Langage QUIMPER

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1 Contracteur

(i) (contractance)
$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$

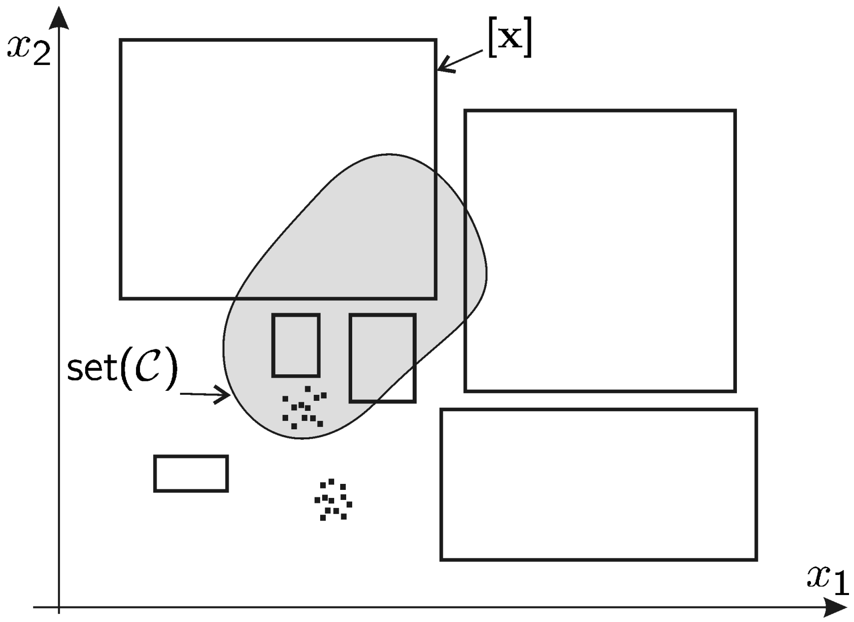
(ii) (consistance)
$(\mathbf{x} \in [\mathbf{x}], \mathcal{C}(\{\mathbf{x}\}) = \{\mathbf{x}\}) \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}])$

L'*ensemble associé* au contracteur $\mathcal{C}$ est défini par,

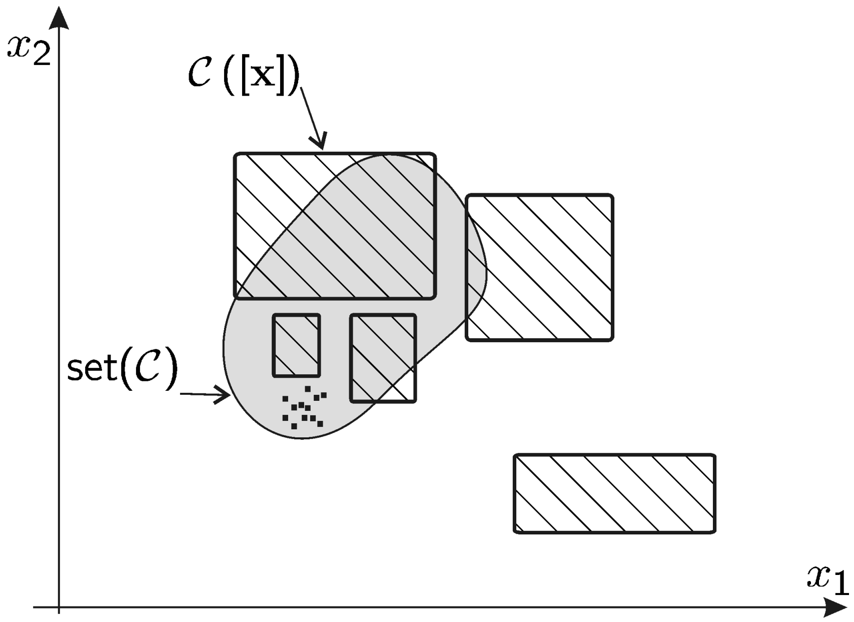
$$\text{set}\,(\mathcal{C}) = \{\mathbf{x} \in \mathbb{R}^n, \mathcal{C}(\{\mathbf{x}\}) = \{\mathbf{x}\}\}\,.$$

Le *domaine gagnable* est

$$\text{dom}\,(\mathcal{C}) = \{\mathbf{x} \in \mathbb{R}^n, \mathcal{C}(\{\mathbf{x}\}) = \emptyset\}\,.$$



x_2
[x]
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
set$(\mathcal{C})$
x_1

$\mathcal{C}$ est *monotone* si	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ est *minimal* si	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathcal{C}([\mathbf{x}]) = [[\mathbf{x}] \cap \text{set}\,(\mathcal{C})]$
$\mathcal{C}$ est *idempotent* si	$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\,(\mathcal{C}([\mathbf{x}])) = \mathcal{C}([\mathbf{x}]),$
$\mathcal{C}$ est *continu* si	$\forall [\mathbf{x}] \in \mathbb{IR}^n, \mathcal{C}\,(\mathcal{C}^\infty([\mathbf{x}])) = \mathcal{C}^\infty([\mathbf{x}])$

2 Fabrication d'un contracteur

Si $x_1 \in [x_1], x_2 \in [x_2], x_3 \in [x_3]$ et si

$$x_3 = x_1 + x_2,$$

alors, il est clair que

$$\begin{array}{l} x_1 \in [x_1] \cap ([x_3] - [x_2]) \\ x_2 \in [x_2] \cap ([x_3] - [x_1]) \\ x_3 \in [x_3] \cap ([x_1] + [x_2]) \end{array}$$

Ainsi, le contracteur

$$\mathcal{C}_1 \begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

est un contracteur minimal et

$$\text{set}\,(\mathcal{C}_1) = \{(x_1, x_2, x_3)\,, x_3 = x_1 + x_2\}\,.$$

Ainsi, le contracteur $\mathcal{C}_1$ permet de représenter l'équation $x_3 = x_1 + x_2$.

Complétude. Nous avons

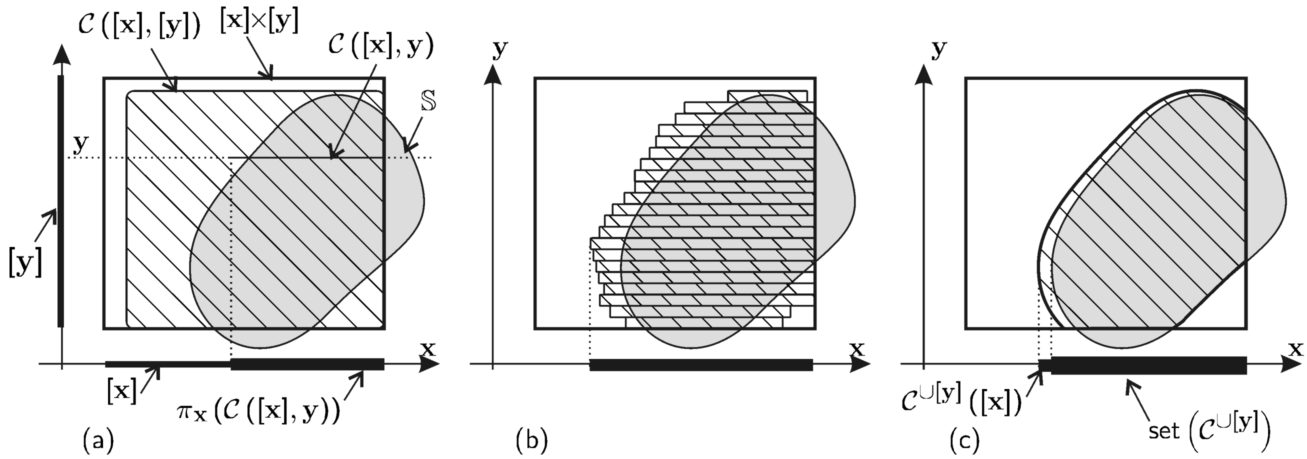
$$\mathcal{C}([\mathbf{x}]) \supset [\mathbf{x}] \cap \mathsf{set}\,(\mathcal{C})\,.$$

3 Opérations sur les contracteurs

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
répétition	$\mathcal{C}^{\infty} \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$
intersection répétée	$\mathcal{C}_1 \sqcap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^{\infty}$
union répétée	$\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^{\infty}$

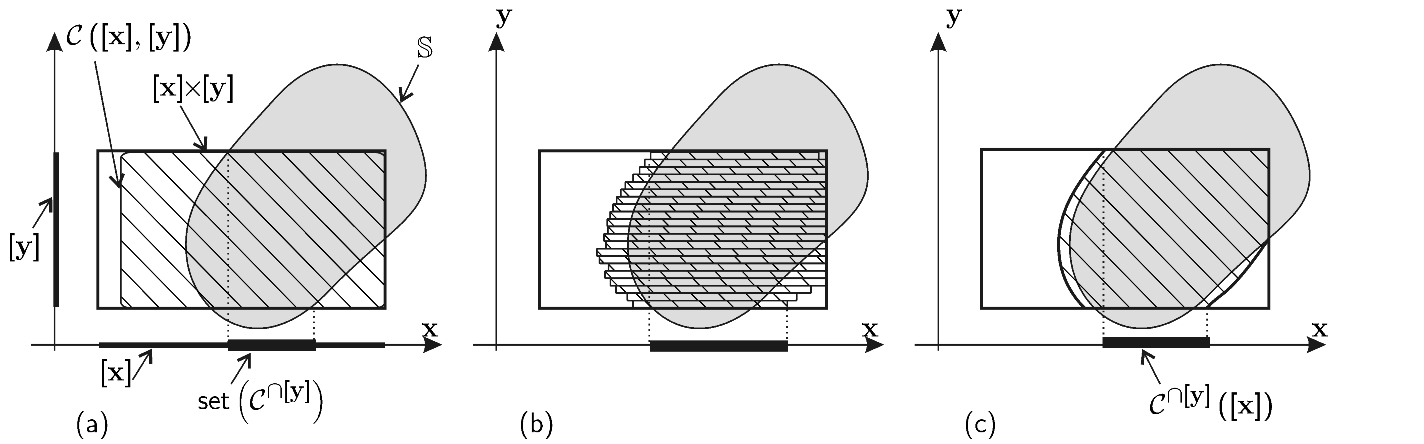
Soit le contracteur $\mathcal{C}([\mathbf{x}],[\mathbf{y}])$, où $[\mathbf{x}] \in \mathbb{R}^n, [\mathbf{y}] \in \mathbb{R}^p$. Définissons le contracteur

$$\mathcal{C}^{\cup[\mathbf{y}]}([\mathbf{x}]) = \left[\bigcup_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}(\mathcal{C}([\mathbf{x}],\mathbf{y}))\right] \quad \text{(union projection)}$$



et le contracteur

$$\mathcal{C}^{\cap[\mathbf{y}]}([\mathbf{x}]) = \bigcap_{\mathbf{y}\in[\mathbf{y}]} \pi_{\mathbf{x}}(\mathcal{C}([\mathbf{x}],\mathbf{y})), \qquad \text{(intersection projection)}$$



Nous avons

$$\mathsf{set}\left(\mathcal{C}^{\cup[\mathbf{y}]}\right) = \{\mathbf{x}, \exists \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \mathsf{set}\left(\mathcal{C}\right)\}$$
$$\mathsf{set}\left(\mathcal{C}^{\cap[\mathbf{y}]}\right) = \{\mathbf{x}, \forall \mathbf{y} \in [\mathbf{y}], (\mathbf{x}, \mathbf{y}) \in \mathsf{set}\left(\mathcal{C}\right)\}$$

La famille de contracteurs $\{\mathcal{C}_1, \dots, \mathcal{C}_m\}$ est dite *complémentaire* si

$$\text{set}\left(\mathcal{C}_1\right) \cap \dots \cap \text{set}\left(\mathcal{C}_m\right) = \emptyset.$$

4 QUIMPER

Quimper is a high-level language for QUick Interval Modeling and Programming in a bounded-ERror context.

Quimper est un langage interprété pour le calcul ensembliste.

Un programme Quimper se décrit par un ensemble de contracteurs complémentaires.

L'exécution d'un programme Quimper construit tout d'abord les contracteurs puis appelle un paveur.

Quimper retourne un sous-pavage par contracteur.

Logiciel libre disposible sur

`http://ibex-lib.org/`

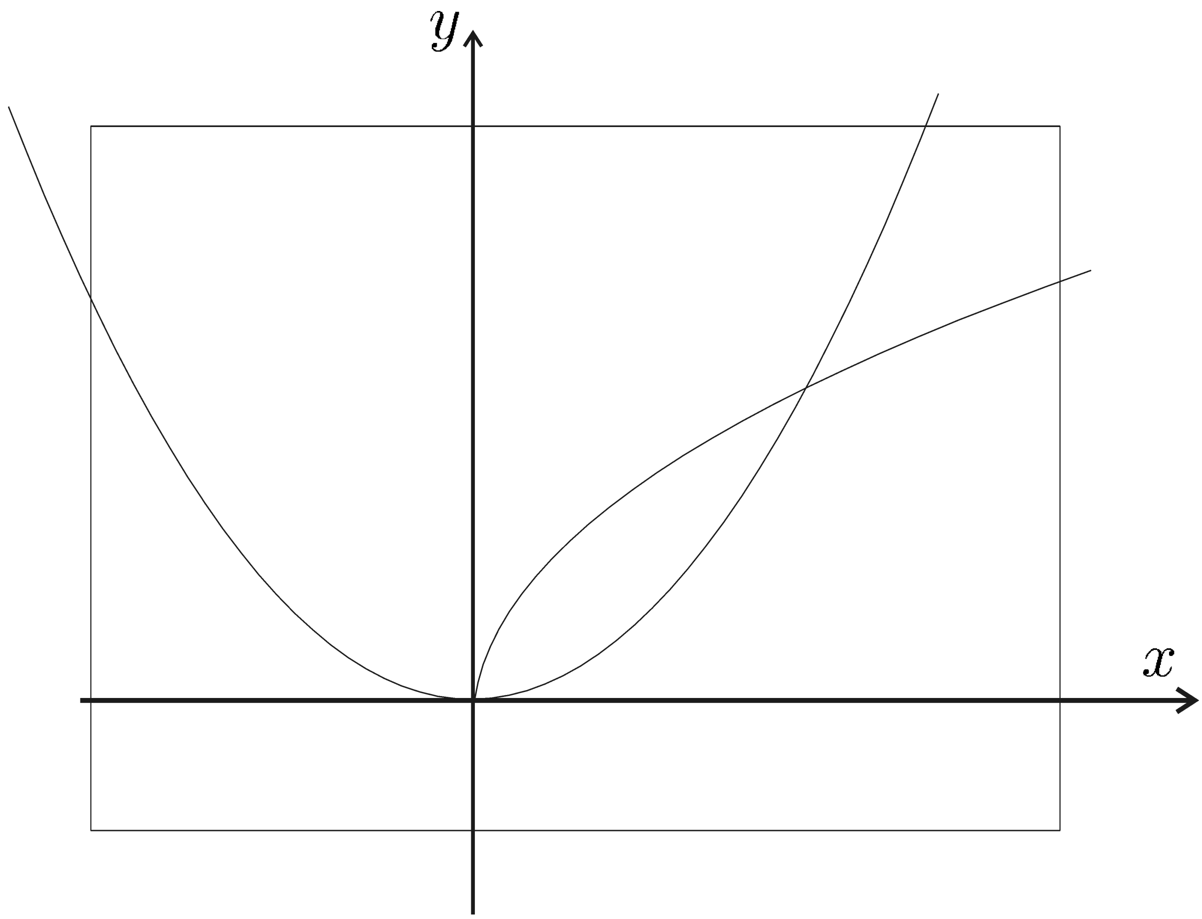
Exemple. Cherchons à résoudre.

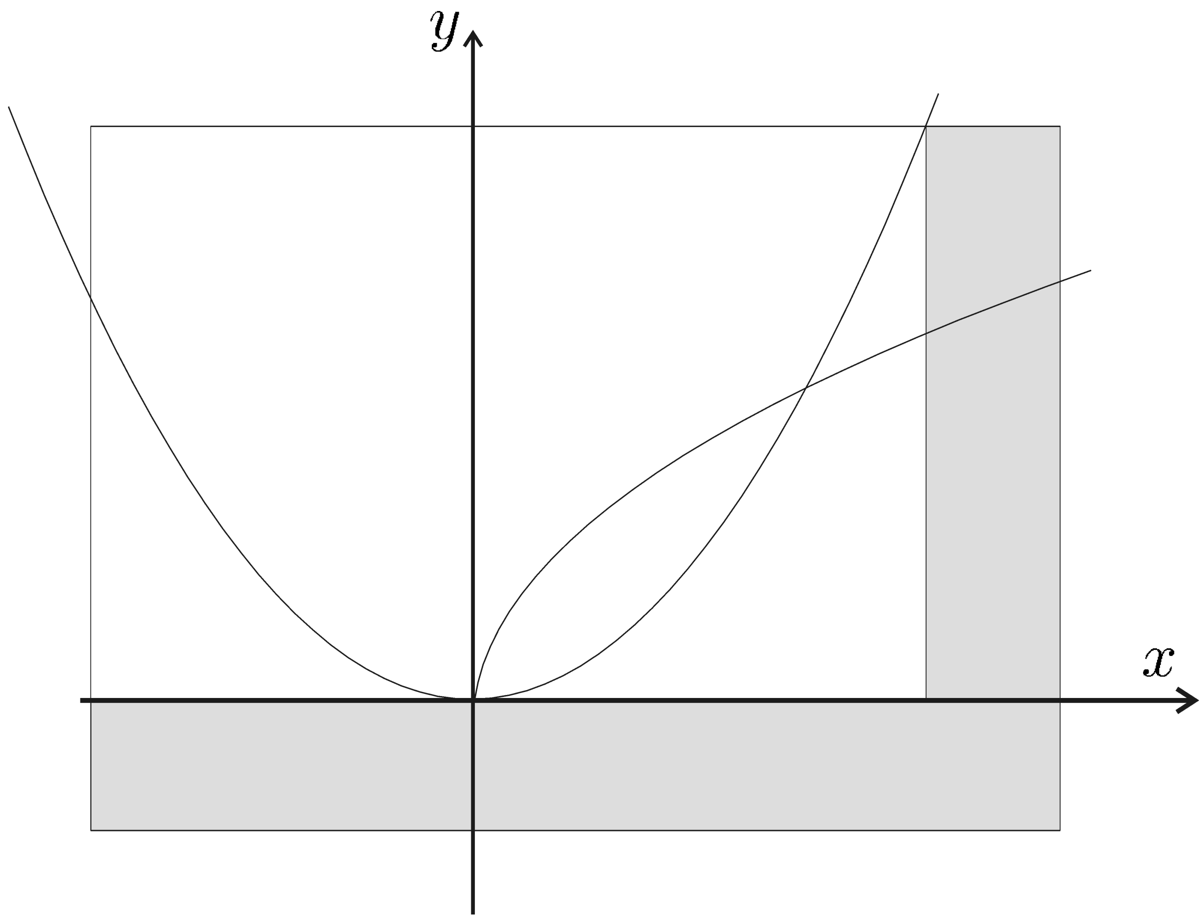
$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

On a deux contracteurs

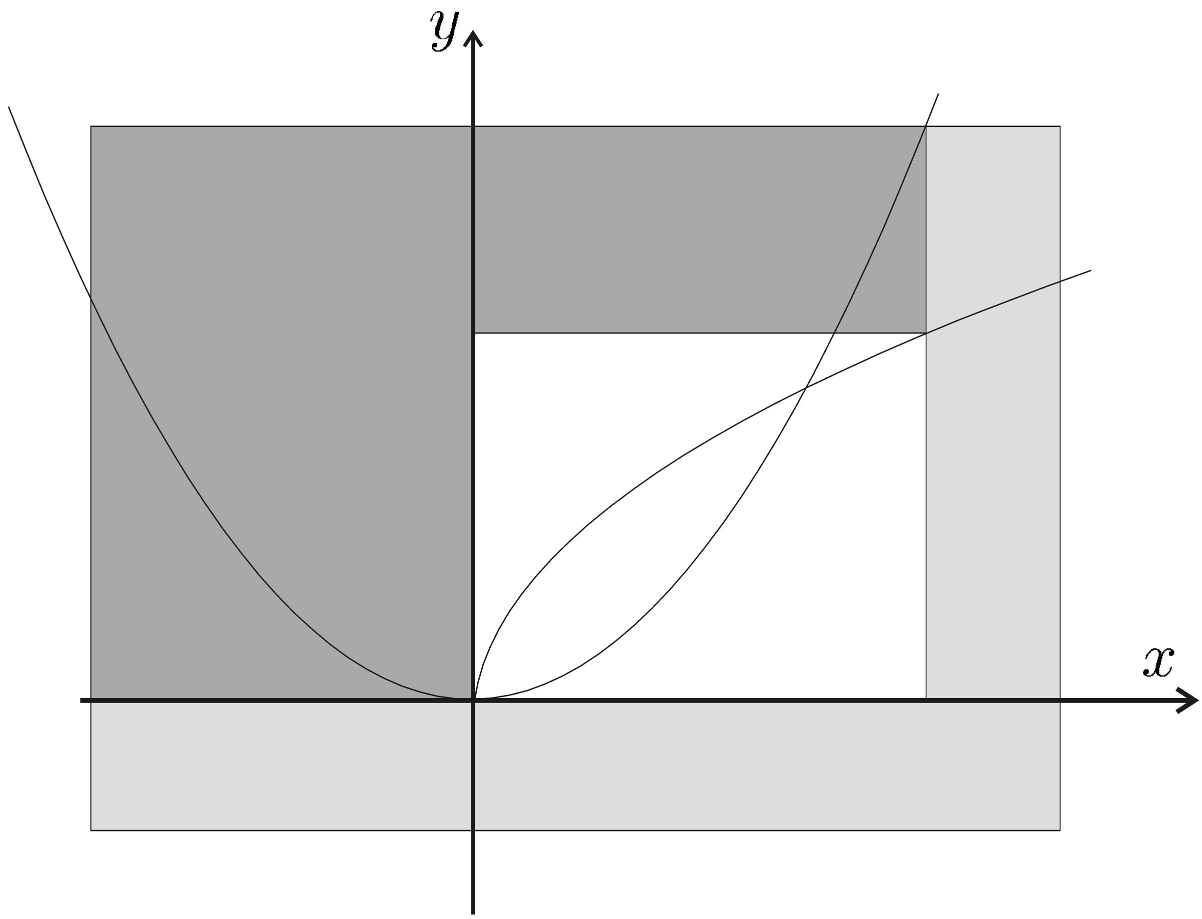
$$\mathcal{C}_1 : \left\{ \begin{array}{l} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{array} \right. \quad \text{associé à } y = x^2$$

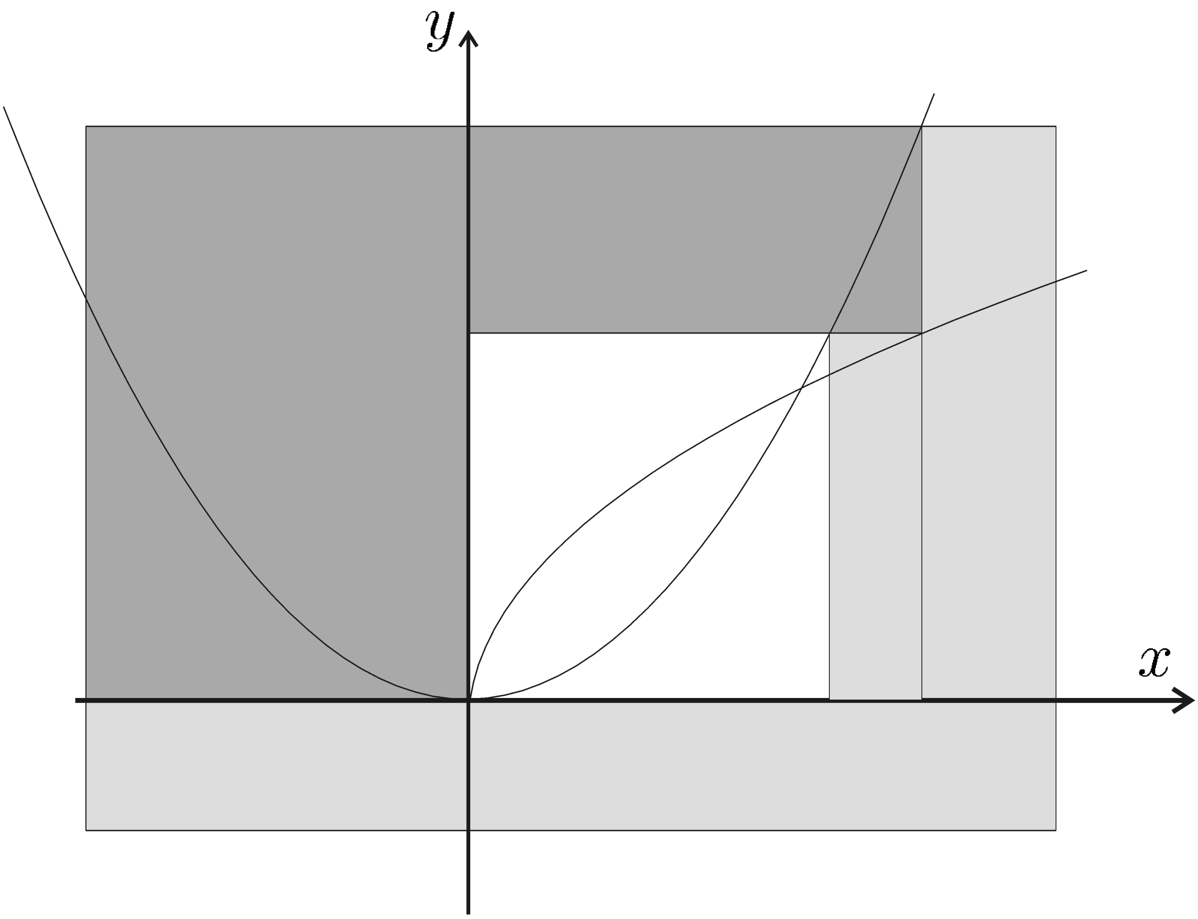
$$\mathcal{C}_2 : \left\{ \begin{array}{l} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{array} \right. \quad \text{associé à } y = \sqrt{x}$$


y
x

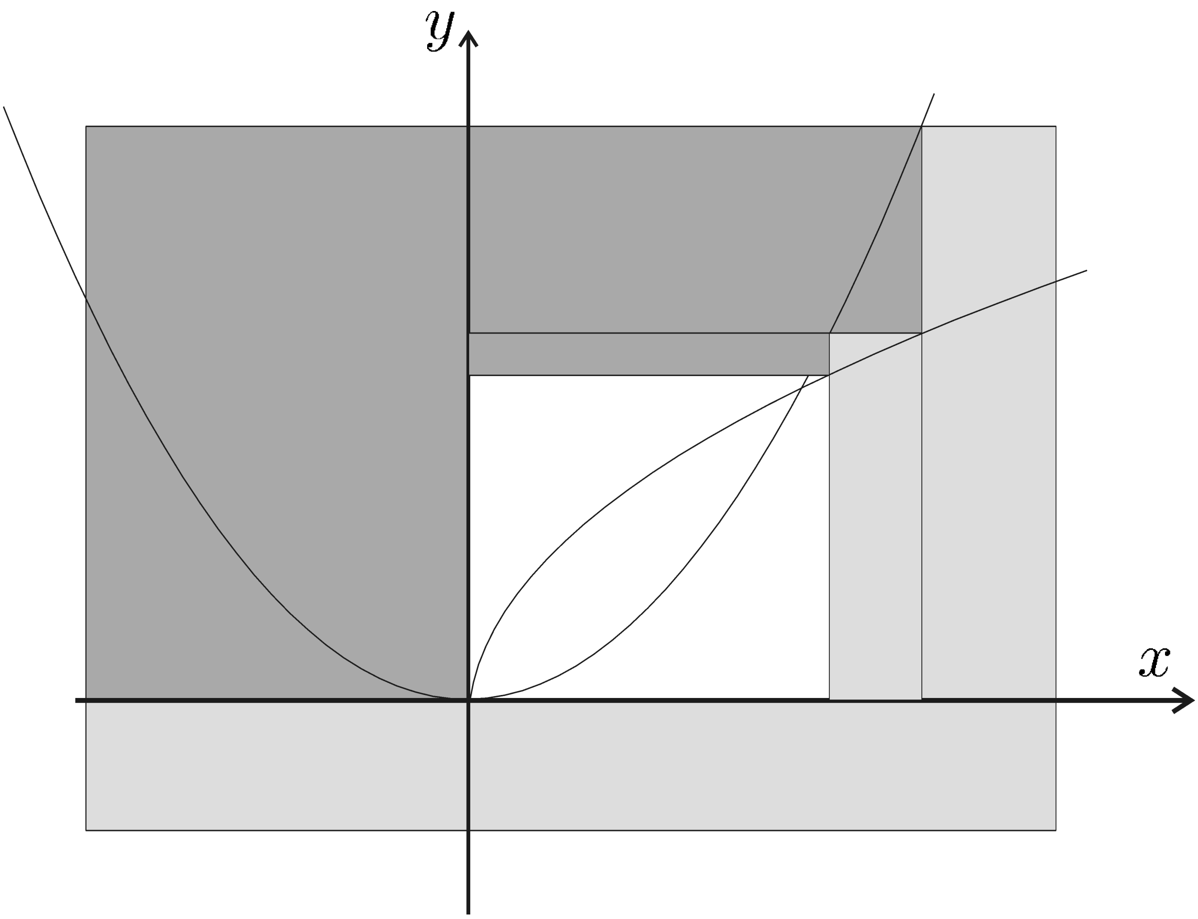


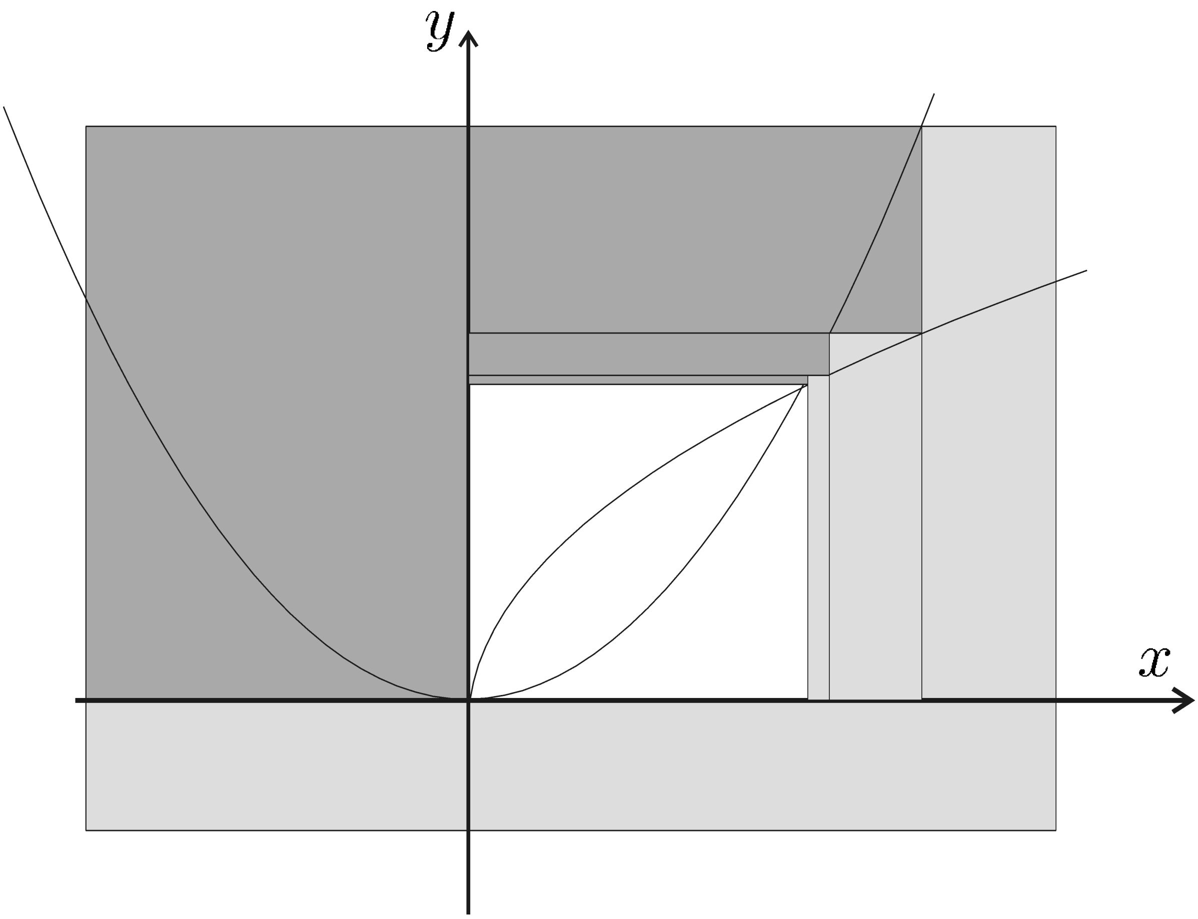
y
x

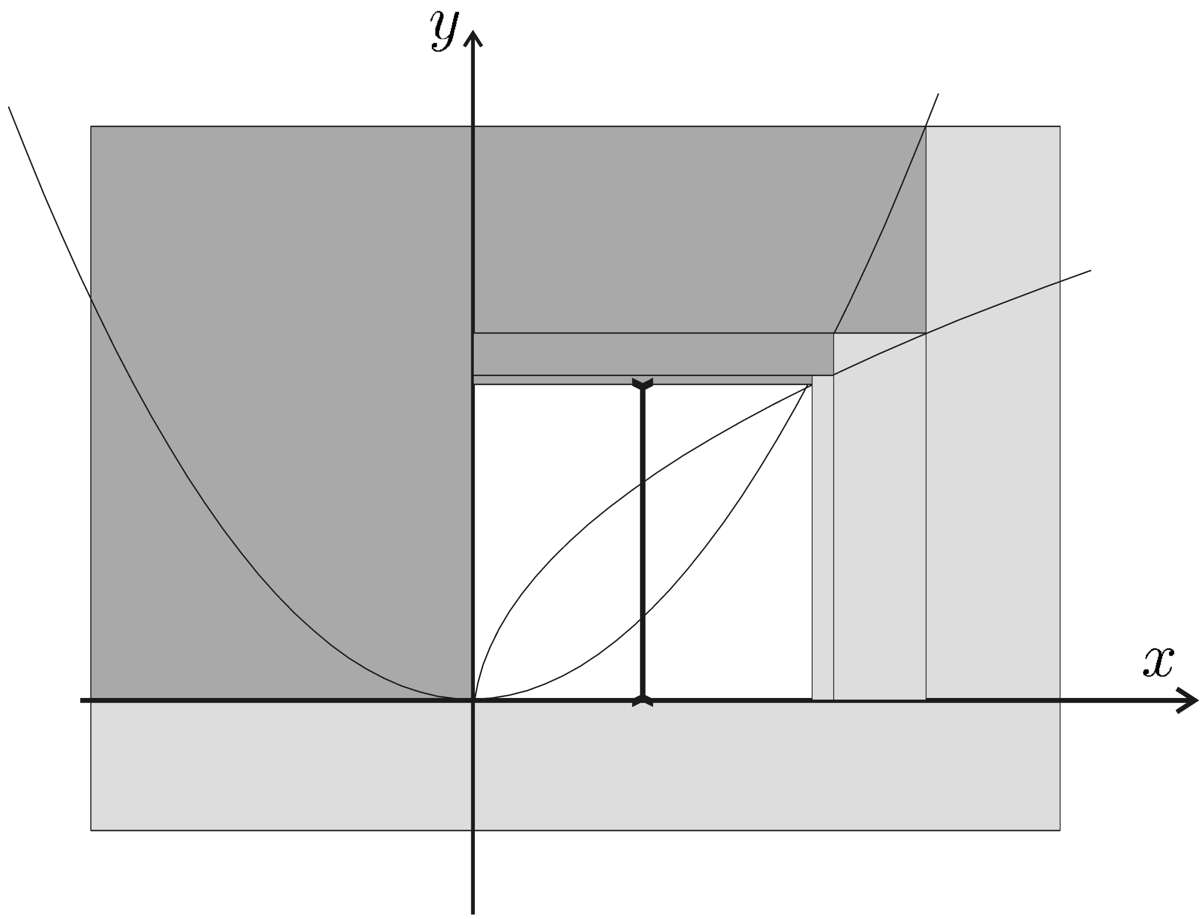

y
x

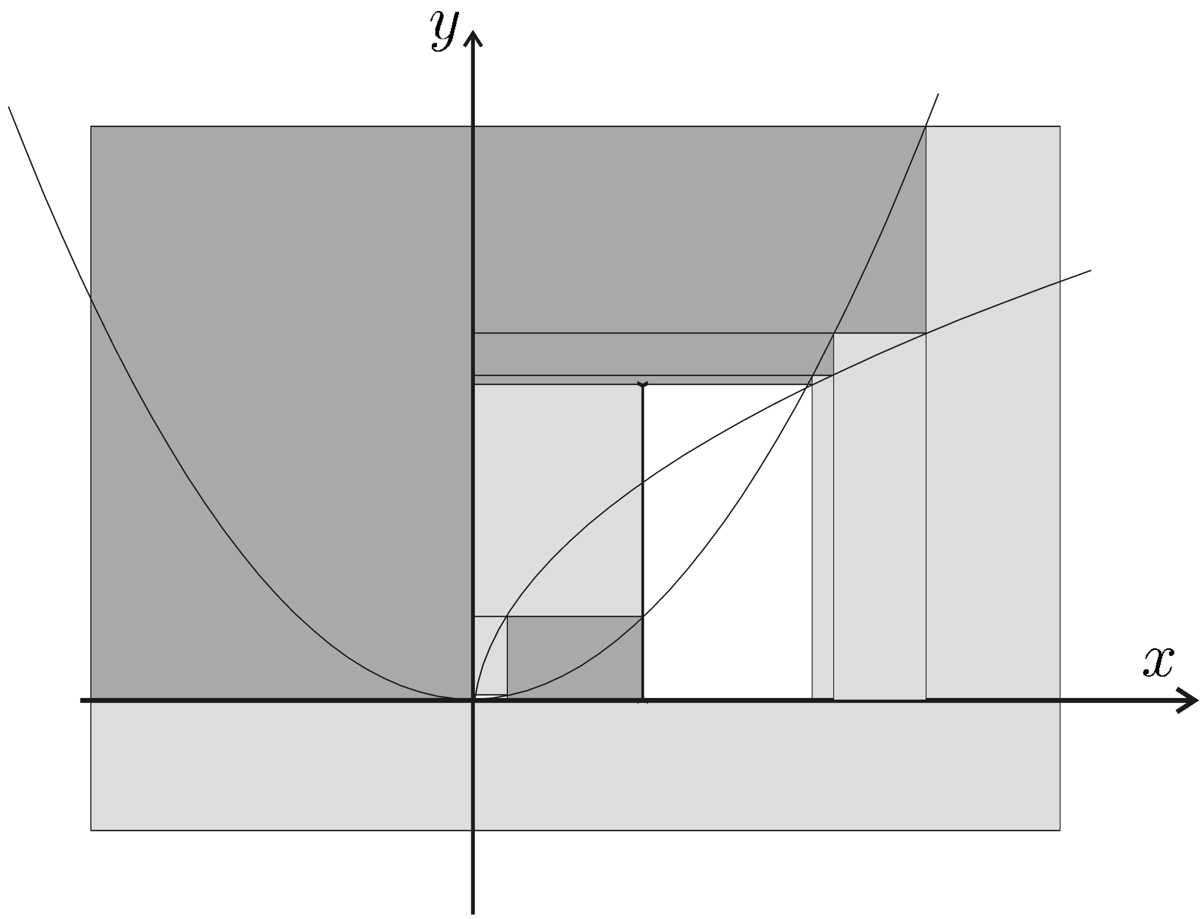


y
x

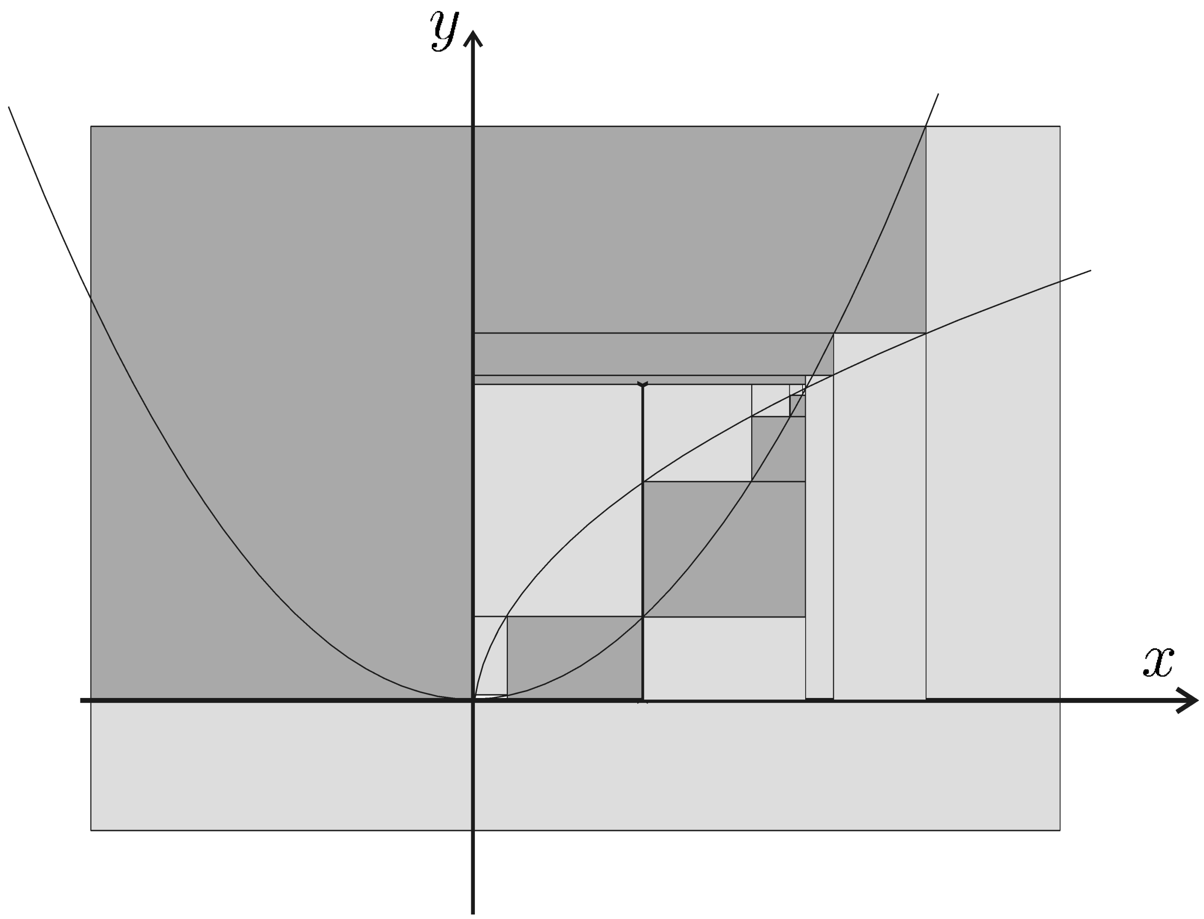

y
x


y
x

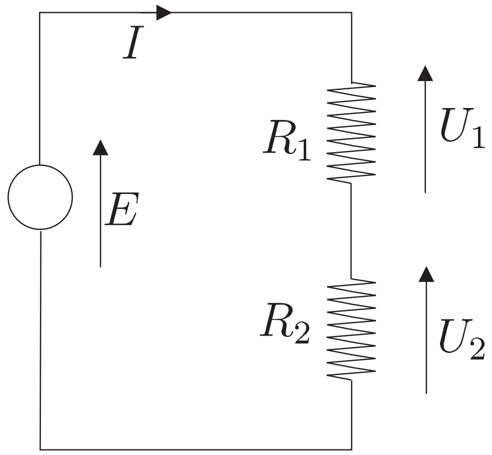

y
x



y
x


y
x

5 Circuit électrique


I
E
R1
U1
R2
U2

Domaines

$$\begin{aligned} E &\in [23V, 26V]; I \in [4A, 8A]; \\ U_1 &\in [10V, 11V]; U_2 \in [14V, 17V]; \\ P &\in [124W, 130W]; R_1 \in [0, \infty[\text{ and } R_2 \in [0, \infty[. \end{aligned}$$

Contraintes

(i) $P = EI$, (ii) $E = (R_1 + R_2)\, I$, (iii) $U_1 = R_1 I$,
(iv) $U_2 = R_2 I$, (v) $E = U_1 + U_2$.

L'ensemble des solutions est

$$\mathbb{S} = \left\{ \begin{pmatrix} E \\ R_1 \\ R_2 \\ I \\ U_1 \\ U_2 \\ P \end{pmatrix} \in \begin{pmatrix} [23, 26] \\ [0, \infty[\\ [0, \infty[\\ [4, 8] \\ [10, 11] \\ [14, 17] \\ [124, 130]; \end{pmatrix}, \left\{ \begin{array}{l} P = EI \\ E = (R_1 + R_2) I \\ U_1 = R_1 I \\ U_2 = R_2 I \\ E = U_1 + U_2 \end{array} \right. \right\}$$

```
variables
 E in [23 ,26];
 I in [4,8];
 U1 in [10,11];
 U2 in [14 ,17];
 P in [124,130];
 R1 in [0 ,1e08 ];
 R2 in [0 ,1e08 ];
contractor_list L
 P=E*I;
 E=(R1+R2)*I;
 U1=R1*I;
 U2=R2*I;
 E=U1+U2;
end
```

```
contractor C
   compose(L);
end
contractor epsilon
   precision(1);
end
```

Quimper renvoie le pavé

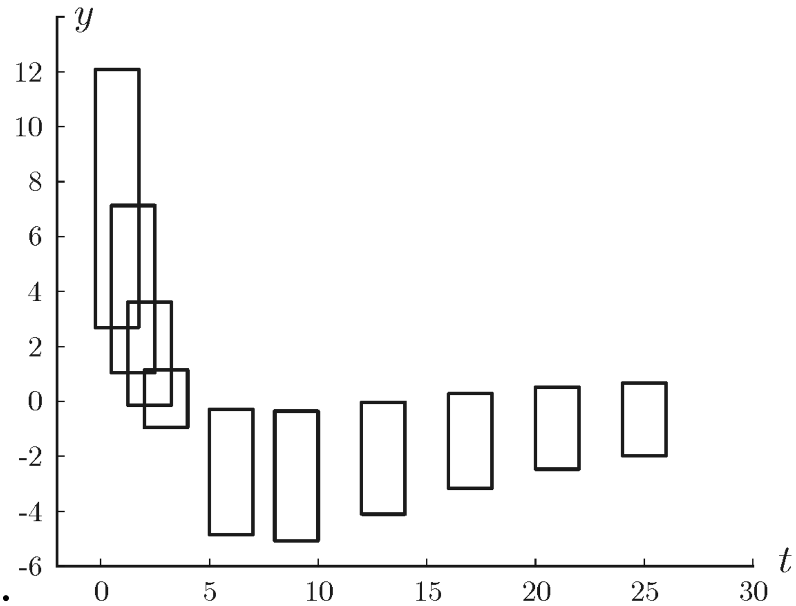
$$[24; 26] \times [1.846; 2.307] \times [2.584; 3.355]$$
$$\times [4.769; 5.417] \times [10; 11] \times [14; 16] \times [124; 130] .$$

c'est-à-dire

$$\begin{array}{ll} E \in [24; 26] , & R_1 \in [1.846; 2.307] , \\ R_2 \in [2.584; 3.355], & I \in [4.769; 5.417] , \\ U_1 \in [10; 11] , & U_2 \in [14; 16] , \\ P \in [124; 130] . & \end{array}$$

6 Problème de l’exponentielle

$$y_m(\mathbf{p}, t) = 20\exp(-p_1 t) - 8\exp(-p_2 t).$$



.

i	$[t_i]$	$[y_i]$
1	$[-0.25, 1.75]$	$[2.7, 12.1]$
2	$[0.5, 2.5]$	$[1.04, 7.14]$
3	$[1.25, 3.25]$	$[-0.13, 3.61]$
4	$[2, 4]$	$[-0.95, 1.15]$
5	$[5, 7]$	$[-4.85, -0.29]$
6	$[8, 10]$	$[-5.06, -0.36]$
7	$[12, 14]$	$[-4.1, -0.04]$
8	$[16, 18]$	$[-3.16, 0.3]$
9	$[20, 22]$	$[-2.5, 0.51]$
10	$[24, 26]$	$[-2, 0.67]$

L'ensemble de vraisemblance est

$$\mathbb{S} = \bigcap_{i\in\{1,\ldots,10\}} \underbrace{\left\{\mathbf{p} \in \mathbb{R}^2 \mid \exists t_i \in [t_i] \mid y_m(\mathbf{p}, t_i) \in [y_i]\right\}}_{\mathbb{S}_i}.$$

Son complémentaire est

$$\bar{\mathbb{S}} = \bigcup_{i\in\{1,\ldots,10\}} \underbrace{\left\{\mathbf{p} \in \mathbb{R}^2 \mid \forall t_i \in [t_i] \mid y_m(\mathbf{p}, t_i) \notin [y_i]\right\}}_{\bar{\mathbb{S}}_i}$$

Si $\mathcal{C}_i\left(\mathbf{p},t_i\right)$ et $\bar{\mathcal{C}}_i\left(\mathbf{p},t_i\right)$ sont deux contracteurs tels que

$$\left\{\begin{array}{lll}\mathsf{set}\left(\mathcal{C}_i\left(\mathbf{p},t_i\right)\right) & = & \left\{\left(\mathbf{p},t_i\right), y_m(\mathbf{p},t_i) \in [y_i]\right\} \\ \mathsf{set}\left(\bar{\mathcal{C}}_i\left(\mathbf{p},t_i\right)\right) & = & \left\{\left(\mathbf{p},t_i\right), y_m(\mathbf{p},t_i) \notin [y_i]\right\},\end{array}\right.$$

on a

$$\begin{array}{lll}\mathsf{set}\left(\mathcal{C}_i^{\cup[t_i]}\right) & = & \mathbb{S}_i \\ \mathsf{set}\left(\bar{\mathcal{C}}_i^{\cap[t_i]}\right) & = & \bar{\mathbb{S}}_i.\end{array}$$

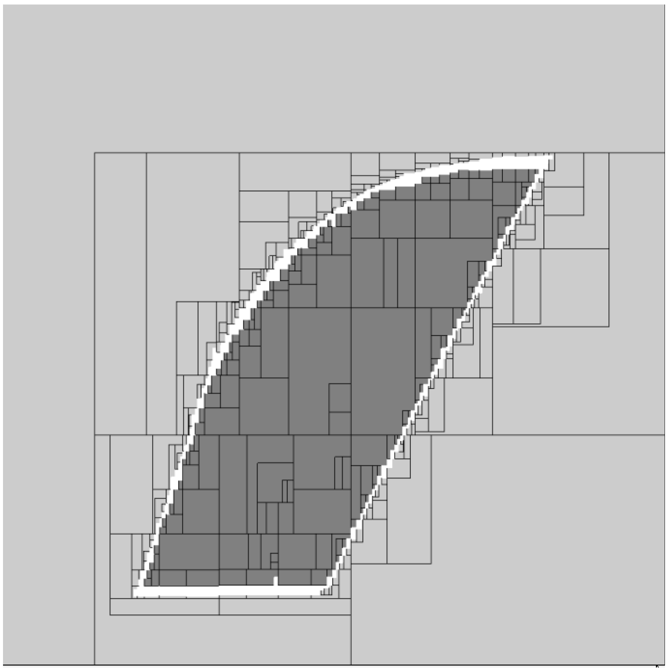
Définissons les deux contracteurs

$$\begin{aligned}\mathcal{C}\left([\mathbf{p}]\right) &= \bigcap_{i\in\{1,\dots,10\}} \mathcal{C}_i^{\cup[t_i]}\left([\mathbf{p}],t_i\right)\\ \bar{\mathcal{C}}\left([\mathbf{p}]\right) &= \bigcup_{i\in\{1,\dots,10\}} \bar{\mathcal{C}}_i^{\cap[t_i]}\left([\mathbf{p}],t_i\right).\end{aligned}$$

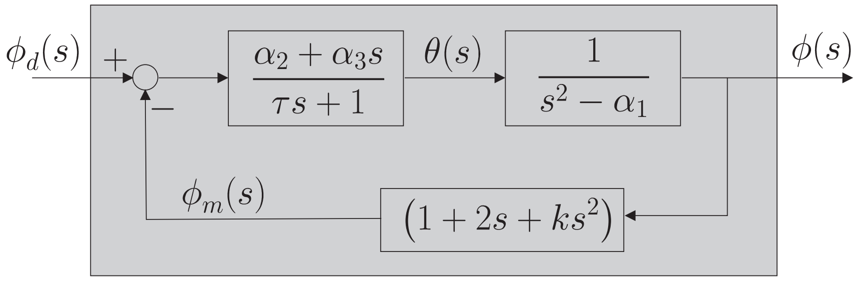
Nous avons $\text{set}(\mathcal{C}) = \mathbb{S}$ et $\text{set}\left(\bar{\mathcal{C}}\right) = \bar{\mathbb{S}}$.

```
constant
  Y[10] = [[2.7,12.1]; [1.04,7.14];
           [-0.13,3.61];[-0.95,1.15];
           [-4.85,-0.29];[-5.06,-0.36];
           [-4.1,-0.04]; [-3.16,0.3];
           [-2.5,0.51]; [-2,0.67]];
variables
  p1 in [0,1.2]; p2 in [0,0.5];
parameters
  t[10] in [[-0.25,1.75]; [0.5,2.5]; [1.25,3.25];
            [2,4]; [5,7]; [8,10]; [12,14];
            [16,18]; [20,22]; [24,26]];
function z=f(p1,p2,t)
  z=20*exp(-p1*t)-8*exp(-p2*t);
end
```

```
contractor outer
   inter (i=1:10,
       proj_union(f(p1,p2,t[i]) in Y[i]),t[i]);
   end
end
contractor inner
   union (i=1:10,
       proj_inter(f(p1,p2,t[i]) notin Y[i]),t[i]);
   end
end
contractor epsilon
   precision(0.01)
en
```



7 Stabilité robuste



Nous avons

$$\begin{aligned} \alpha_1 &\in [8.8; 9.2]\,, \alpha_2 \in [2.8; 3.2]\,, \alpha_3 \in [0.8; 1.2]\,, \\ \tau &\in [1.8; 2.2]\,, k \in [-3.2; -2.8] \end{aligned}$$

Le polynôme caractéristique est

$$P(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

avec

$$\begin{aligned} a_3 &= \tau + \alpha_3 k, \\ a_2 &= \alpha_2 k + 2\alpha_3 + 1, \\ a_1 &= \alpha_3 - \alpha_1 \tau + 2\alpha_2 \\ a_0 &= -\alpha_1 + \alpha_2 \end{aligned}$$

La table de Routh est

a_3	a_1
a_2	a_0
$\frac{a_2 a_1 - a_3 a_0}{a_2}$	0
a_0	0

Or, nous avons

$$b_1,\ b_2,\ b_3,\ b_4 \text{ de même signe}$$
$$\Leftrightarrow \left\{ \begin{array}{lll} \min(b_1, b_2, b_3, b_4) & > & 0 \text{ or} \\ \max(b_1, b_2, b_3, b_4) & < & 0. \end{array} \right.$$

Nous avons donc stabilité robuste si

$$\begin{array}{l}
\exists \alpha_1 \in [8.8; 9.2]\,, \exists \alpha_2 \in [2.8; 3.2]\,, \exists \alpha_3 \in [0.8; 1.2]\,,\\
\exists \tau \in [1.8; 2.2]\,, \exists k \in [-3.2; -2.8],\\
a_3 = \tau + \alpha_3 k\ ;\ a_2 = \alpha_2 k + 2\alpha_3 + 1\ ;\ a_1 = \alpha_3 - \alpha_1 \tau + 2\alpha_2,\\
a_0 = -\alpha_1 + \alpha_2\ ;\ b = \frac{a_2 a_1 - a_3 a_0}{a_2};\\
\left\{ \begin{array}{lll} \text{min}\left(a_3, a_2, \frac{a_2 a_1 - a_3 a_0}{a_2}, a_0\right) & \leq & 0 \text{ et} \\ \text{max}\left(a_3, a_2, \frac{a_2 a_1 - a_3 a_0}{a_2}, a_0\right) & \geq & 0 \end{array} \right.
\end{array}$$

n'a aucune solution.

```
variables
 alpha1 in [8.8,9.2];
 alpha2 in [2.8,3.2];
 alpha3 in [0.8,1.2];
 tau in [1.8,2.2];
 k in [-3.2,-2.8];
 r in [-1e08,0];
 b1 in [-1e08,0];
 b2 in [0,-1e08];
 a3,a2,a1,a0,b;
```

```
contractor_list L
 a3=tau+alpha3*k;
 a2=alpha2*k+2*alpha3+1;
 a1=alpha3-alpha1*tau+2*alpha2;
 a0=alpha2-alpha1;
 b1=min(a3,a2,(a2*a1-a3*a0)/a2,a0);
 b2=max(a3,a2,(a2*a1-a3*a0)/a2,a0);
end
contractor C
   compose(L)
end
```