Contractors and QUIMPER language

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1 Contractors

(i) (contractance)
$\forall[\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$

(ii) (consistency)
$(\mathbf{x} \in [\mathbf{x}], \mathcal{C}(\{\mathbf{x}\}) = \{\mathbf{x}\}) \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}])$

(iii) (weak continuity)
$\mathcal{C}(\{\mathbf{x}\}) = \emptyset \Rightarrow (\exists \varepsilon > 0, \forall [\mathbf{x}] \subset B(\mathbf{x}, \varepsilon), \mathcal{C}([\mathbf{x}]) = \emptyset)$

where $B(\mathbf{x}, \varepsilon)$ is the ball with center $\mathbf{x}$ and radius ε.

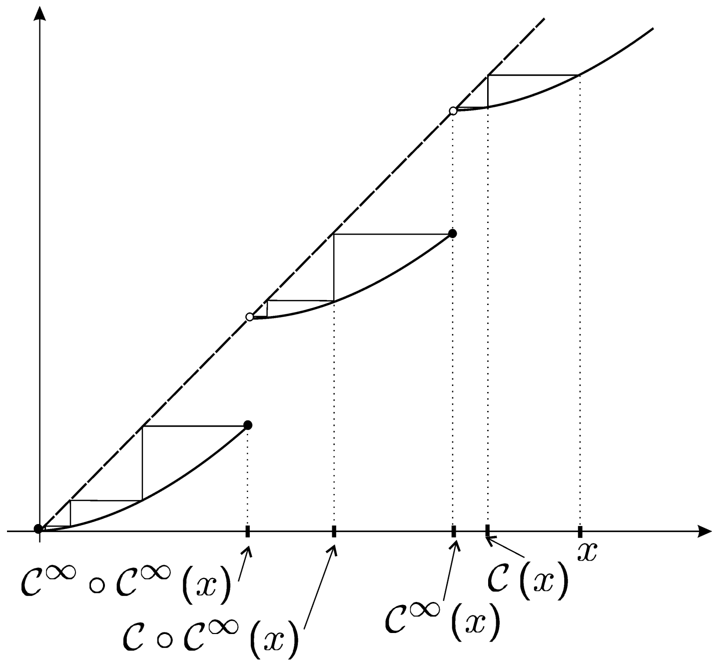
The operator $\mathcal{C}_1 : \mathbb{IR} \rightarrow \mathbb{IR}$ defined by

$$\mathcal{C}_1([a,b]) : \begin{cases} = [a, \frac{a+b}{2}] & \text{if } a \neq b \\ = \emptyset & \text{if } a = b \end{cases}$$

does not satisfy the weak continuity condition and thus, it is not a contractor.

Question :

$$\begin{aligned} \mathcal{C}_1 \circ \mathcal{C}_1 \circ \cdots \circ \mathcal{C}_1([2,3]) &= \{2\} \text{ or} \\ \mathcal{C}_1 \circ \mathcal{C}_1 \circ \cdots \circ \mathcal{C}_1([2,3]) &= \emptyset \text{ ?} \end{aligned}$$



Question : How many fixed points do we have?

The set (or constraint) associated to the contractor $\mathcal{C}$ is,

$$\text{set}\,(\mathcal{C}) = \{\mathbf{x} \in \mathbb{R}^n, \mathcal{C}(\{\mathbf{x}\}) = \{\mathbf{x}\}\}\,.$$

Question. Is set($\mathcal{C}$) always closed?

$\mathcal{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is *minimal* if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathcal{C}([\mathbf{x}]) = [[\mathbf{x}] \cap \mathsf{set}\,(\mathcal{C})]$
$\mathcal{C}$ is *idempotent* if	$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\,(\mathcal{C}([\mathbf{x}])) = \mathcal{C}([\mathbf{x}]),$
$\mathcal{C}$ is *continuous* if	$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\,(\mathcal{C}^{\infty}([\mathbf{x}])) = \mathcal{C}^{\infty}([\mathbf{x}])$

Completeness. We have

$$\mathcal{C}([\mathbf{x}]) \supset [\mathbf{x}] \cap \mathsf{set}\,(\mathcal{C})\,.$$

Example. A *precision contractor* is defined by

$$\mathcal{C}_\varepsilon([\mathbf{x}]) : \begin{cases} = [\mathbf{x}] & \text{if } w([\mathbf{x}]) > \varepsilon \\ = \emptyset & \text{otherwise} \end{cases}$$

where $\varepsilon > 0$.

Questions

$\mathcal{C}_\varepsilon$ is monotonic ?
$\mathcal{C}_\varepsilon$ idempotent ?
$\mathcal{C}_\varepsilon$ is not minimal ?
What is set($\mathcal{C}_\varepsilon$) ?.
For $\varepsilon = 0$, is $\mathcal{C}_\varepsilon$ a contractor ?

2 Operations on contractors

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
repetition	$\mathcal{C}^{\infty} \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$
repeted intersection	$\mathcal{C}_1 \sqcap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^{\infty}$
repeted union	$\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^{\infty}$

Questions:

$$
\begin{aligned}
\mathcal{C}_1 \circ \mathcal{C}_2 &= \mathcal{C}_2 \circ \mathcal{C}_1?\\
\mathcal{C}_1 \cap \mathcal{C}_2 &= \mathcal{C}_2 \cap \mathcal{C}_1?\\
\mathcal{C}_1 \sqcap \mathcal{C}_2 &= \mathcal{C}_2 \sqcap \mathcal{C}_1?\\
\mathcal{C}_1 \cup \mathcal{C}_2 &= \mathcal{C}_2 \cup \mathcal{C}_1?\\
\mathcal{C}_1 \sqcup \mathcal{C}_2 &= \mathcal{C}_2 \sqcup \mathcal{C}_1?
\end{aligned}
$$

Inclusion between contractors

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}_1([\mathbf{x}]) \subset \mathcal{C}_2([\mathbf{x}]).$$

Proposition : If the contractor $\mathcal{C}$ is a monotonic and continuous, the set of all steady boxes is a lattice with respect to $\subset$. Moreover

$$\mathcal{C}^{\infty}([\mathbf{x}]) = \sup_{\subset} \{[\mathbf{a}], \mathcal{C}([\mathbf{a}]) = [\mathbf{a}]\},$$

i.e., $\mathcal{C}^{\infty}([\mathbf{x}])$ corresponds to the largest steady box in $[\mathbf{x}]$.

Theorem : The set of all idempotent, monotonic et continuous, equipped with $\subset$ relation, is a complete lattice. The two internal operators are $\mathcal{C}_1 \sqcup \mathcal{C}_2$ and $\mathcal{C}_1 \sqcap \mathcal{C}_2$.

Question:

What is the smallest element is $\mathcal{C}^\perp$?

What is the largest element $\mathcal{C}^\top$?

Is the lattice distributive, i.e.,

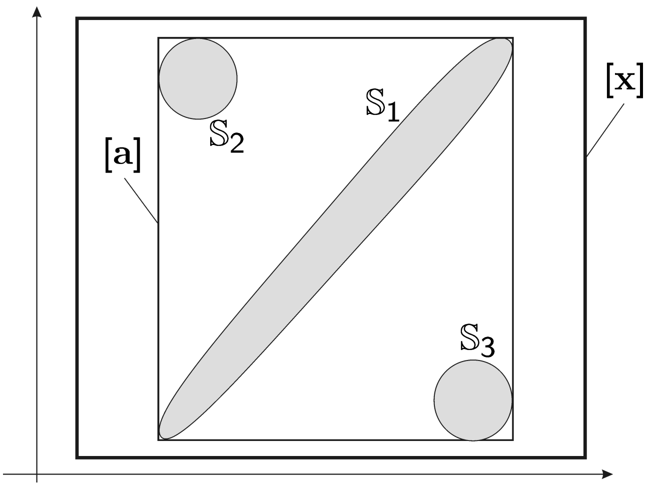
$$(\mathcal{C}_1 \sqcap (\mathcal{C}_2 \sqcup \mathcal{C}_3)) = (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqcup (\mathcal{C}_1 \sqcap \mathcal{C}_3)?$$

No the lattice is only sup-distributive:

$$(\mathcal{C}_1 \sqcap (\mathcal{C}_2 \sqcup \mathcal{C}_3)) \supset (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqcup (\mathcal{C}_1 \sqcap \mathcal{C}_3) \,.$$

Counter-example. If $\mathcal{C}_i([\mathbf{x}]) = [\mathbb{S}_i \cap [\mathbf{x}]]$, we have

$$\begin{array}{rcl} (\mathcal{C}_1 \sqcap (\mathcal{C}_2 \sqcup \mathcal{C}_3))([\mathbf{x}]) & = & [\mathbf{a}] \\ (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqcup (\mathcal{C}_1 \sqcap \mathcal{C}_3)([\mathbf{x}]) & = & \emptyset. \end{array}$$



Some properties

$\mathsf{set}\,(\mathcal{C}_1 \sqcap \mathcal{C}_2) = \mathsf{set}\,(\mathcal{C}_1 \cap \mathcal{C}_2) = \mathsf{set}\,(\mathcal{C}_1) \cap \mathsf{set}\,(\mathcal{C}_2) = \mathsf{set}\,(\mathcal{C}_1 \circ \mathcal{C}_2$

$\mathsf{set}\,(\mathcal{C}_1 \sqcup \mathcal{C}_2) = \mathsf{set}\,(\mathcal{C}_1) \cup \mathsf{set}\,(\mathcal{C}_2)$

$\mathsf{set}\left(\mathcal{C}_1^\infty\right) = \mathsf{set}\,(\mathcal{C}_1)$

$\mathcal{C}_1 \sqcap \mathcal{C}_2 \subset \mathcal{C}_1 \circ \mathcal{C}_2 \subset \mathcal{C}_1 \cap \mathcal{C}_2$ (if $\mathcal{C}_1$ is monotonic)

2.1 Principle of unique repetition

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are monotonic (but not idempotent), then

$$\begin{array}{lrcl} \text{(i)} & \mathcal{C}_1^\infty \sqcap \mathcal{C}_2^\infty & = & \mathcal{C}_1 \sqcap \mathcal{C}_2 \\ \text{(ii)} & \mathcal{C}_1^\infty \sqcup \mathcal{C}_2^\infty & = & \mathcal{C}_1 \sqcup \mathcal{C}_2 \end{array}$$

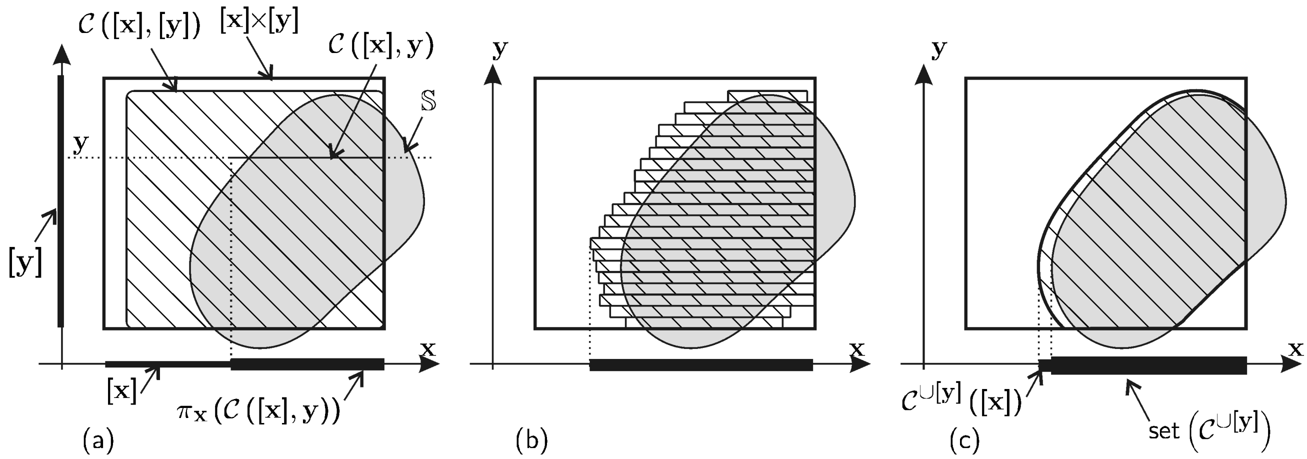
Application

$$\begin{aligned} \left(\left(\mathcal{C}_1^\infty \sqcup \mathcal{C}_2^\infty\right) \sqcap \mathcal{C}_3^\infty\right) &\overset{\text{(ii)}}{=} \left((\mathcal{C}_1 \cup \mathcal{C}_2)^\infty \sqcap \mathcal{C}_3^\infty\right) \\ &\overset{\text{(i)}}{=} ((\mathcal{C}_1 \cup \mathcal{C}_2) \cap \mathcal{C}_3)^\infty \,. \end{aligned}$$

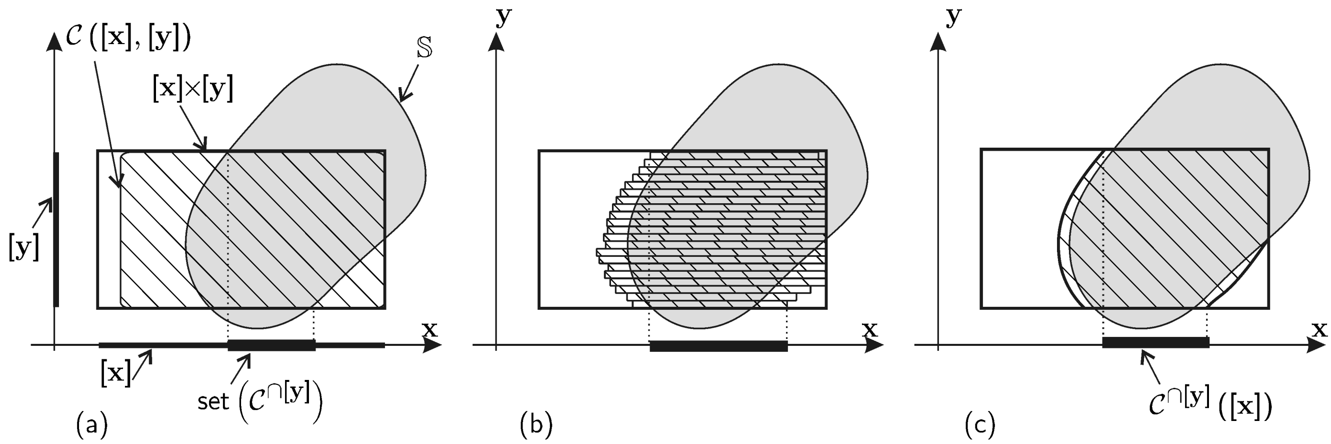
2.2 Projections

Consider the contractor $\mathcal{C}([\mathbf{x}],[\mathbf{y}])$, where $[\mathbf{x}] \in \mathbb{IR}^n$, $[\mathbf{y}] \in \mathbb{IR}^p$. Define :

$$\mathcal{C}^{\cup[\mathbf{y}]}([\mathbf{x}]) = \left[\bigcup_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}\left(\mathcal{C}([\mathbf{x}],\mathbf{y})\right)\right] \quad \text{(union projection)}$$



$$\mathcal{C}^{\cap[\mathbf{y}]}([\mathbf{x}]) = \bigcap_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}(\mathcal{C}([\mathbf{x}],\mathbf{y})), \quad \text{(intersection projection)}$$



We have

(i) $\mathcal{C}^{\cap[\mathbf{y}]} \subset \mathcal{C}^{\cup[\mathbf{y}]}$,

(ii) $\mathcal{C}^{\cup[\mathbf{y}]}$ and $\mathcal{C}^{\cap[\mathbf{y}]}$ are contractors

(iii) $\text{set}\left(\mathcal{C}^{\cup[\mathbf{y}]}\right) = \{\mathbf{x}, \exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \text{set}(\mathcal{C})\}$

(iv) $\text{set}\left(\mathcal{C}^{\cap[\mathbf{y}]}\right) = \{\mathbf{x}, \forall \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \text{set}(\mathcal{C})\}$

The collection of contractor $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ is *complementary* if

$$\text{set}(\mathcal{C}_1) \cap \cdots \cap \text{set}(\mathcal{C}_m) = \emptyset.$$

3 QUIMPER

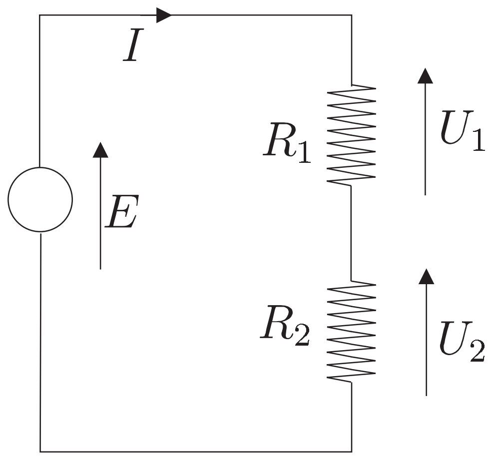
Quimper is an interpreted language for set computation.

A Quimper program describes a collection of complementary contractors.

An execution of a Quimper first builds the contractors and then runs a paver.

Quimper returns subpavings each of them associated with one contractor.

4 Electric circuit



Domains

$$\begin{aligned} E &\in [23V, 26V]; I \in [4A, 8A]; \\ U_1 &\in [10V, 11V]; U_2 \in [14V, 17V]; \\ P &\in [124W, 130W]; R_1 \in [0, \infty[\text{ and } R_2 \in [0, \infty[. \end{aligned}$$

Constraints

(i) $P = EI$, (ii) $E = (R_1 + R_2)\, I$, (iii) $U_1 = R_1 I$, (iv) $U_2 = R_2 I$, (v) $E = U_1 + U_2$.

The solution set is

$$\mathbb{S} = \left\{ \begin{pmatrix} E \\ R_1 \\ R_2 \\ I \\ U_1 \\ U_2 \\ P \end{pmatrix} \in \begin{pmatrix} [23, 26] \\ [0, \infty[\\ [0, \infty[\\ [4, 8] \\ [10, 11] \\ [14, 17] \\ [124, 130]; \end{pmatrix}, \left\{ \begin{array}{l} P = EI \\ E = (R_1 + R_2)\, I \\ U_1 = R_1 I \\ U_2 = R_2 I \\ E = U_1 + U_2 \end{array} \right. \right\}$$

```
variables
 E in [23 ,26];
 I in [4,8];
 U1 in [10,11];
 U2 in [14 ,17];
 P in [124,130];
 R1 in [0 ,1e08 ];
 R2 in [0 ,1e08 ];
contractor_list L
 P=E*I;
 E=(R1+R2)*I;
 U1=R1*I;
 U2=R2*I;
 E=U1+U2;
end
contractor C
   compose(L);
end
contractor epsilon
   precision(1);
end
```

Quimper returns the box

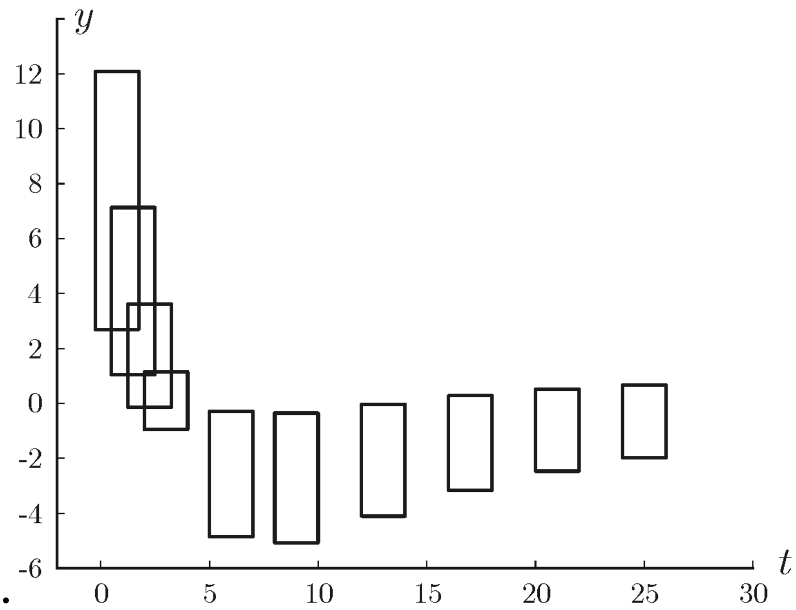
$$[24; 26] \times [1.846; 2.307] \times [2.584; 3.355]$$
$$\times [4.769; 5.417] \times [10; 11] \times [14; 16] \times [124; 130] .$$

or equivalently

$$\begin{array}{ll} E \in [24; 26] , & R_1 \in [1.846; 2.307] , \\ R_2 \in [2.584; 3.355], & I \in [4.769; 5.417] , \\ U_1 \in [10; 11] , & U_2 \in [14; 16] , \\ P \in [124; 130] . & \end{array}$$

5 Exponential problem

$$y_m(\mathbf{p}, t) = 20\,\mathsf{exp}(-p_1 t) - 8\,\mathsf{exp}(-p_2 t).$$



.

i	$[t_i]$	$[y_i]$
1	$[-0.25, 1.75]$	$[2.7, 12.1]$
2	$[0.5, 2.5]$	$[1.04, 7.14]$
3	$[1.25, 3.25]$	$[-0.13, 3.61]$
4	$[2, 4]$	$[-0.95, 1.15]$
5	$[5, 7]$	$[-4.85, -0.29]$
6	$[8, 10]$	$[-5.06, -0.36]$
7	$[12, 14]$	$[-4.1, -0.04]$
8	$[16, 18]$	$[-3.16, 0.3]$
9	$[20, 22]$	$[-2.5, 0.51]$
10	$[24, 26]$	$[-2, 0.67]$

Feasible set for the parameters

$$\mathbb{S} = \bigcap_{i \in \{1,\dots,10\}} \underbrace{\left\{ \mathbf{p} \in \mathbb{R}^2 \mid \exists t_i \in [t_i] \mid y_m(\mathbf{p}, t_i) \in [y_i] \right\}}_{\mathbb{S}_i}.$$

Its complementary set is

$$\begin{aligned}
\mathbb{R}^2 \backslash \mathbb{S} &= \bigcup_{i \in \{1,\dots,10\}} \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \forall t_i \in [t_i] \mid y_m(\mathbf{p}, t_i) \notin [y_i] \right\} \\
&\subset \bigcup_{i \in \{1,\dots,10\}} \underbrace{\left\{ \mathbf{p} \in \mathbb{R}^2 \mid \forall t_i \in [t_i] \mid y_m(\mathbf{p}, t_i) \notin \mathrm{int}\left([y_i]\right) \right\}}_{\bar{\mathbb{S}}_i} \\
&= \bar{\mathbb{S}}
\end{aligned}$$

If $\mathcal{C}_i(\mathbf{p}, t_i)$ and $\bar{\mathcal{C}}_i(\mathbf{p}, t_i)$ are two contractors such that

$$\left\{ \begin{array}{lll} \text{set}\left(\mathcal{C}_i(\mathbf{p}, t_i)\right) & = & \{(\mathbf{p}, t_i), y_m(\mathbf{p}, t_i) \in [y_i]\} \\ \text{set}\left(\bar{\mathcal{C}}_i(\mathbf{p}, t_i)\right) & = & \{(\mathbf{p}, t_i), y_m(\mathbf{p}, t_i) \notin \text{int}([y_i])\}, \end{array} \right.$$

we have

$$\begin{array}{lll} \text{set}\left(\mathcal{C}_i^{\cup[t_i]}\right) & = & \mathbb{S}_i \\ \text{set}\left(\bar{\mathcal{C}}_i^{\cap[t_i]}\right) & = & \bar{\mathbb{S}}_i. \end{array}$$

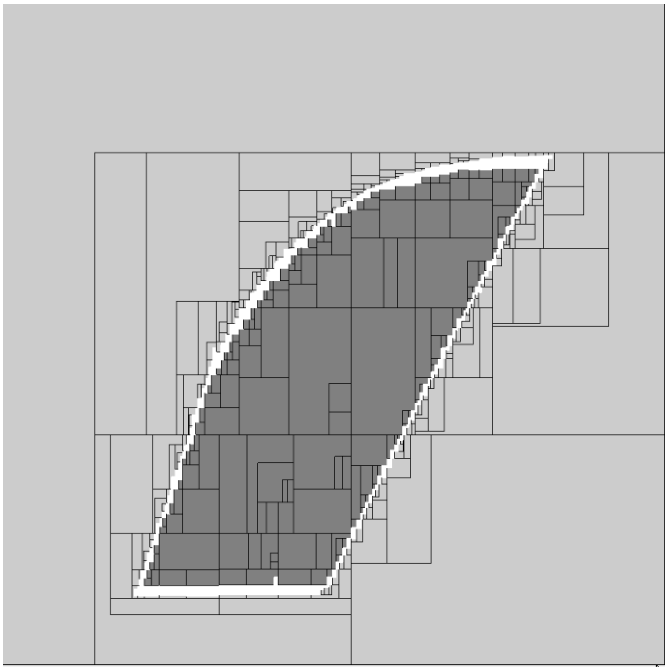
Define

$$\begin{aligned}\mathcal{C}\left([\mathbf{p}]\right) &= \bigcap_{i\in\{1,\dots,10\}} \mathcal{C}_i^{\cup[t_i]}\left([\mathbf{p}],t_i\right)\\ \bar{\mathcal{C}}\left([\mathbf{p}]\right) &= \left[\bigcup_{i\in\{1,\dots,10\}} \bar{\mathcal{C}}_i^{\cap[t_i]}\left([\mathbf{p}],t_i\right)\right].\end{aligned}$$

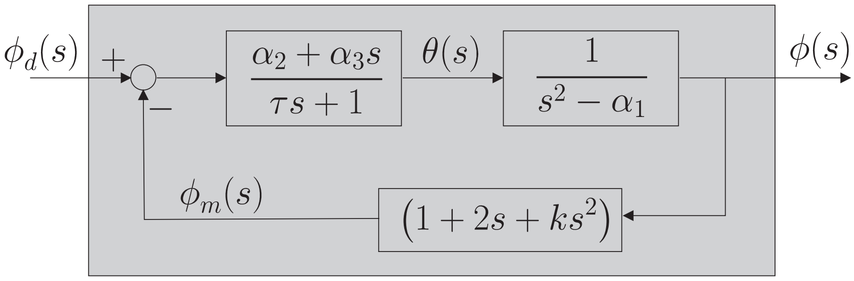
We have $\text{set}(\mathcal{C}) = \mathbb{S}$ and $\text{set}\left(\bar{\mathcal{C}}\right) = \bar{\mathbb{S}}$.

```
constant
  Y[10] = [[2.7,12.1]; [1.04,7.14];
           [-0.13,3.61];[-0.95,1.15];
           [-4.85,-0.29];[-5.06,-0.36];
           [-4.1,-0.04]; [-3.16,0.3];
           [-2.5,0.51]; [-2,0.67]];
variables
  p1 in [0,1.2]; p2 in [0,0.5];
parameters
  t[10] in [[-0.25,1.75]; [0.5,2.5]; [1.25,3.25];
            [2,4]; [5,7]; [8,10]; [12,14];
            [16,18]; [20,22]; [24,26]];
function z=f(p1,p2,t)
  z=20*exp(-p1*t)-8*exp(-p2*t);
end
contractor outer
   inter (i=1:10,
       proj_union(f(p1,p2,t[i]) in Y[i]),t[i]);
   end
end
contractor inner
```

```
   union (i=1:10,
       proj_inter(f(p1,p2,t[i]) notin Y[i]),t[i]);
   end
end
contractor epsilon
   precision(0.01)
end
```



6 Robust stability



We have

$$\begin{aligned} \alpha_1 &\in [8.8; 9.2]\,, \alpha_2 \in [2.8; 3.2]\,, \alpha_3 \in [0.8; 1.2]\,, \\ \tau &\in [1.8; 2.2]\,, k \in [-3.2; -2.8] \end{aligned}$$

The characteristic polynomial is

$$P(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

with

$$\begin{aligned} a_3 &= \tau + \alpha_3 k, \\ a_2 &= \alpha_2 k + 2\alpha_3 + 1, \\ a_1 &= \alpha_3 - \alpha_1 \tau + 2\alpha_2 \\ a_0 &= -\alpha_1 + \alpha_2 \end{aligned}$$

.

The Routh table is

a_3	a_1
a_2	a_0
$\frac{a_2 a_1 - a_3 a_0}{a_2}$	0
a_0	0

Now, we have

$$
\begin{aligned}
& b_1,\ b_2,\ b_3,\ b_4 \text{ have the same sign} \\
\Leftrightarrow\ & \left\{ \begin{array}{lcl} \min\left(b_1, b_2, b_3, b_4\right) & > & 0 \text{ or} \\ \max\left(b_1, b_2, b_3, b_4\right) & < & 0. \end{array} \right.
\end{aligned}
$$

The robust stability is proven if

$$
\begin{array}{l}
\exists\alpha_1 \in [8.8; 9.2], \exists\alpha_2 \in [2.8; 3.2], \exists\alpha_3 \in [0.8; 1.2], \\
\exists\tau \in [1.8; 2.2], \exists k \in [-3.2; -2.8], \\
a_3 = \tau + \alpha_3 k \ ; \ a_2 = \alpha_2 k + 2\alpha_3 + 1 \ ; \ a_1 = \alpha_3 - \alpha_1\tau + 2\alpha_2, \\
a_0 = -\alpha_1 + \alpha_2 \ ; \ b = \frac{a_2 a_1 - a_3 a_0}{a_2}; \\
\left\{ \begin{array}{lcl}
\min\left(a_3, a_2, \frac{a_2 a_1 - a_3 a_0}{a_2}, a_0\right) & \leq & 0 \text{ et} \\
\max\left(a_3, a_2, \frac{a_2 a_1 - a_3 a_0}{a_2}, a_0\right) & \geq & 0
\end{array} \right.
\end{array}
$$

has no solution.

```
variables
 alpha1 in [8.8,9.2];
 alpha2 in [2.8,3.2];
 alpha3 in [0.8,1.2];
 tau in [1.8,2.2];
 k in [-3.2,-2.8];
 r in [-1e08,0];
 b1 in [-1e08,0];
 b2 in [0,-1e08];
 a3,a2,a1,a0,b;
contractor_list L
 a3=tau+alpha3*k;
 a2=alpha2*k+2*alpha3+1;
 a1=alpha3-alpha1*tau+2*alpha2;
 a0=alpha2-alpha1;
 b1=min(a3,a2,(a2*a1-a3*a0)/a2,a0);
 b2=max(a3,a2,(a2*a1-a3*a0)/a2,a0);
end
contractor C
   compose(L)
end
```