

Computing capture tubes

Würzburg, September 22, 2014

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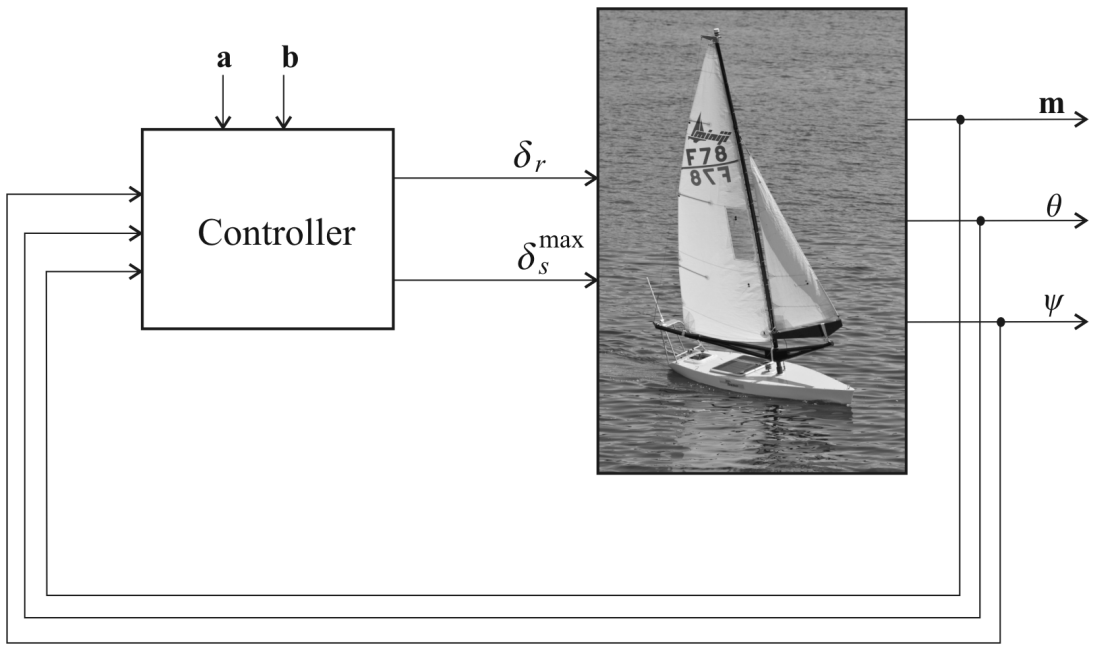
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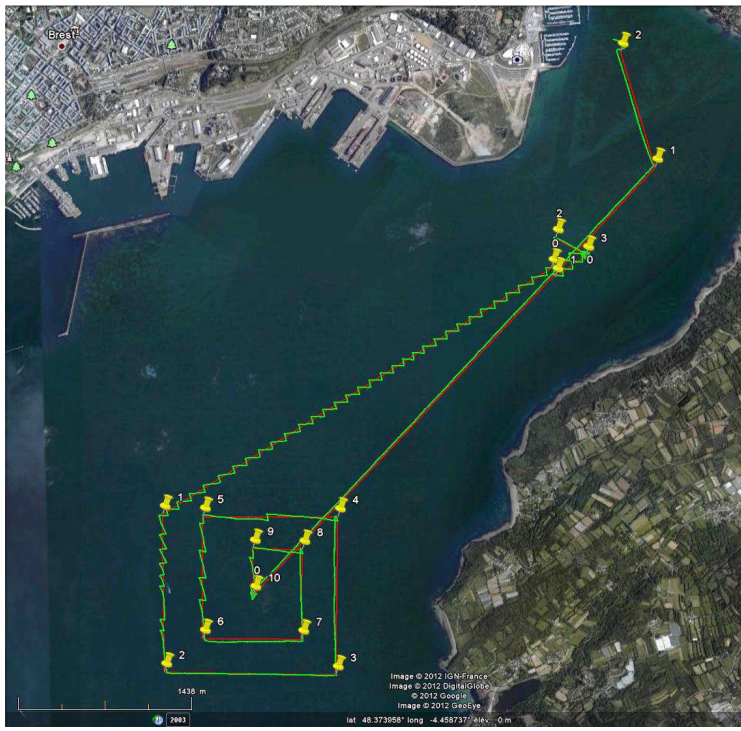
SCAN 2014.

1 V-stability

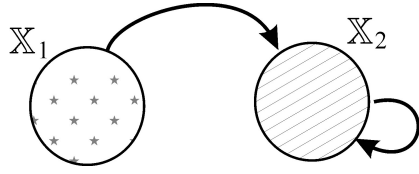


Vaimos (IFREMER and ENSTA)





$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$



X_1 : outside the corridor.

X_2 : inside the corridor.

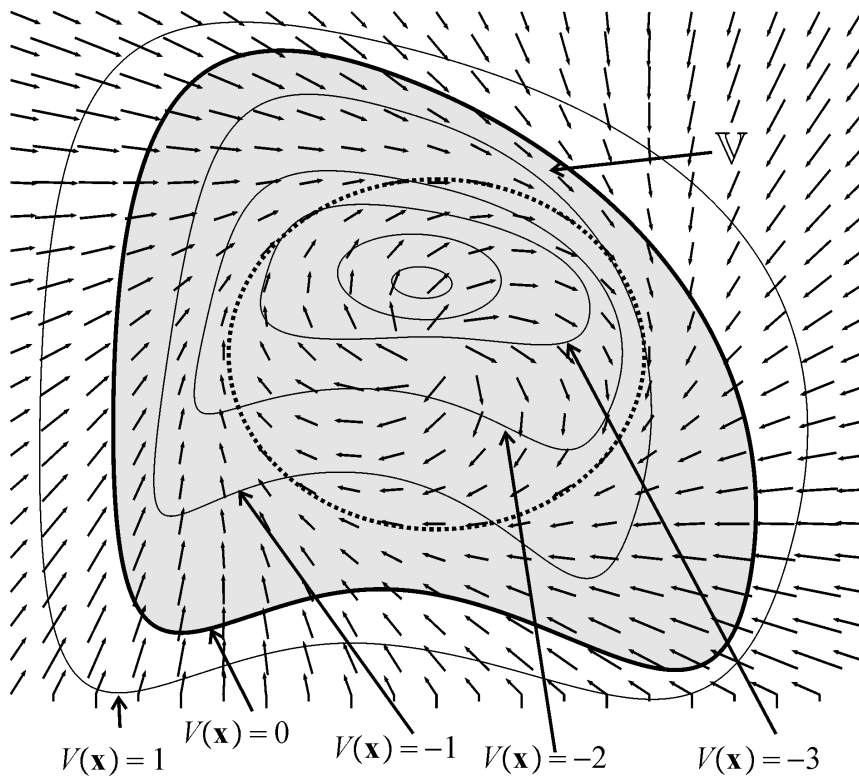
Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$. The system is V -stable if

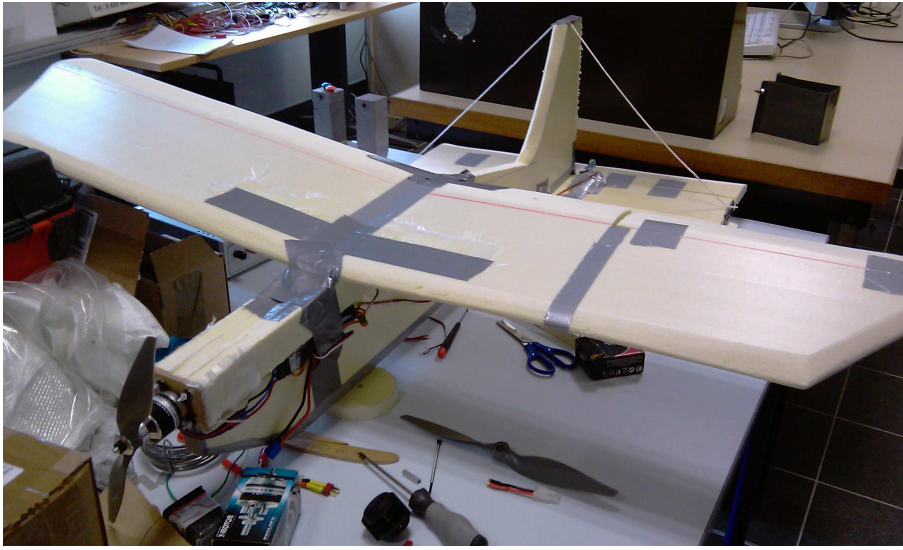
$$\left(V(\mathbf{x}) \geq 0 \Rightarrow \dot{V}(\mathbf{x}) < 0 \right).$$

Since

$$\dot{V}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$$

Checking the V -stability can be done using interval analysis.

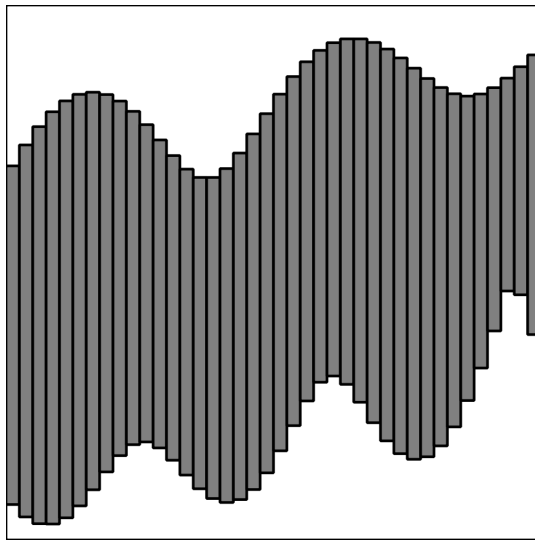




Non-holonomic system

2 Tubes

A tube is a function which associates to any $t \in \mathbb{R}$ a subset of \mathbb{R}^n .



In the machine a tube can be represented by two stair functions

Example of tubes

$$[f](t) = [1, 2] \cdot t + \sin([1, 3] \cdot t)$$

$$[g](t) = [a_0] + [a_1]t + [a_2]t^2 + [a_3]t^3$$

$$\int_0^t [g](\tau) d\tau = [a_0]t + [a_1]\frac{t^2}{2} + [a_2]\frac{t^3}{3} + [a_3]\frac{t^4}{4}.$$

3 Capture tubes

Consider the time dependant system

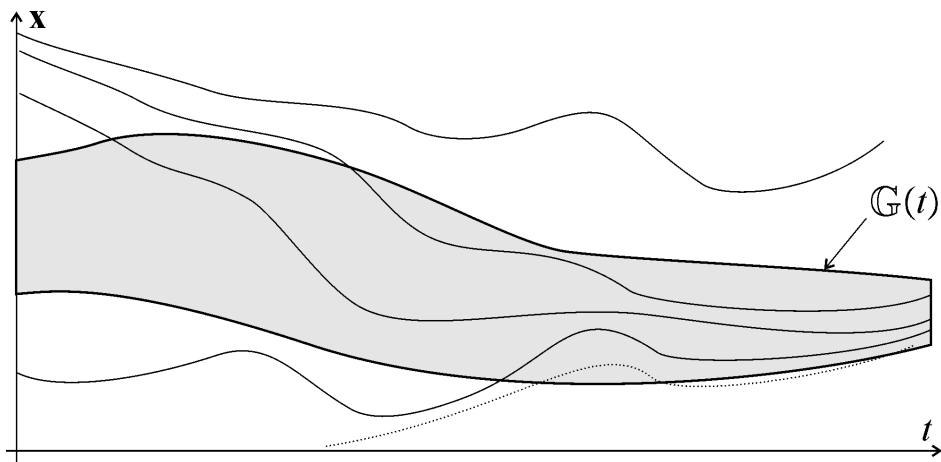
$$\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

and a *tube*

$$\mathbb{G}(t) \subset \mathbb{R}^n, t \in \mathbb{R}.$$

The tube $\mathbb{G}(t)$ is said to be a *capture tube* if

$$\mathbf{x}(t) \in \mathbb{G}(t), \tau > 0 \Rightarrow \mathbf{x}(t + \tau) \in \mathbb{G}(t + \tau).$$



Theorem. Consider the tube

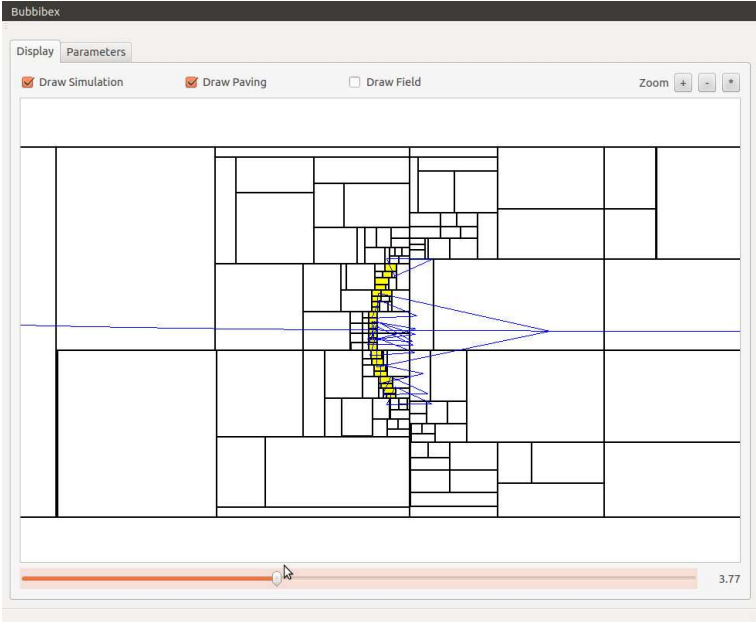
$$\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}$$

where $\mathbf{g} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m$. If the *cross out* condition

$$\left\{ \begin{array}{l} \underbrace{\frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t)}_{\dot{g}_i(\mathbf{x}, t)} \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0} \end{array} \right.$$

is inconsistent for all (\mathbf{x}, t, i) , then $\mathbb{G}(t)$ is a capture tube for $\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

A software Bubbibex (using Ibex) made by students from ENSTA Bretagne uses interval analysis to prove the inconsistency.



4 Test-case

Robot

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{\theta} = -\theta. \end{cases}$$

Target $(x_d, y_d) = (t, 0)$. We choose the control

$$u_1 = -x + t, \quad u_2 = -y.$$

The closed loop system satisfies

$$\begin{cases} \dot{x} = -x + t \\ \dot{y} = -y \\ \dot{\theta} = -\theta. \end{cases}$$

Target tube. The tube we want is

$$\mathbb{G}(t) = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\},$$

with

$$\begin{cases} g_1(\mathbf{x}, t) = (x_1 - t)^2 + x_2^2 - r^2 \\ g_2(\mathbf{x}, t) = (\cos x_3 - 1)^2 + \sin^2 x_3 - 0.2. \end{cases}$$

For $r = 4$, Bubbibex proves that $\mathbb{G}(t)$ is a capture tube.

For $r < 1$, some trajectories leave $\mathbb{G}(t)$ forever.

5 Lattice and capture tubes

Consider $\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

If \mathbb{T} is the set of tubes and \mathbb{T}_c is the set of all capture tubes of \mathcal{S} then (\mathbb{T}_c, \subset) is a sublattice of (\mathbb{T}, \subset) .

We have indeed

$$\begin{cases} G_1(t) \in \mathbb{T}_c \\ G_2(t) \in \mathbb{T}_c \end{cases} \Rightarrow \begin{cases} G_1(t) \cap G_2(t) \in \mathbb{T}_c \\ G_1(t) \cup G_2(t) \in \mathbb{T}_c \end{cases}$$

6 Computing capture tubes

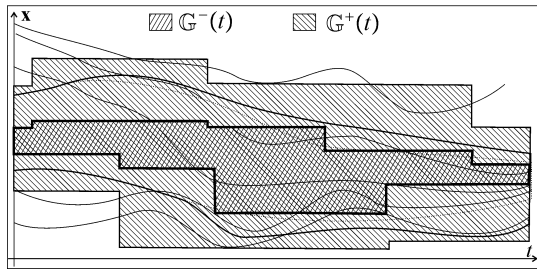
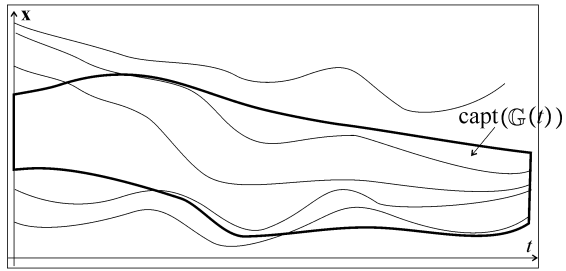
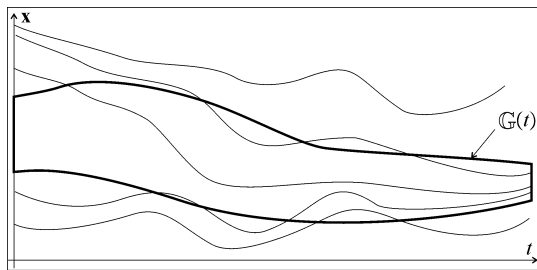
If $\mathbb{G}(t) \in \mathbb{T}$, define

$$\text{capt}(\mathbb{G}(t)) = \bigcap \{ \overline{\mathbb{G}}(t) \in \mathbb{T}_c \mid \mathbb{G}(t) \subset \overline{\mathbb{G}}(t) \}.$$

This set is the smallest capture tube enclosing $\mathbb{G}(t)$.

Problem. Given $\mathbb{G}(t) \in \mathbb{T}$, compute an interval $[\mathbb{G}^-(t), \mathbb{G}^+(t)] \in \mathbb{IT}$ such that

$$\text{capt}(\mathbb{G}(t)) \in [\mathbb{G}^-(t), \mathbb{G}^+(t)].$$



Flow. The flow associated with the system $\mathcal{S}_f : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is a function $\phi_{t_0, t_1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \Rightarrow \phi_{t_0, t_1}(\mathbf{x}(t_0)) = \mathbf{x}(t_1).$$

Proposition. For the system $\mathcal{S}_f : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ and the tube $\mathbb{G}(t)$, we have

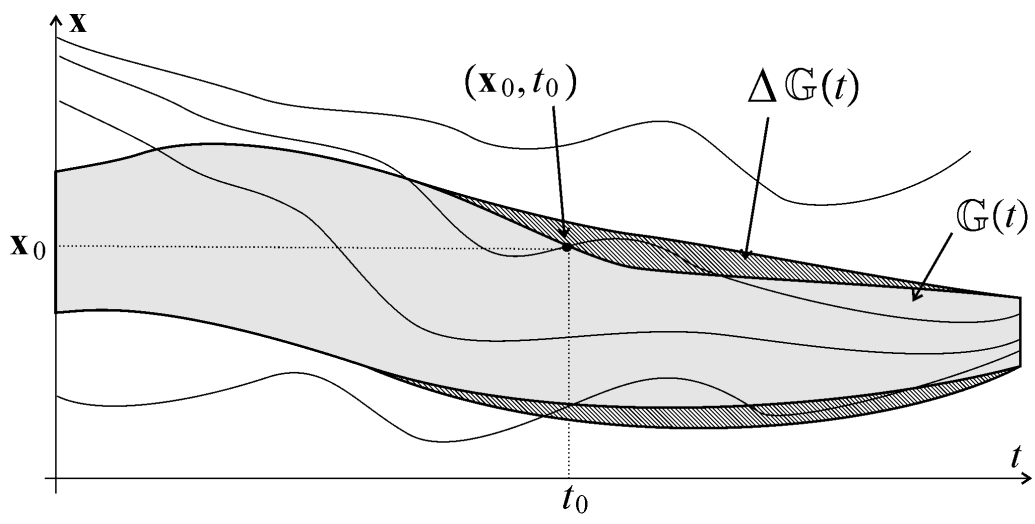
$$\text{capt}(\mathbb{G}(t)) = \mathbb{G}(t) \cup \Delta\mathbb{G}(t),$$

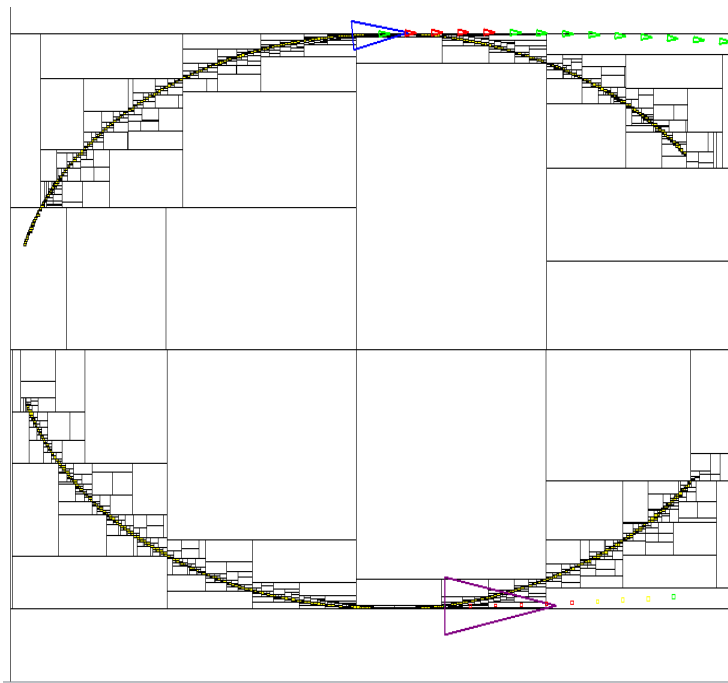
with

$$\Delta\mathbb{G}(t) = \{(\mathbf{x}, t) \mid \exists (\mathbf{x}_0, t_0) \text{ satisfying the cross out condition } t \geq t_0, \phi_{t_0, t}(\mathbf{x}_0) \notin \mathbb{G}(t)\}$$

Recall the cross out condition:

$$\left\{ \begin{array}{l} \frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t) \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq 0 \end{array} \right.$$





7 Test case 1

Consider the system

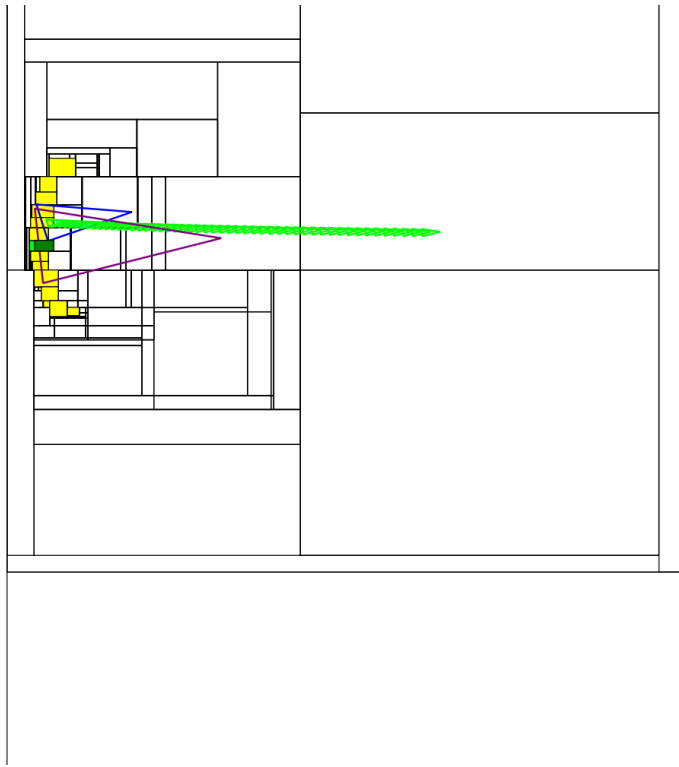
$$\mathcal{S} : \begin{cases} \dot{x} &= -x + t \\ \dot{y} &= -y \\ \dot{\theta} &= -\theta. \end{cases}$$

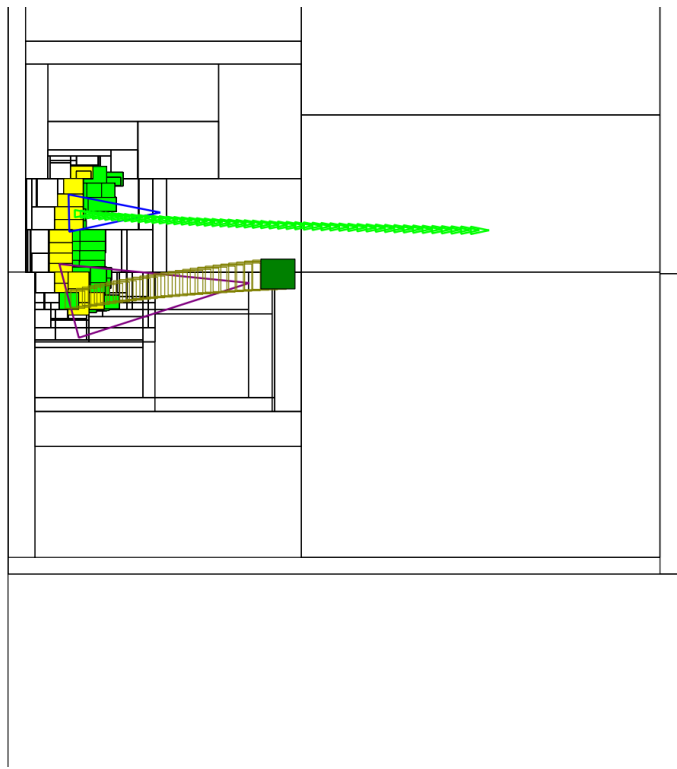
and the tube

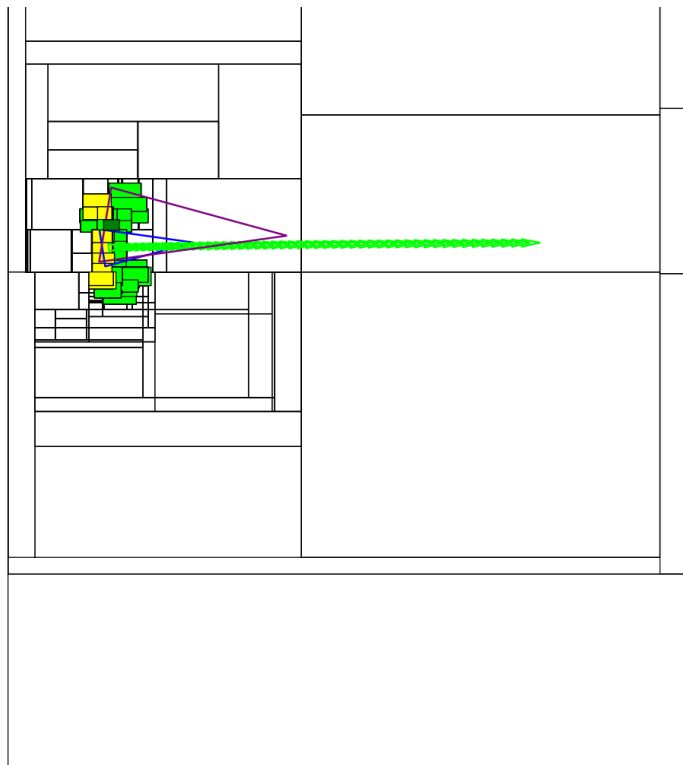
$$\mathbb{G}(t) : \begin{cases} g_1(\mathbf{x}, t) &= (x_1 - t)^2 + x_2^2 - r(t) \\ g_2(\mathbf{x}, t) &= (\cos x_3 - 1)^2 + \sin^2 x_3 - 0.2. \end{cases}$$

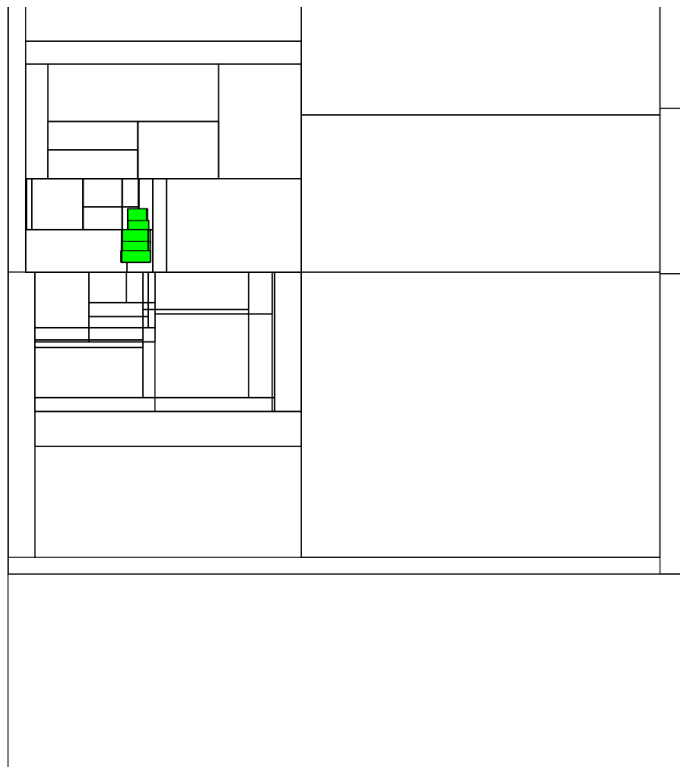
$$r(t) = 0.2 \cdot (t + 1)^2.$$

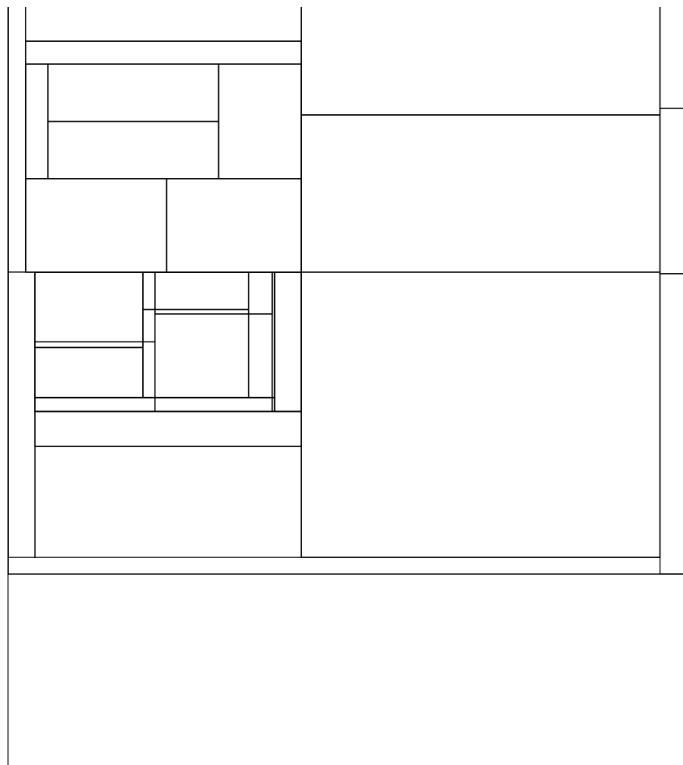
Some trajectories leave and come back to $\mathbb{G}(t)$.











8 Test case 2

Consider the robot (Dubin's car)

$$\begin{cases} \dot{x} &= \cos \theta \\ \dot{y} &= \sin \theta \\ \dot{\theta} &= u \end{cases}$$

where $u \in [-2, 2]$.

To move toward the target (x_d, y_d) , we take the controller:

$$\begin{cases} \mathbf{n} &= \frac{1}{\sqrt{(x_d-x)^2+(y_d-y)^2}} \begin{pmatrix} x_d - x \\ y_d - y \end{pmatrix} + \frac{2}{\sqrt{\dot{x}_d^2+\dot{y}_d^2}} \begin{pmatrix} \dot{x}_d \\ \dot{y}_d \end{pmatrix} \\ \theta_d &= \text{atan2}(\mathbf{n}) \\ u &= -2 \cdot \sin(\theta - \theta_d). \end{cases}$$

Target

$$\begin{cases} x_d(t) = \rho_x \cos t \\ y_d(t) = \rho_y \sin t. \end{cases}$$

For the derivative, we get

$$\begin{cases} \dot{x}_d(t) = -\rho_x \sin t \\ \dot{y}_d(t) = \rho_y \cos t. \end{cases}$$

Target tube. We want the robot to stay inside the set

$$\mathbb{G}(t) = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\},$$

with

$$\begin{cases} g_1(\mathbf{x}, t) = (x - x_d)^2 + (y - y_d)^2 - \rho^2 \\ g_2(\mathbf{x}, t) = \left(\cos \theta - \frac{n_x}{\|\mathbf{n}\|}\right)^2 + \left(\sin \theta - \frac{n_y}{\|\mathbf{n}\|}\right)^2 - \alpha^2. \end{cases}$$

Resolution. We used the solver Bubbibex.

The tube is proved to be unsafe.

For given parameters, Bubbibex is able to compute the margin (i.e., $\text{width}\left(\left[\mathbb{G}^-(t), \mathbb{G}^+(t)\right]\right)$).

References

L. Jaulin and F. Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. *IEEE Transaction on Robotics*, Volume 27, Issue 5.