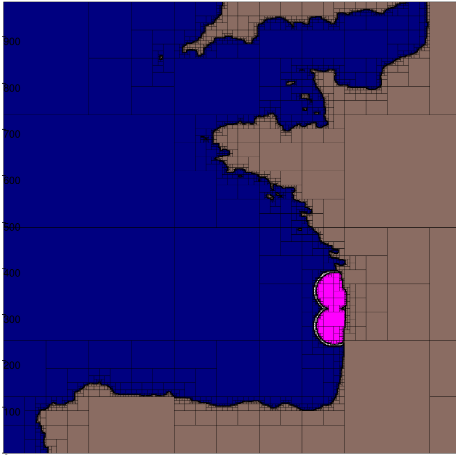
$$\mathbb{X}(t) = \mathbb{G} \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt)) \cap \bigcap_i \overline{\mathbb{V}(\mathbf{a}_i)}$$

where $\mathbb{X}(0) = \mathbb{G}$. The secure zone is

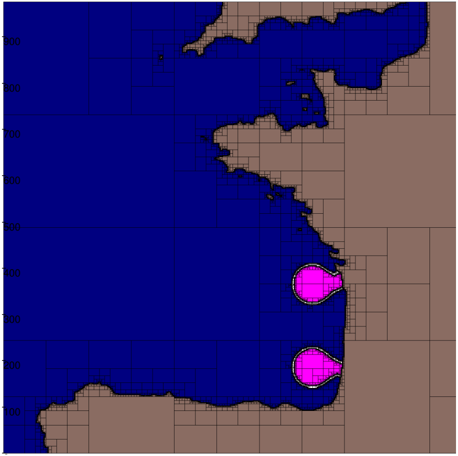
$$\mathbb{S}(t) = \overline{\mathbb{X}(t)}.$$



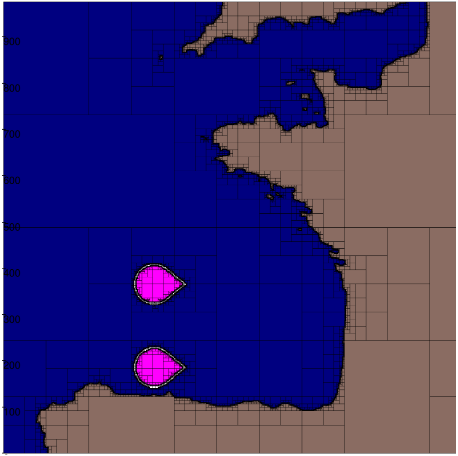
Set $\mathbb{G}$ in blue



Magenta: $\bigcup_i \mathbb{V}(\mathbf{a}_i)$ Blue: $\mathbb{G} \cap \bigcap_i \overline{\mathbb{V}(\mathbf{a}_i)}$



Blue: $\mathbb{X}(t) = \mathbb{G} \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt))) \cap \bigcap_i \overline{\mathbb{V}(\mathbf{a}_i)}$.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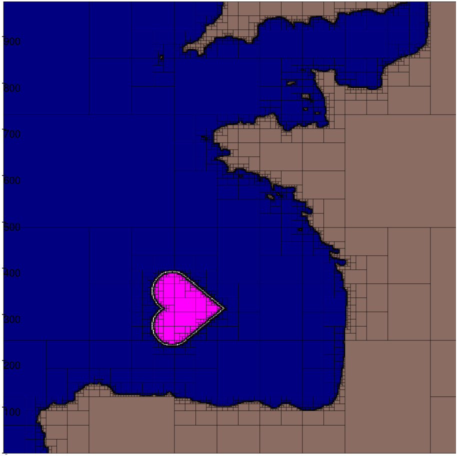
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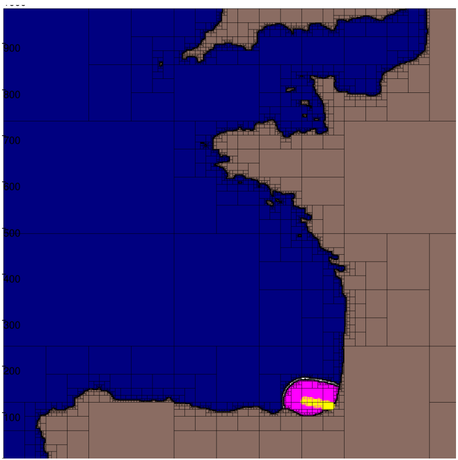
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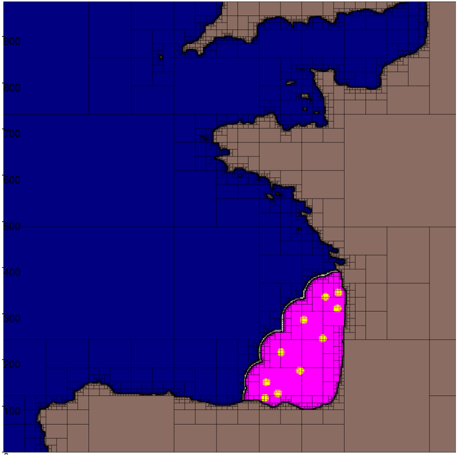
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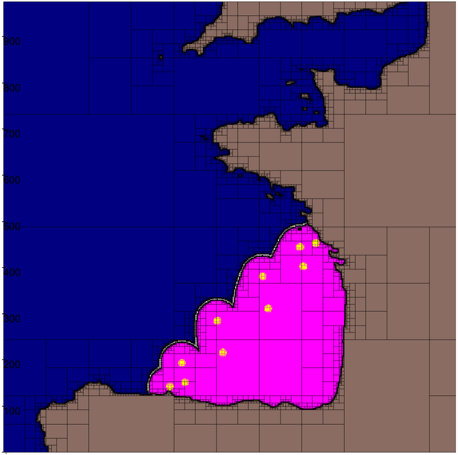
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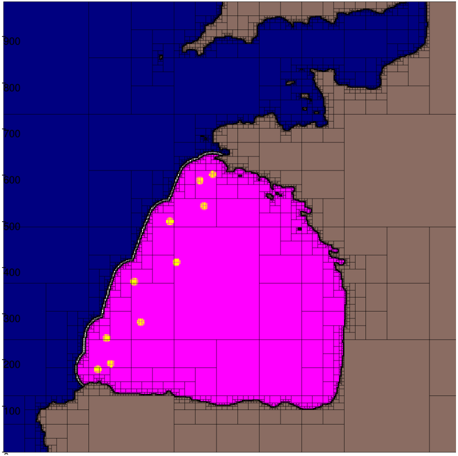
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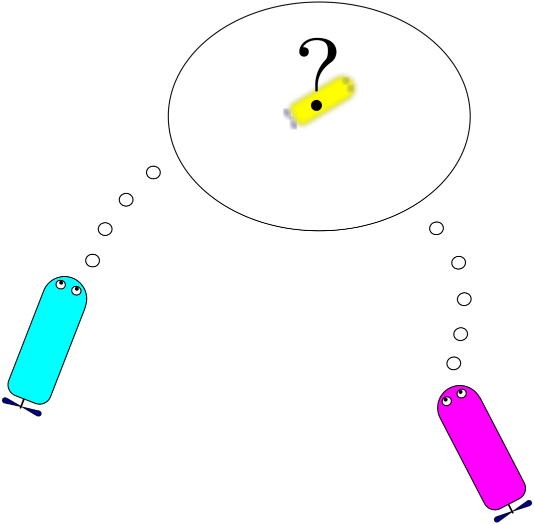
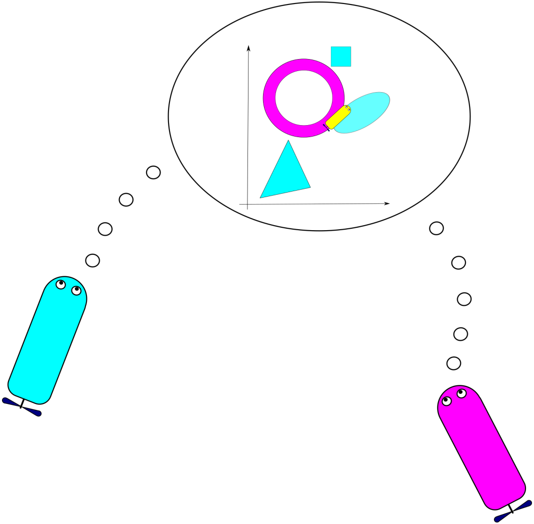
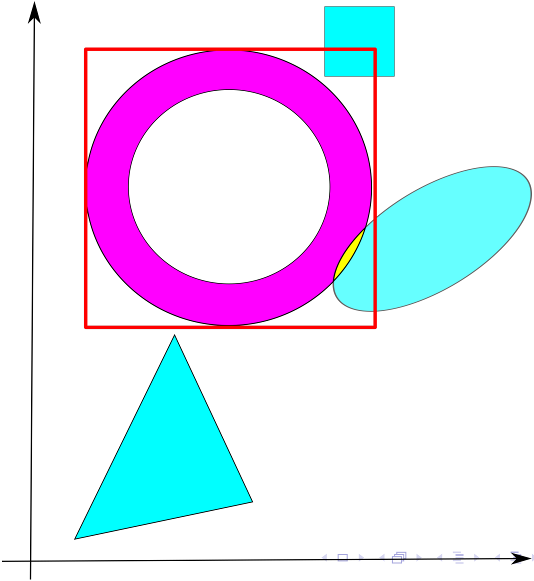
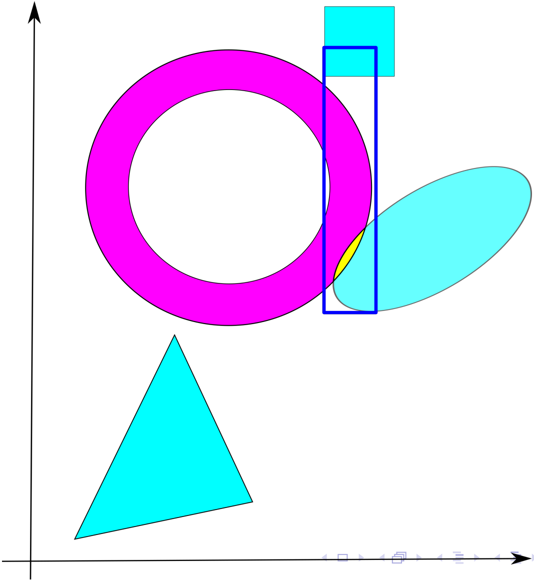
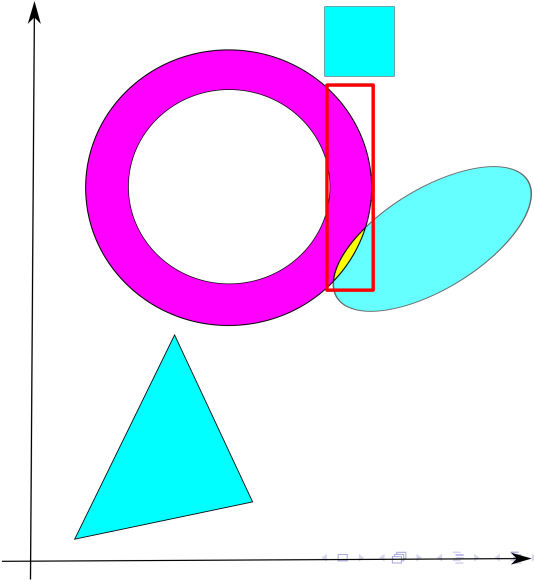
3. Distributed solving

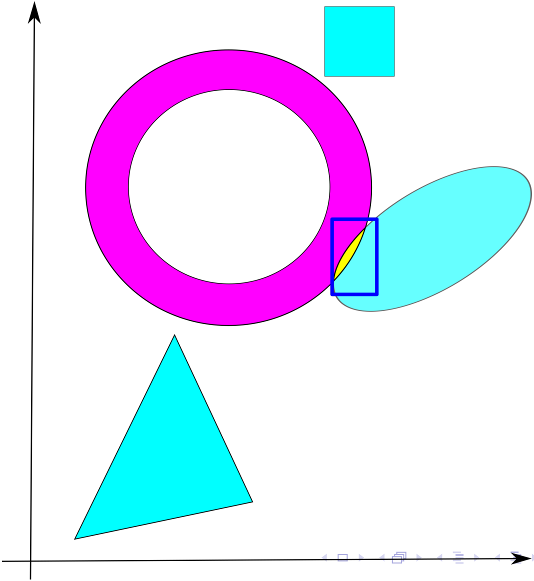








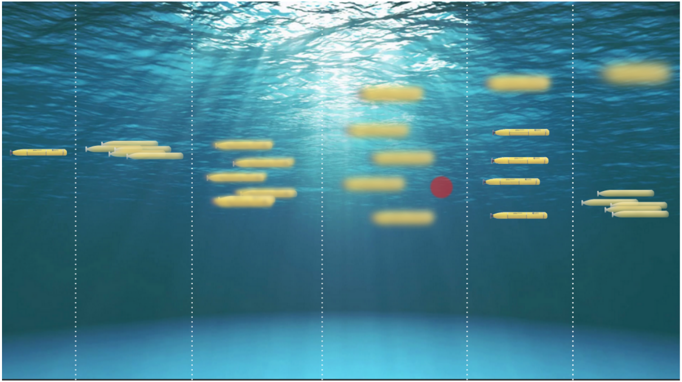




4. Prior validation

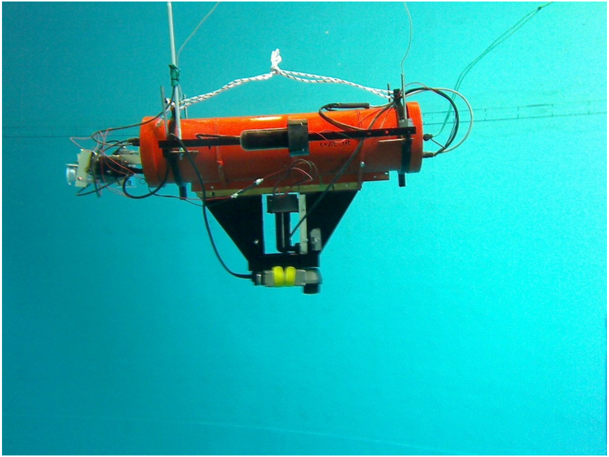
We want to check that

- The robots will not be lost
- A significant zone will be secured
- The secured zone will be known by us



A robot is a mechanical system equipped with

- actuators
- sensors
- an intelligence
- a memory.



Saucisse (ENSTA Bretagne)

Reachability

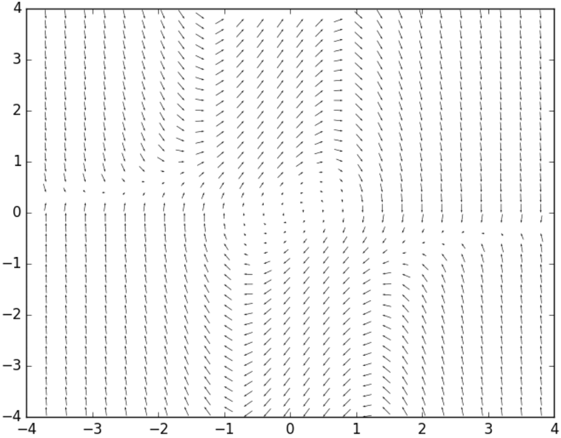
A dynamical system [Newton 1690]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

Example. Van der Pol oscillator

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\left(x_1^2 - 1\right) x_2 - x_1 \end{cases}$$

$$\mathbf{x}(0) \in \mathbb{X}_0$$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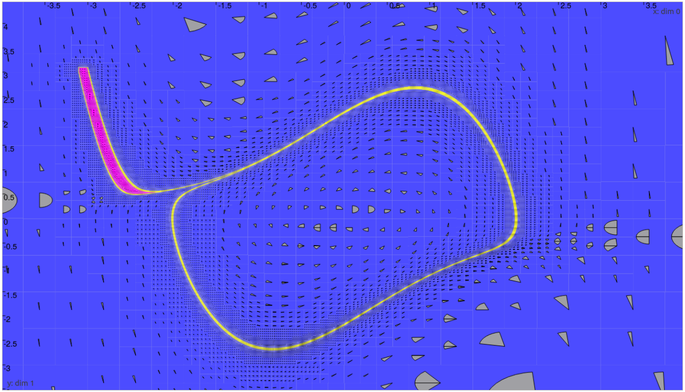

x: dim 0
y: dim 1

Vehicle

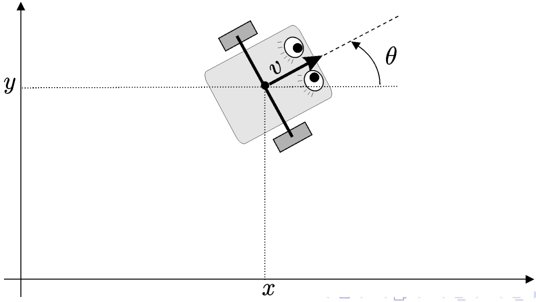
A **vehicle** is a dynamical system with actuators

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}).$$

Example. The Dubin's car (1957).

$$\begin{cases} \dot{x} & = & \cos\theta \\ \dot{y} & = & \sin\theta \\ \dot{\theta} & = & u \end{cases}$$

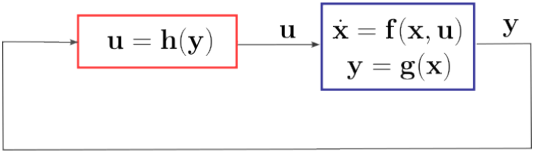
with $u \in [-1, 1]$.



Intelligence

A robot is a vehicle with sensors, actuators and an intelligence:

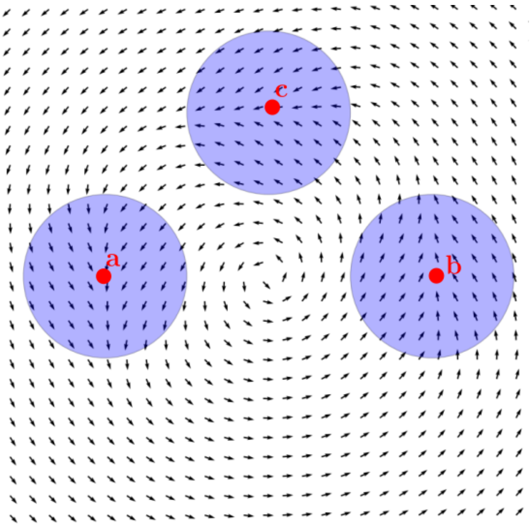
$$\begin{array}{rcll} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x},\mathbf{u}) & \text{(evolution)} \\ \mathbf{y} & = & \mathbf{g}(\mathbf{x}) & \text{(observation)} \\ \mathbf{u} & = & \mathbf{h}(\mathbf{y})\,. & \text{(control)} \end{array}$$



We have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{g}(\mathbf{x}))) = \psi(\mathbf{x})$$

Thus an intelligent vehicle is a dynamical system.


c
a
b

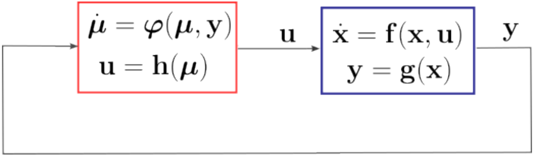
A robot has memory

A robot is a intelligent vehicle with memory

$$
\begin{array}{rcll}
\dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) & \text{(ontic evolution)} \\
\mathbf{y}(t) & = & \mathbf{g}(\mathbf{x}(t)) & \text{(observation)} \\
\mu_{k+1} & = & \varphi(\mu_k,\mathbf{y}(t_k)) & \text{(epistemic evolution)} \\
\mathbf{u}(t_k) & = & \mathbf{h}(\mu_k) & \text{(control)}
\end{array}
$$

With an analog controller

$$\begin{array}{rcll}
\dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) & \text{(ontic evolution)} \\
\mathbf{y}(t) & = & \mathbf{g}(\mathbf{x}(t)) & \text{(observation)} \\
\dot{\mu}(t) & = & \varphi(\mu(t),\mathbf{y}(t)) & \text{(epistemic evolution)} \\
\mathbf{u}(t) & = & \mathbf{h}(\mu(t)). & \text{(control)}
\end{array}$$

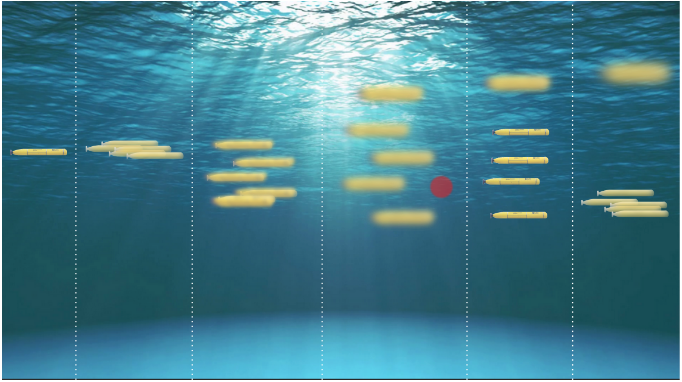


$$\begin{array}{lcll} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{h}(\mu(t))) & \text{(ontic evolution)} \\ \dot{\mu}(t) & = & \varphi(\mu(t), \mathbf{g}(\mathbf{x}(t))) & \text{(epistemic evolution)} \end{array}$$

The global state is $\mathbf{z} = (\mathbf{x}, \mu)$. We have

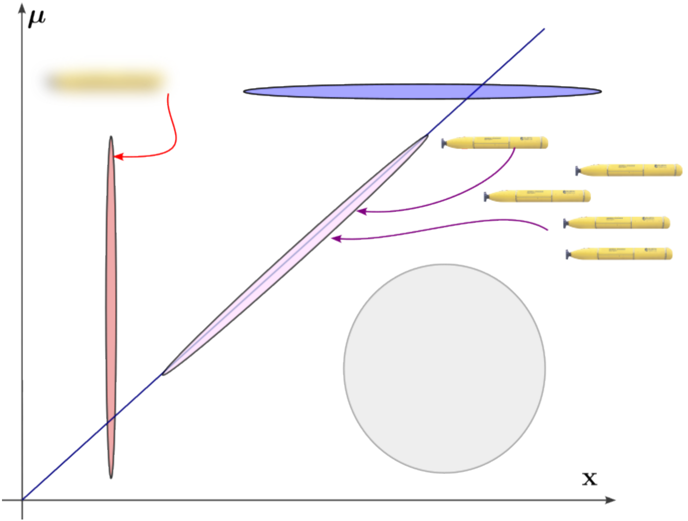
$$\dot{\mathbf{z}} = \psi(\mathbf{z})$$

An intelligent vehicle with memory is thus a dynamical system.



State of the robot : $\mathbf{z} = (\mathbf{x}, \mu)$, with

- $\mathbf{x}$: the ontic state
- μ: the epistemic state


μ
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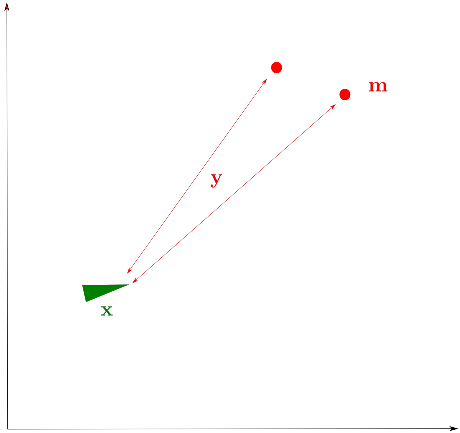
Perception : We measure $\mathbf{x}$
Communication : we measure μ

Swarm

https://youtu.be/xlgp9P0SY1Y
t=1:40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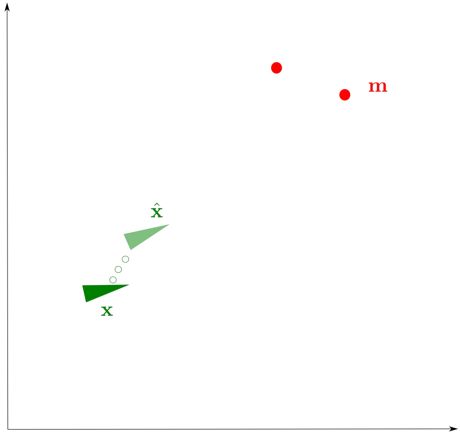
$$\begin{array}{rcl} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \mathbf{y} & = & \mathbf{g}(\mathbf{x}) \end{array}$$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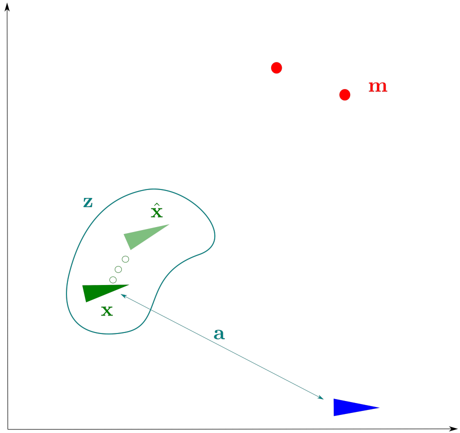
If we set $\mathbf{z} = (\mathbf{x}, \hat{\mathbf{x}})$, we get

$$\dot{\mathbf{z}} = \psi(\mathbf{z})$$

Assume that we can observe the motion of the robot

$$\begin{array}{rcl} \dot{\mathbf{z}} & = & \psi(\mathbf{z}) \\ \mathbf{a} & = & \eta(\mathbf{x}) \end{array}$$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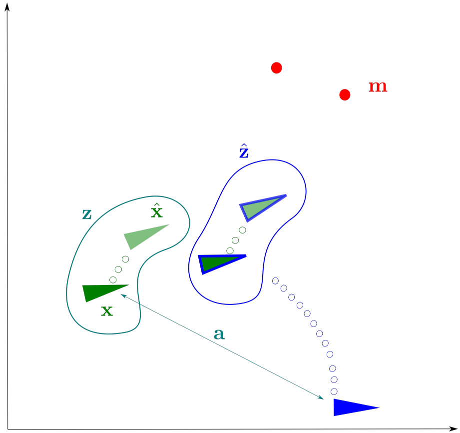
$\hat{x}$
x
a

We can build an observer for $\mathbf{z}$:

$$\begin{aligned} \dot{\mathbf{z}} &= \psi(\mathbf{z}) \\ \mathbf{a} &= \eta(\mathbf{x}) \\ \dot{\hat{\mathbf{z}}} &= \hat{\psi}(\mathbf{a}, \hat{\mathbf{z}}) \end{aligned}$$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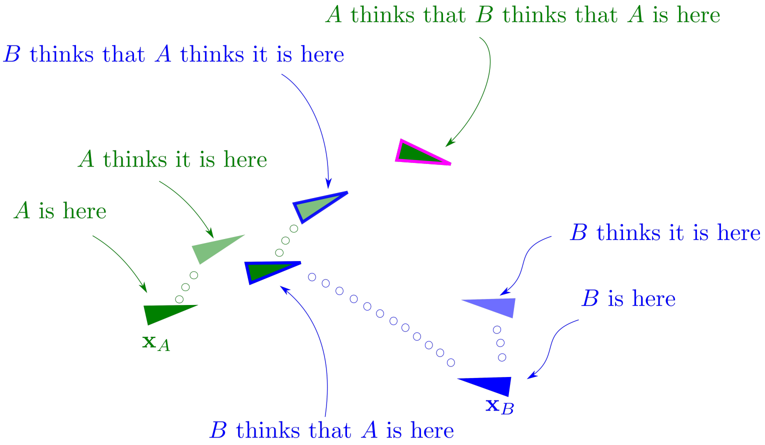
Distributed knowledge


A thinks that B thinks that A is here
B thinks that A thinks it is here
A thinks it is here
A is here
B thinks it is here
B is here
$\mathbf{x}_A$
$\mathbf{x}_B$
B thinks that A is here

References

1. Interval analysis [4, 2, 3]
2. Exploration : [6]
3. Secure a zone [5]
4. Swarm localization [1]

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