Proving the stability of navigation cycles

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SCAN 2021

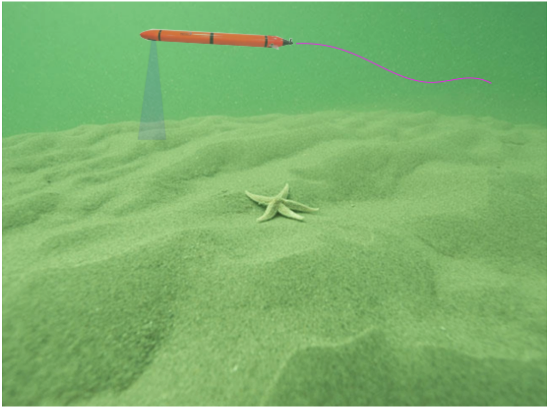
Video of the presentation: https://youtu.be/tZ0ApWd1pF0

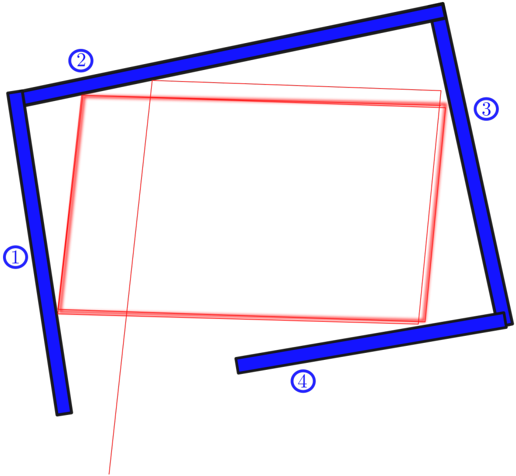
Stable cycles

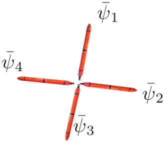
With Julien Damers, Simon Rohou, etc

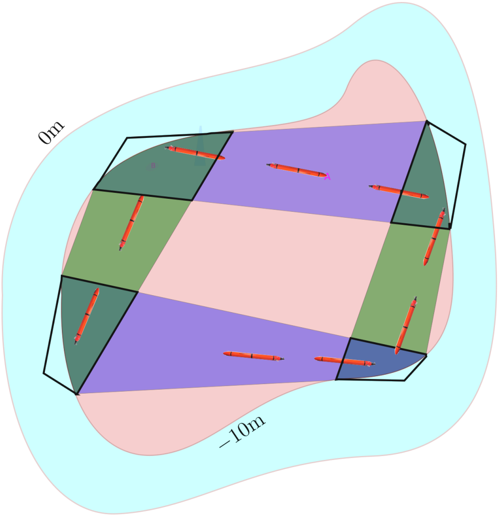
Submeeting 2018

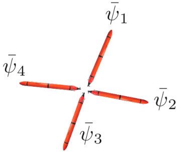
Cycles

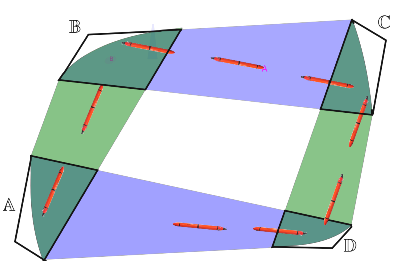


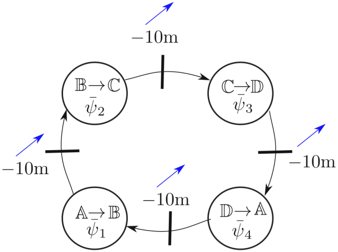

2
3
1
4


$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


0m
A
−10m


$\bar{\psi}_1$
$\bar{\psi}_4$
$\bar{\psi}_2$
$\bar{\psi}_3$


B
C
A
D

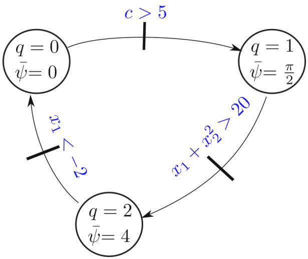

−10m
B→C
$\bar{\psi}_2$
C→D
$\bar{\psi}_3$
−10m
−10m
−10m
A→B
$\bar{\psi}_1$
D→A
$\bar{\psi}_4$

Test-case

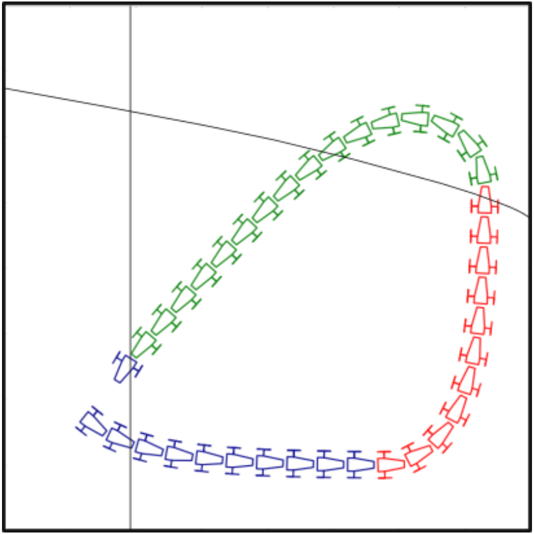
Consider the robot

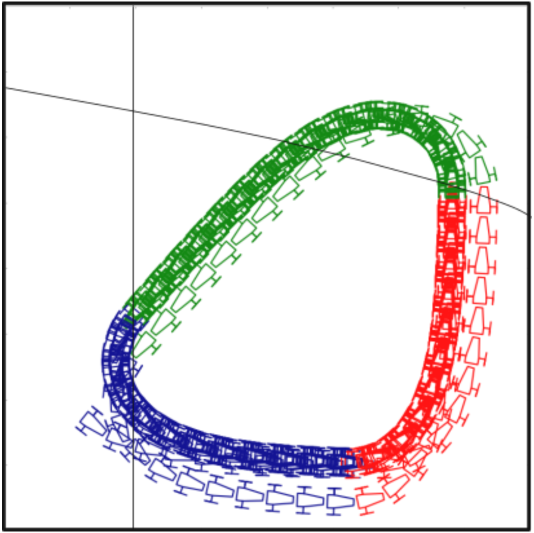
$$\begin{cases} \dot{x}_1 & = & \cos x_3 \\ \dot{x}_2 & = & \sin x_3 \\ \dot{x}_3 & = & u \end{cases}$$

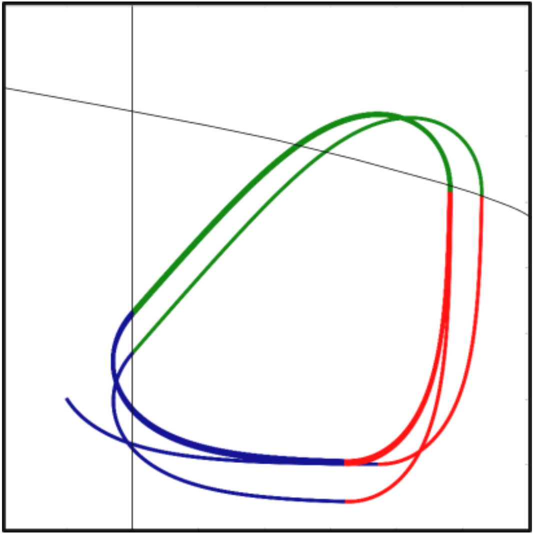
with the heading control $u = \sin(\bar{\psi} - x_3)$.



The simulation can be tested here :
https://replit.com/@aulin/poincare-car







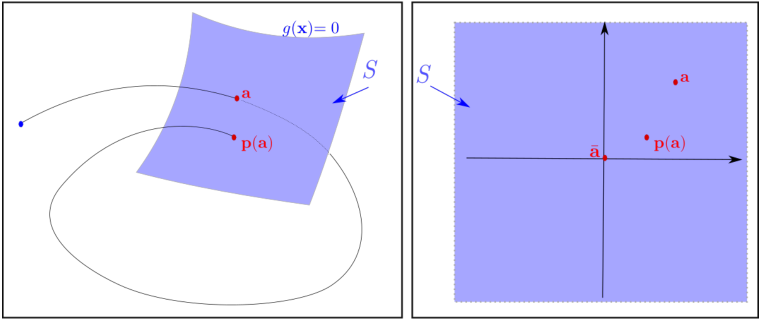
Stability with poincaré map

System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
Poincaré section: $g(\mathbf{x}) = 0$
Transversality: $g(\mathbf{x}) = 0 \Rightarrow \left(\frac{\partial g}{\partial \mathbf{x}} \cdot \mathbf{f}\right)(\mathbf{x}) \neq 0$

Define $\mathscr{G} = g^{-1}(0)$.

$$\mathbf{p} : \begin{array}{ccc} \mathscr{G} & \to & \mathscr{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where $\mathbf{p}(\mathbf{a})$ is the point of $\mathscr{G}$ such that the trajectory initialized at $\mathbf{a}$ intersects $\mathscr{G}$ for the first time.

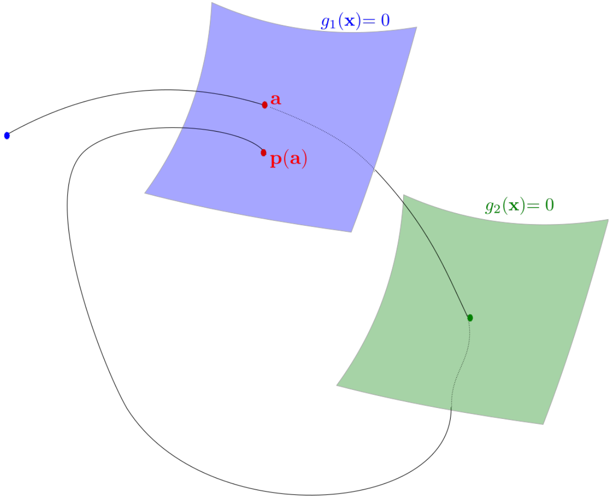

g(x)= 0
S
a
p(a)
S
a
ā
p(a)

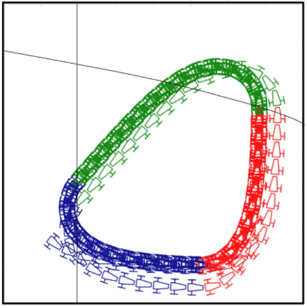
The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$

With hybrid systems

Systems: $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}), i \in \{1, \dots, m\}$
Section i: $g_i(\mathbf{x}) = 0$
Transversality: $g_i(\mathbf{x}) = 0 \Rightarrow \left(\frac{\partial g_i}{\partial \mathbf{x}} \cdot \mathbf{f}_i \right)(\mathbf{x}) \neq 0$
Automaton: $g_i(\mathbf{x}) = 0 \Rightarrow i := \mathrm{mod}(i+1, m)$


$g_1(\mathbf{x}) = 0$
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$
$g_2(\mathbf{x}) = 0$



After 10 laps, we get approximately $\bar{\mathbf{a}} = (-2, 2.3, -2.29)$.

To conclude about the stability, we compute the Jacobian matrix

$$\mathbf{J}(\bar{\mathbf{a}}) = \frac{d\mathbf{p}}{d\mathbf{a}}(\bar{\mathbf{a}}) =$$

$$\begin{pmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \frac{\partial p_1}{\partial x_3} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \frac{\partial p_2}{\partial x_3} \\ \frac{\partial p_3}{\partial x_1} & \frac{\partial p_3}{\partial x_2} & \frac{\partial p_3}{\partial x_3} \end{pmatrix}$$

Using a non validated numerical method, we get

$$\begin{pmatrix} \frac{\partial p_2}{\partial x_2}(\bar{\mathbf{a}}) & \frac{\partial p_2}{\partial x_3}(\bar{\mathbf{a}}) \\ \frac{\partial p_3}{\partial x_2}(\bar{\mathbf{a}}) & \frac{\partial p_3}{\partial x_3}(\bar{\mathbf{a}}) \end{pmatrix} \simeq \begin{pmatrix} 0.045 & 2.66 \\ 0.0 & 0.02 \end{pmatrix}.$$

The eigen values are approximately

$$\{0.06, 0.009\}$$

Both are in the unit circle. We conclude that the limit cycle is stable.

A validated apporach is presented in [2][5].
It is based on interval analysis and guaranteed ODE solver (CAPD [4], [6]).
Several numerical examples are given in the thesis of A. Bourgois [1].

Validated stability

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

The flow $\mathbf{\Phi}(\mathbf{x}_0, t)$ is defined as the solution at time t for the initial vector $\mathbf{x}_0$.
Define $\mathbf{A}(\mathbf{x}_0, t) = \frac{\partial \mathbf{\Phi}(\mathbf{x}_0, t)}{\partial \mathbf{x}_0}$. We have the *variational equation*

$$\dot{\mathbf{A}} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{A}$$

with $\mathbf{A}(0) = \mathbf{I}$.

Example [3]: Van der Pol system.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 \end{pmatrix}.$$

We have

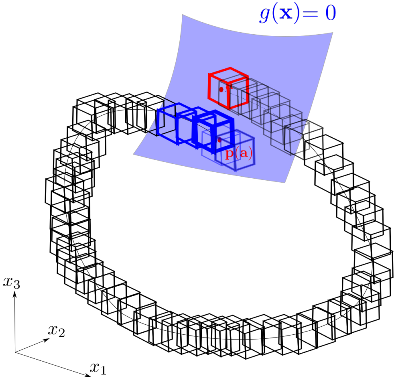
$$\begin{pmatrix} \dot{a}_{11} & \dot{a}_{12} \\ \dot{a}_{21} & \dot{a}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2x_1x_2 - 1 & 1 - x_1^2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

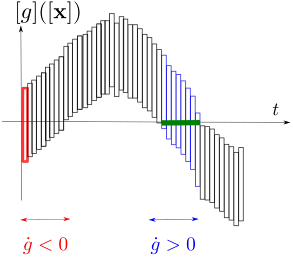
with

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

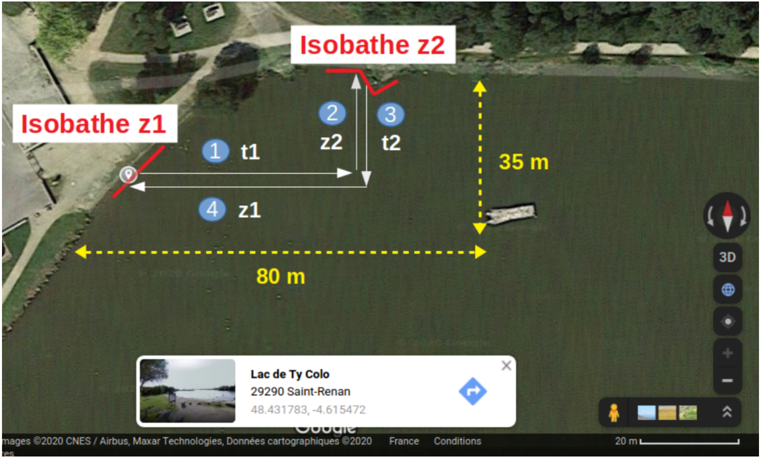
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{a}_{11} \\ \dot{a}_{12} \\ \dot{a}_{21} \\ \dot{a}_{22} \end{pmatrix} = \begin{pmatrix} x_2 \\ (1-x_1^2)x_2 - x_1 \\ a_{21} \\ a_{22} \\ (-2x_1x_2-1)\,a_{11} + \left(1-x_1^2\right)a_{21} \\ (-2x_1x_2-1)\,a_{12} + \left(1-x_1^2\right)a_{22} \end{pmatrix}$$

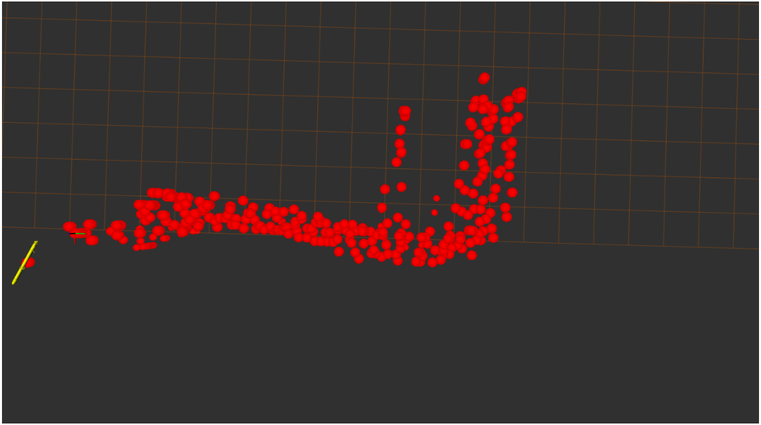
Using an interval ODE solver, we get an enclosure for $\mathbf{x}(t), \mathbf{A}(t)$, for a given initial box $[\mathbf{x}_0]$.
We can also get the time at which the system crosses the surface $g(\mathbf{x}) = 0$.

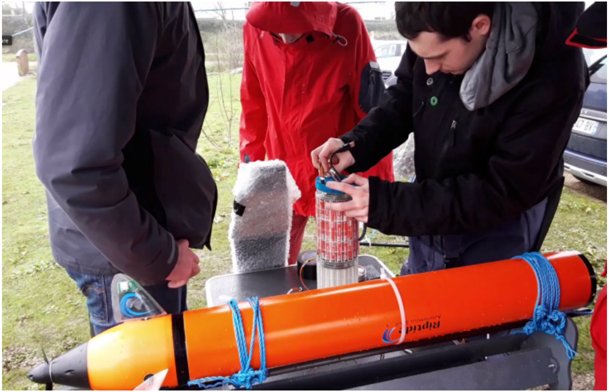

$g(\mathbf{x}) = 0$
p(a)
x_3
x_2
x_1


$[g]([\mathbf{x}])$
t
$\dot{g} < 0$
$\dot{g} > 0$

Experiment


Isobathe z2
Isobathe z1
1 t1
2 z2
3 t2
4 z1
35 m
80 m
3D
Lac de Ty Colo
29290 Saint-Renan
48.431783, -4.615472
mages ©2020 CNES / Airbus, Maxar Technologies, Données cartographiques ©2020 France Conditions
20 m





Stable cycles

Youtube

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A new method for integrating ODE based on monotonicity.
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