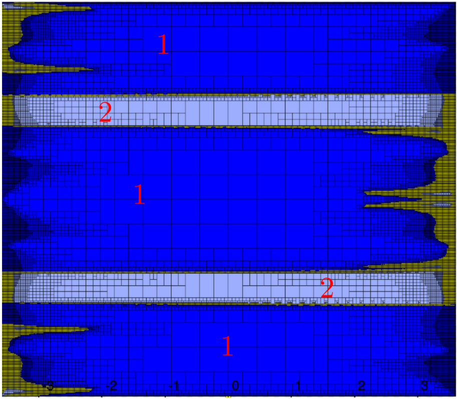
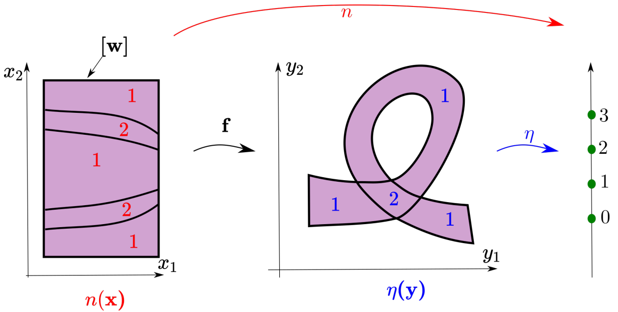
# Partial borders and injective covering

Brest (virtual)
2021, July 26




How to avoid these unclassified yellow boxes?

# Problem


$n$
$[\mathbf{w}]$
$x_2$
1
2
1
2
1
$x_1$
$n(\mathbf{x})$
$\mathbf{f}$
$y_2$
1
1
2
1
$y_1$
$\eta(\mathbf{y})$
$\eta$
3
2
1
0

Given a box $[\mathbf{w}]$, a continuous function $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^2$. We define two functions $\eta:\mathbb{R}^2\to\mathbb{N}$, $n:\mathbb{R}^2\to\mathbb{N}$ as

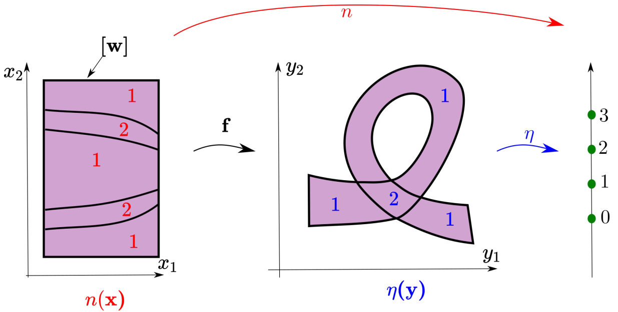
$$\eta(\mathbf{y})=\mathrm{card}\left\{\mathbf{f}^{-1}(\{\mathbf{y}\})\cap[\mathbf{w}]\right\}$$

$$n(\mathbf{x})=\eta(\mathbf{f}(\mathbf{x}))$$

**Proposition** (initial). The function $n$ changes on the set $\mathbf{f}^{-1} \circ \mathbf{f}(\partial[\mathbf{w}])$.

**Proposition** (new). The function $n$ changes on the set

$$\mathbf{f}^{-1} \circ \mathbf{f}(\partial[\mathbf{w}]) \cap \mathrm{int}([\mathbf{w}]).$$


n
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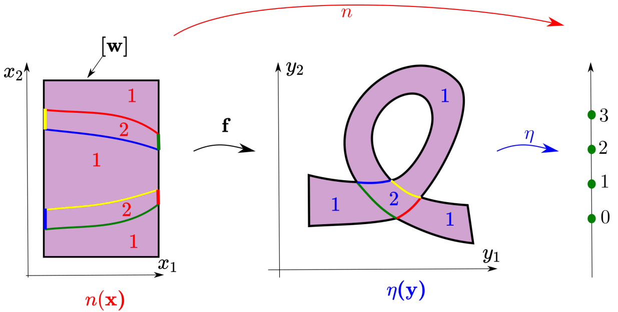
$n$
$[\mathbf{w}]$
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$n(\mathbf{x})$
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$\eta$
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2
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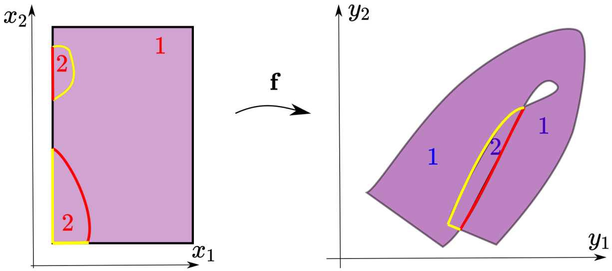
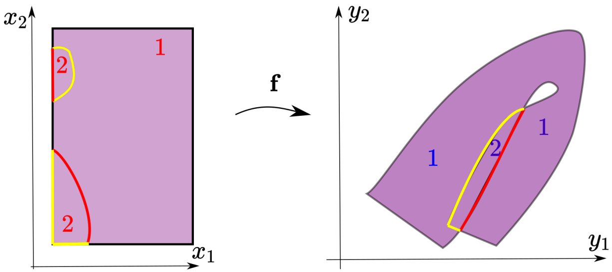
0


$x_2$
1
2
2
$x_1$
f
$y_2$
1
2
1
$y_1$

**Assumptions**.
Given $[\mathbf{y}]$ we can get $\eta([\mathbf{y}])$ using the winding number.
We have an inclusion function $[\mathbf{f}]$ for $\mathbf{f}$.
For all $\mathbf{x} \in [\mathbf{w}]$, $\det \mathbf{J_f}(\mathbf{x}) > 0$ (local injectivity)

**Problem**. Characterize the function $n(\mathbf{x})$.

# Injective covering


$x_2$
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$x_1$
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2
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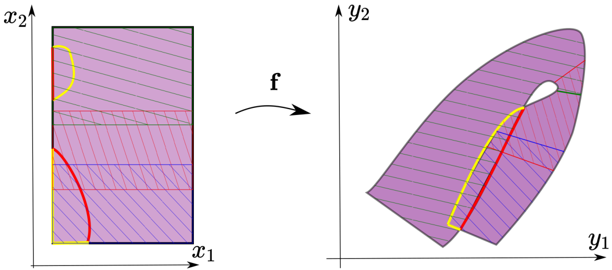
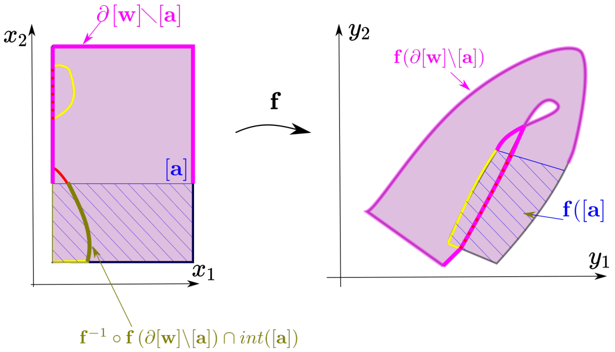
$y_1$




Injective covering of the waterfall


$\partial[\mathbf{w}]\setminus[\mathbf{a}]$
$x_2$
$x_1$
$[\mathbf{a}]$
$\mathbf{f}$
$y_2$
$y_1$
$\mathbf{f}(\partial[\mathbf{w}]\setminus[\mathbf{a}])$
$\mathbf{f}([\mathbf{a}]$
$\mathbf{f}^{-1} \circ \mathbf{f}\,(\partial[\mathbf{w}]\setminus[\mathbf{a}]) \cap int([\mathbf{a}])$

**Proposition**. If $[\mathbf{a}](i), i \in \{1,\ldots,p\}$ is an injective covering of $[\mathbf{w}]$. Then

$$\mathbf{f}^{-1} \circ \mathbf{f}(\partial[\mathbf{w}]) \cap \mathrm{int}([\mathbf{w}]) = \bigcup_{i \in \{1,\ldots,p\}} \mathbf{f}^{-1} \circ \mathbf{f}(\partial[\mathbf{w}] \backslash [\mathbf{a}](i)) \cap \mathrm{int}\big([\mathbf{a}](i)\big)$$

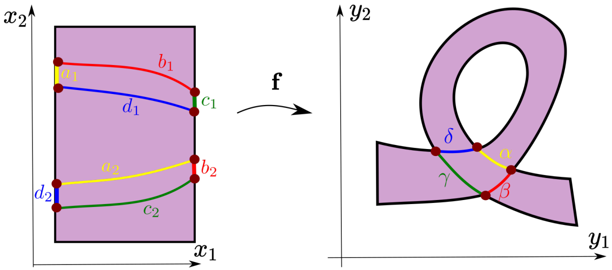
**Proof** $(\supset)$.

$$
\begin{array}{ll}
 & \mathbf{x} \in \bigcup_{i\in\{1,\ldots,p\}} \mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\backslash[\mathbf{a}](i)) \cap \mathsf{int}([\mathbf{a}](i)) \\
\Rightarrow & \exists i, \mathbf{x} \in \mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\backslash[\mathbf{a}](i)) \cap \mathsf{int}([\mathbf{a}](i)) \\
\Rightarrow & \mathbf{x} \in \mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]) \cap \mathsf{int}([\mathbf{w}])
\end{array}
$$

**Proof** ($\subset$).

$$
\begin{array}{ll}
 & \mathbf{x}\in\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}])\cap\mathsf{int}([\mathbf{w}]) \\
\Rightarrow & \exists i,\mathbf{x}\in\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}])\cap\mathsf{int}([\mathbf{a}](i)) \\
\Rightarrow & \mathbf{x}\in\left(\left(\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\backslash[\mathbf{a}](i))\cup\left(\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\cap[\mathbf{a}](i))\right)\right)\right) \\
 & \quad\cap\mathsf{int}([\mathbf{a}](i)) \\
\Rightarrow & \mathbf{x}\in\left(\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\backslash[\mathbf{a}](i))\cap\mathsf{int}([\mathbf{a}](i))\right) \\
 & \quad\cup\underbrace{\left(\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\cap[\mathbf{a}](i))\cap\mathsf{int}([\mathbf{a}](i))\right)}_{=\emptyset\text{(partial injectivity)}} \\
\Rightarrow & \mathbf{x}\in\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\backslash[\mathbf{a}](i))\cap\mathsf{int}([\mathbf{a}](i)) \\
\Rightarrow & \mathbf{x}\in\bigcup_{i\in\{1,\dots,p\}}\mathbf{f}^{-1}\circ\mathbf{f}(\partial[\mathbf{w}]\backslash[\mathbf{a}](i))\cap\mathsf{int}([\mathbf{a}](i))
\end{array}
$$

# Sewing borders




$$\mathscr{I}([\mathbf{w}]) = \mathbf{f}(\partial[\mathbf{w}]) \setminus \partial\mathbf{f}([\mathbf{w}]) = \alpha+\beta+\gamma+\delta$$
$$\mathscr{S}([\mathbf{w}]) = \mathbf{f}^{-1}(\mathscr{I}([\mathbf{w}])) \cap \partial[\mathbf{w}] = a_1+c_1+b_2+d_2$$

$\mathscr{S}([\mathbf{w}])$ is computed by inverting crossing points.

**Proposition**. If $[\mathbf{a}](i), i \in \{1, \ldots, p\}$ is an injective covering of $[\mathbf{w}]$. Then

$$\mathbf{f}^{-1} \circ \mathbf{f}(\partial[\mathbf{w}]) \cap \mathrm{int}([\mathbf{w}]) = \bigcup_{i \in \{1,\ldots,p\}} \mathbf{f}^{-1} \circ \mathbf{f}\left(\mathscr{S}([\mathbf{w}]) \backslash [\mathbf{a}](i)\right) \cap \mathrm{int}\big([\mathbf{a}](i)\big)$$