Calcul par intervalles pour l'étude de la stabilité dynamique

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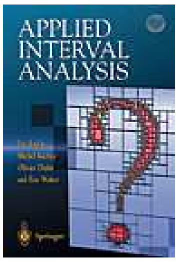
4 octobre 2012, Orléans



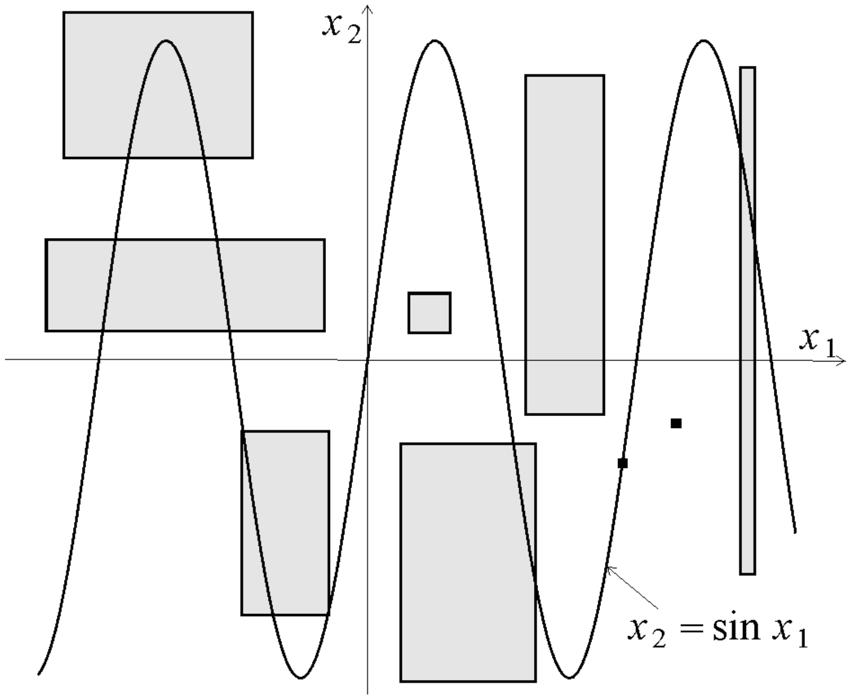
1 Contractors

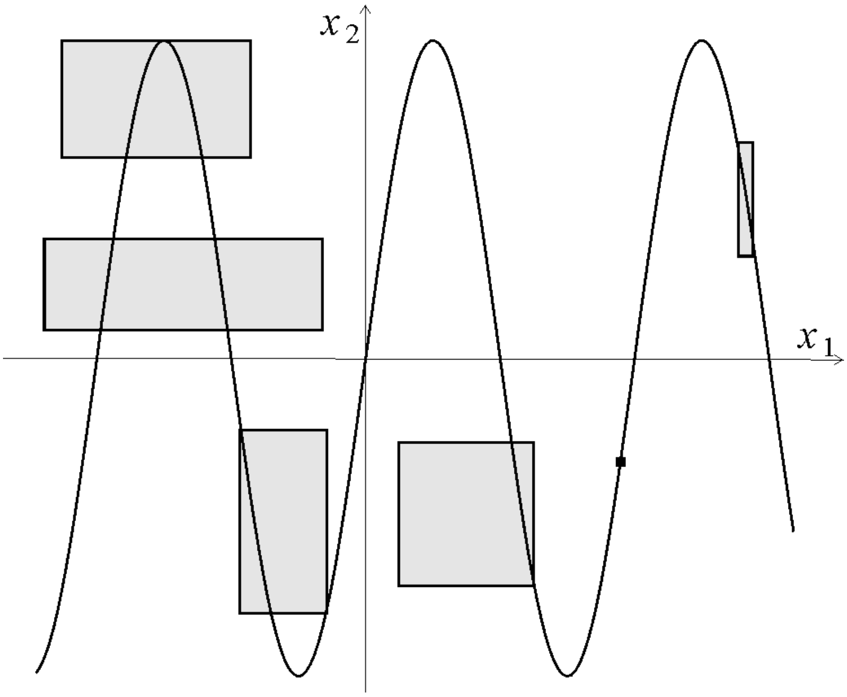
The operator $\mathcal{C} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\left\{ \begin{array}{ll} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{array} \right.$$

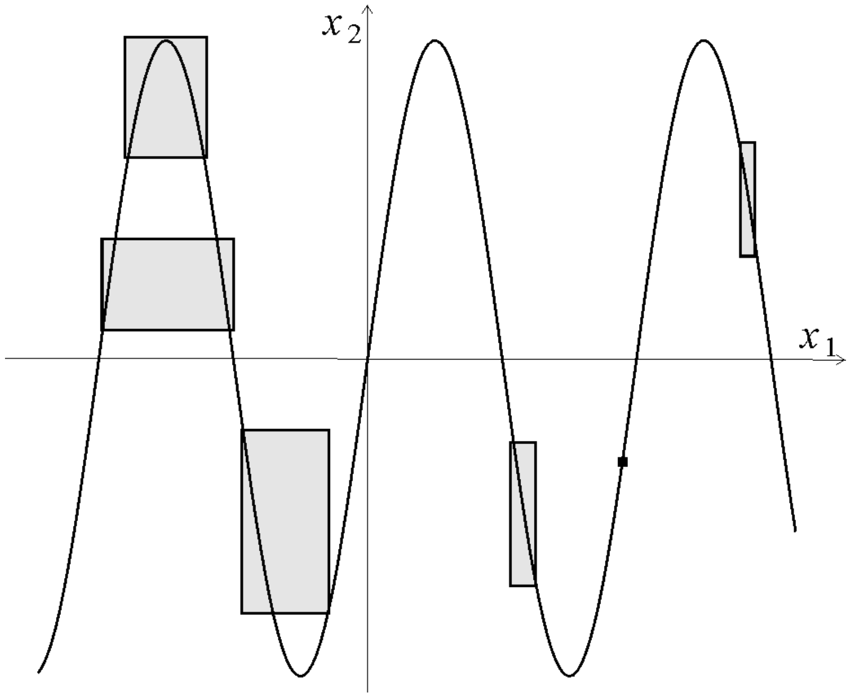
Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$


x_2
x_1
$x_2 = \sin x_1$



Forward contraction



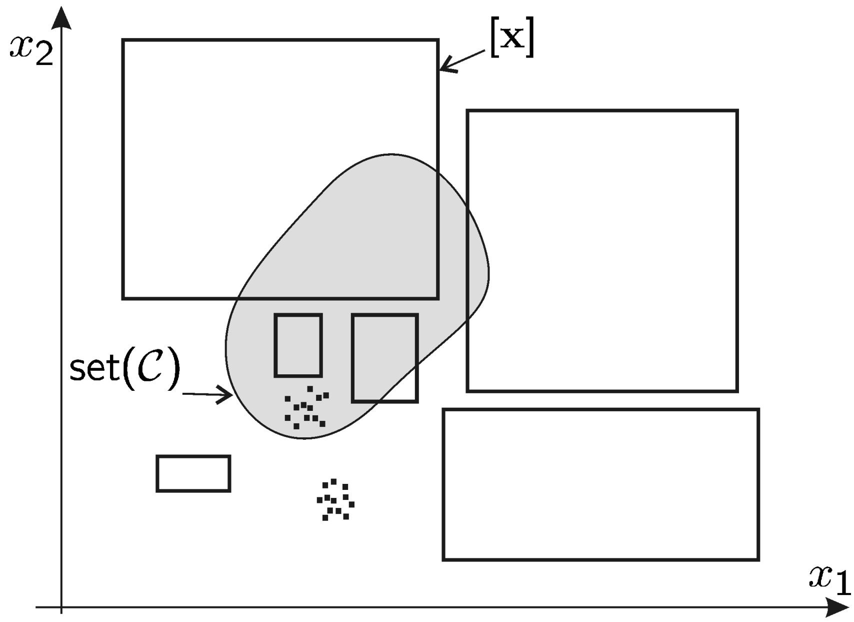
Backward contraction

More generally, $\mathcal{C} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* if

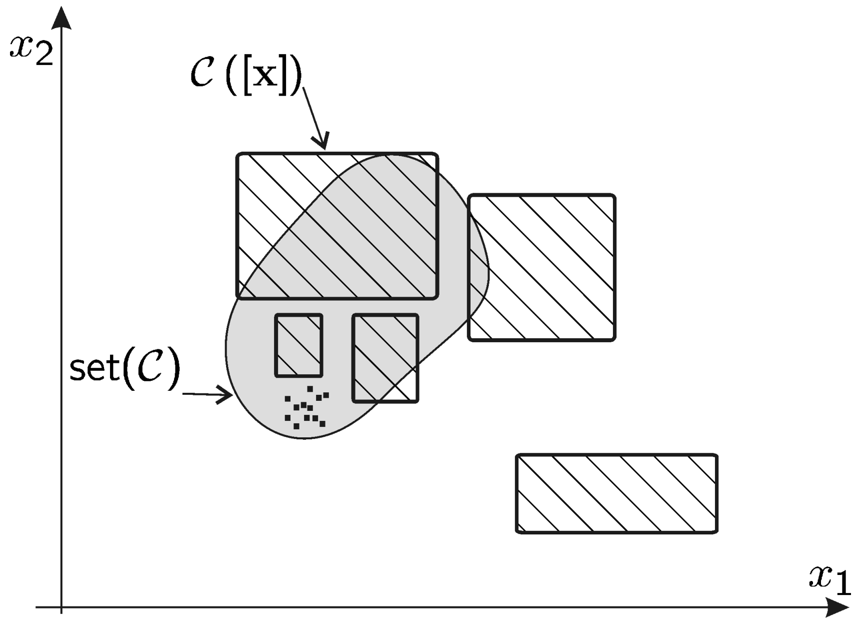
(i) $\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$ (contractance)
(ii) $(\mathbf{a} \in [\mathbf{x}], \mathcal{C}(\{\mathbf{a}\}) = \{\mathbf{a}\}) \Rightarrow \mathbf{a} \in \mathcal{C}([\mathbf{x}])$ (consistence)

The set associated to $\mathcal{C}$ is

$$\text{set}(\mathcal{C}) = \{\mathbf{a} \in \mathbb{R}^n, \mathcal{C}(\{\mathbf{a}\}) = \{\mathbf{a}\}\}.$$



x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1



x_2
$\mathcal{C}([\mathbf{x}])$
set($\mathcal{C}$)
x_1

$\mathcal{C}$ is *monotonic* if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is *minimal* if	$\mathcal{C}([\mathbf{x}]) = [[\mathbf{x}] \cap \mathsf{set}\,(\mathcal{C})]$
$\mathcal{C}$ is *idempotent* if	$\mathcal{C}\left(\mathcal{C}([\mathbf{x}])\right) = \mathcal{C}([\mathbf{x}])$
$\mathcal{C}$ is *continuous* if	$\mathcal{C}\left(\mathcal{C}^{\infty}([\mathbf{x}])\right) = \mathcal{C}^{\infty}([\mathbf{x}])$.

Contractor algebra

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
reiteration	$\mathcal{C}^\infty \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

Dealing with outliers

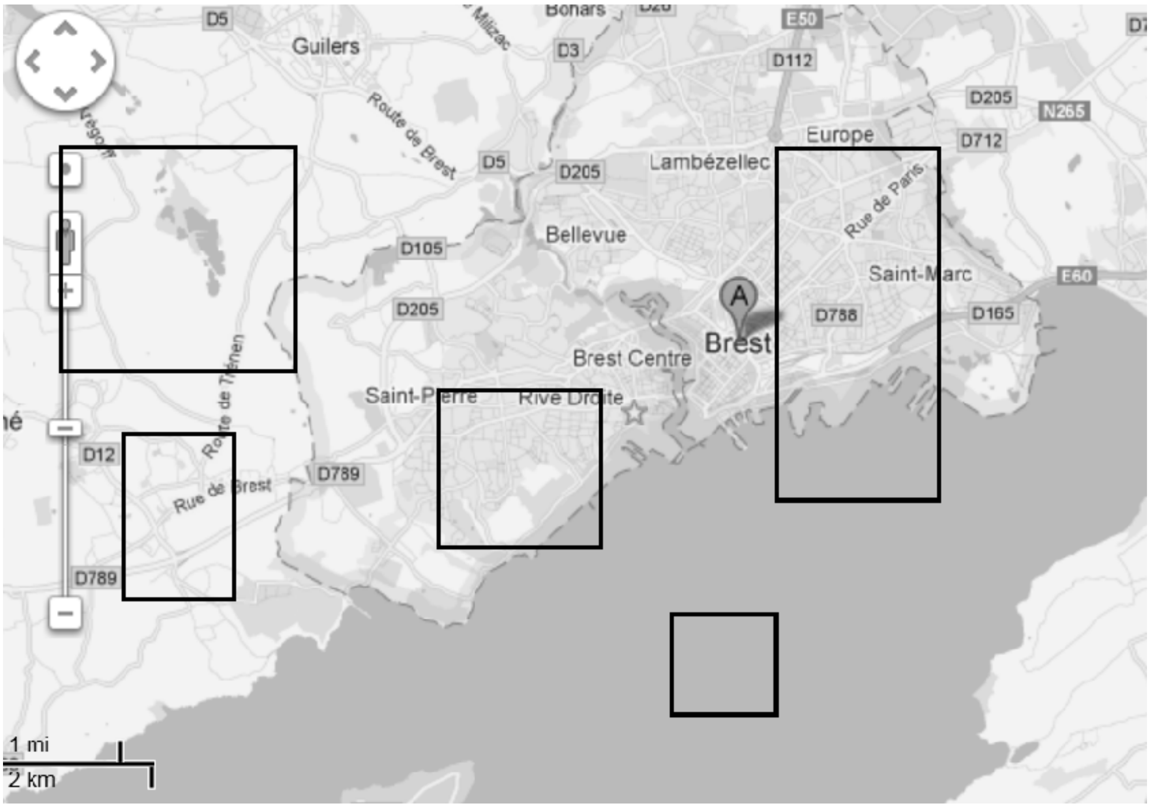
$$\mathcal{C} = (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_2 \cap \mathcal{C}_3) \cup (\mathcal{C}_1 \cap \mathcal{C}_3)$$

Properties

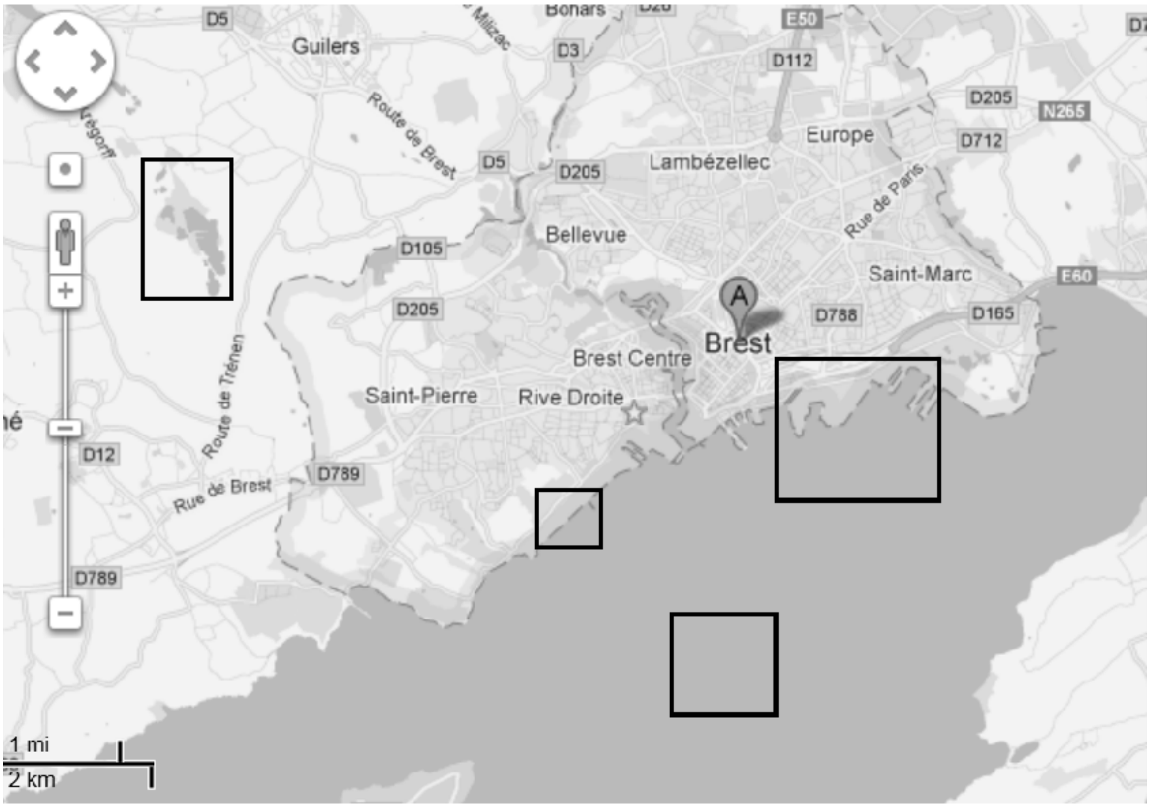
$$\begin{array}{rcl}
(\mathcal{C}_1^\infty \cap \mathcal{C}_2^\infty)^\infty & = & (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty \\
(\mathcal{C}_1 \cap (\mathcal{C}_2 \cup \mathcal{C}_3)) & \supset & (\mathcal{C}_1 \cap \mathcal{C}_2) \cup (\mathcal{C}_1 \cap \mathcal{C}_3) \\
\left\{ \begin{array}{l} \mathcal{C}_1 \text{ minimal} \\ \mathcal{C}_2 \text{ minimal} \end{array} \right. & \Rightarrow & \mathcal{C}_1 \cup \mathcal{C}_2 \text{ minimal}
\end{array}$$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.



D5
Guilers
Bohars
D3
E50
D112
D205
N265
Route de Brest
D5
D205
Lambézellec
Europe
D712
Bellevue
D105
Rue de Paris
Saint-Marc
E60
D205
D788
D165
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Tréhen
D12
Rue de Brest
D789
D789
1 mi
2 km


Guilers
Route de Brest
Lambézellec
Europe
Bellevue
Rue de Paris
Saint-Marc
Brest
Brest Centre
Saint-Pierre
Rive Droite
Route de Trénen
Rue de Brest
D5
D3
E50
D112
D205
N265
D712
D105
E60
D788
D165
D12
D789
1 mi
2 km

Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$
\begin{array}{llll}
x_3 = x_1 + x_2 \Rightarrow & x_3 \in & [x_3] \cap ([x_1] + [x_2]) & \text{// forward} \\
x_1 = x_3 - x_2 \Rightarrow & x_1 \in & [x_1] \cap ([x_3] - [x_2]) & \text{// backward} \\
x_2 = x_3 - x_1 \Rightarrow & x_2 \in & [x_2] \cap ([x_3] - [x_1]) & \text{// backward}
\end{array}
$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C}\begin{pmatrix} [x_1] \\ [x_2] \\ [x_3] \end{pmatrix} = \begin{pmatrix} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{pmatrix}$$

Forward-backward contractor

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2]\,,$$

we have the following contractor:

algorithm $\mathcal{C}$ (inout $[x_1], [x_2], [x_3]$)	
$[a] = [x_1] + [x_2]$	$// \; a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	$// \; b = a \cdot x_3$
$[b] = [b] \cap [1, 2]$	$// \; b \in [1, 2]$
$[x_3] = [x_3] \cap \frac{[b]}{[a]}$	$// \; x_3 = \frac{b}{a}$
$[a] = [a] \cap \frac{[b]}{[x_3]}$	$// \; a = \frac{b}{x_3}$
$[x_1] = [x_1] \cap [a] - [x_2]$	$// \; x_1 = a - x_2$
$[x_2] = [x_2] \cap [a] - [x_1]$	$// \; x_2 = a - x_1$

2 Solver

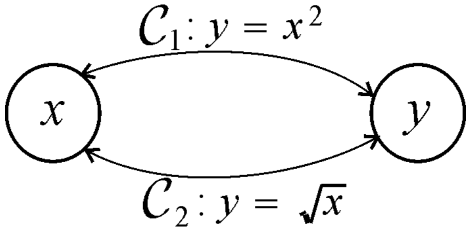
Example 1. Solve the system

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

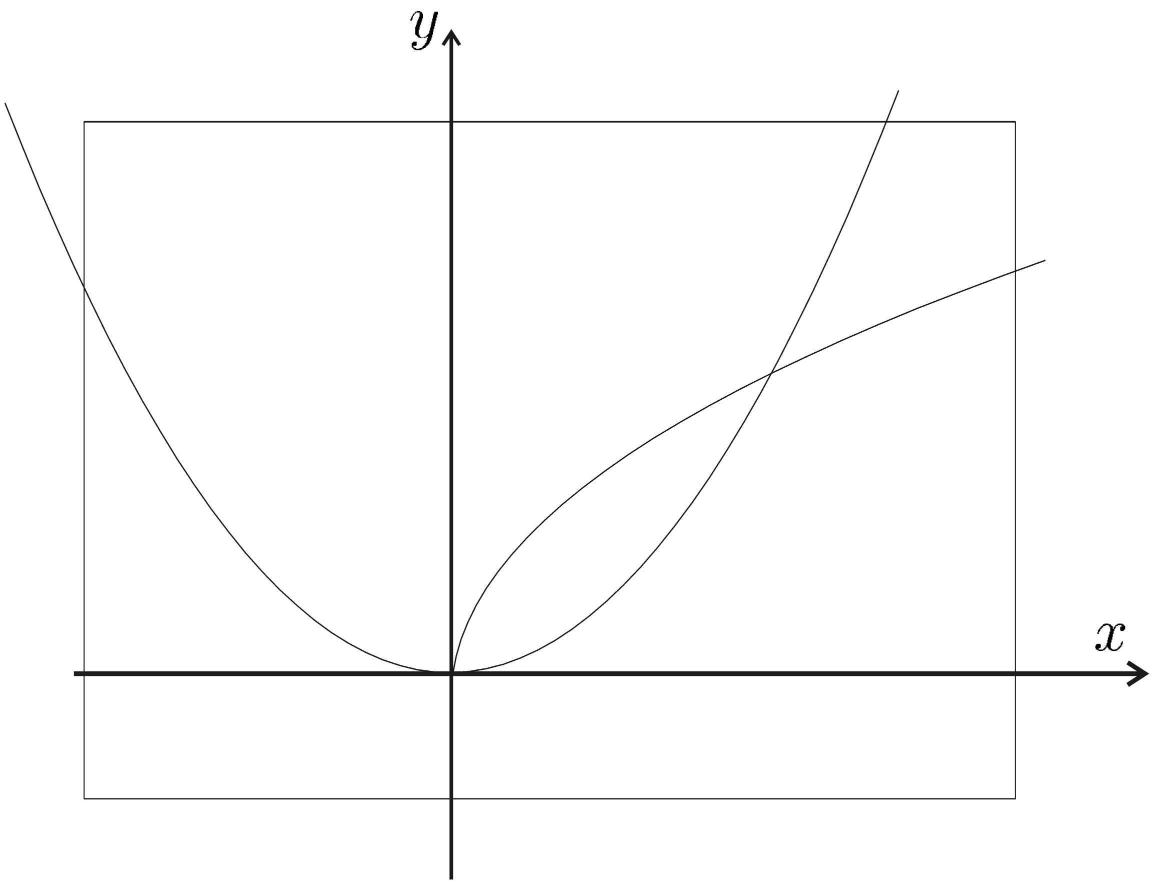
We build two contractors

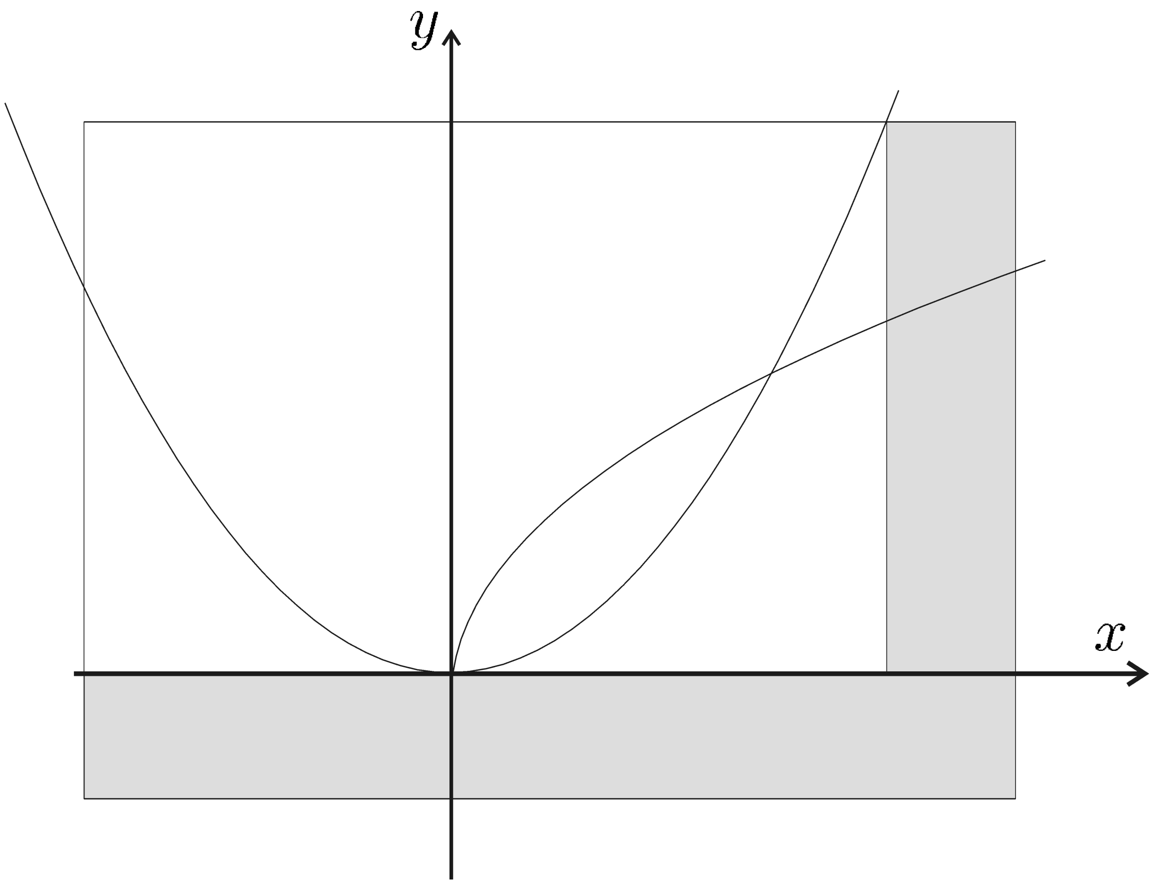
$$\mathcal{C}_1 : \left\{ \begin{array}{l} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{array} \right. \quad \text{associated to } y = x^2$$

$$\mathcal{C}_2 : \left\{ \begin{array}{l} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{array} \right. \quad \text{associated to } y = \sqrt{x}$$

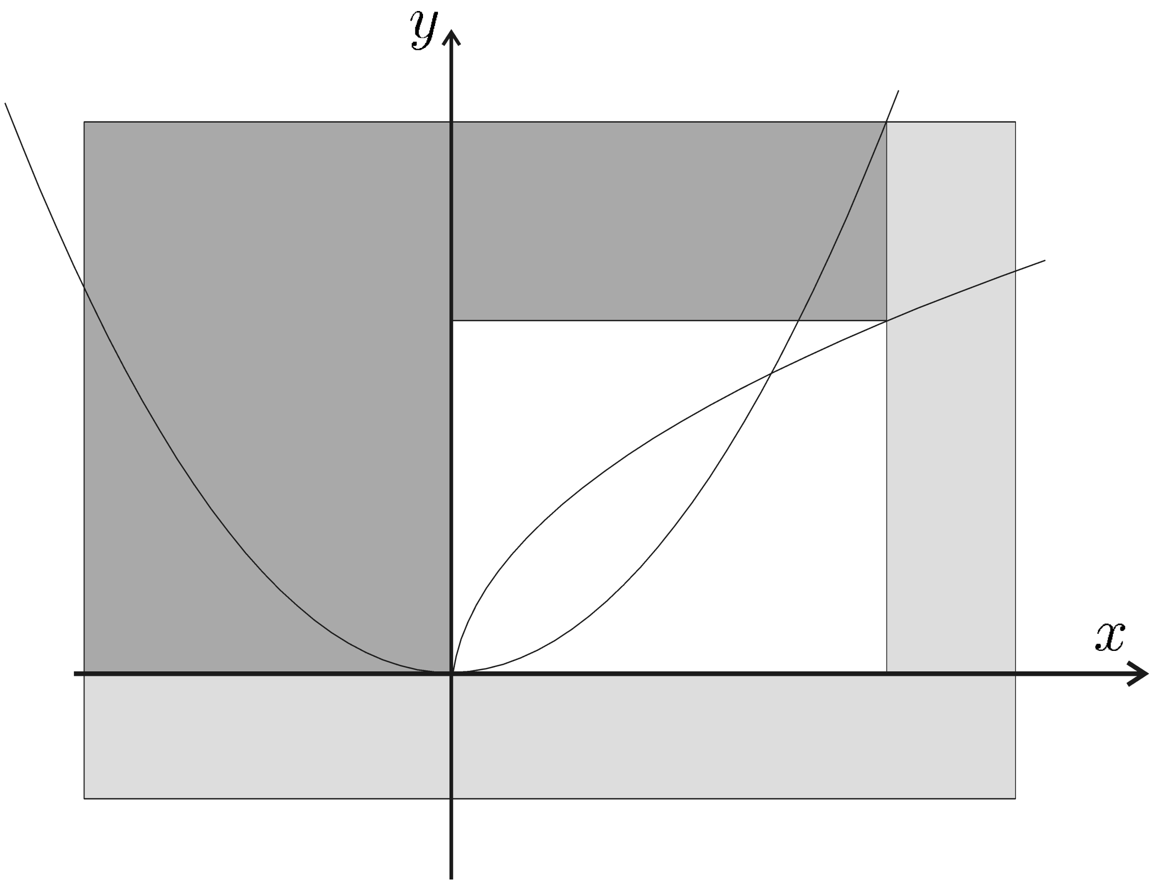


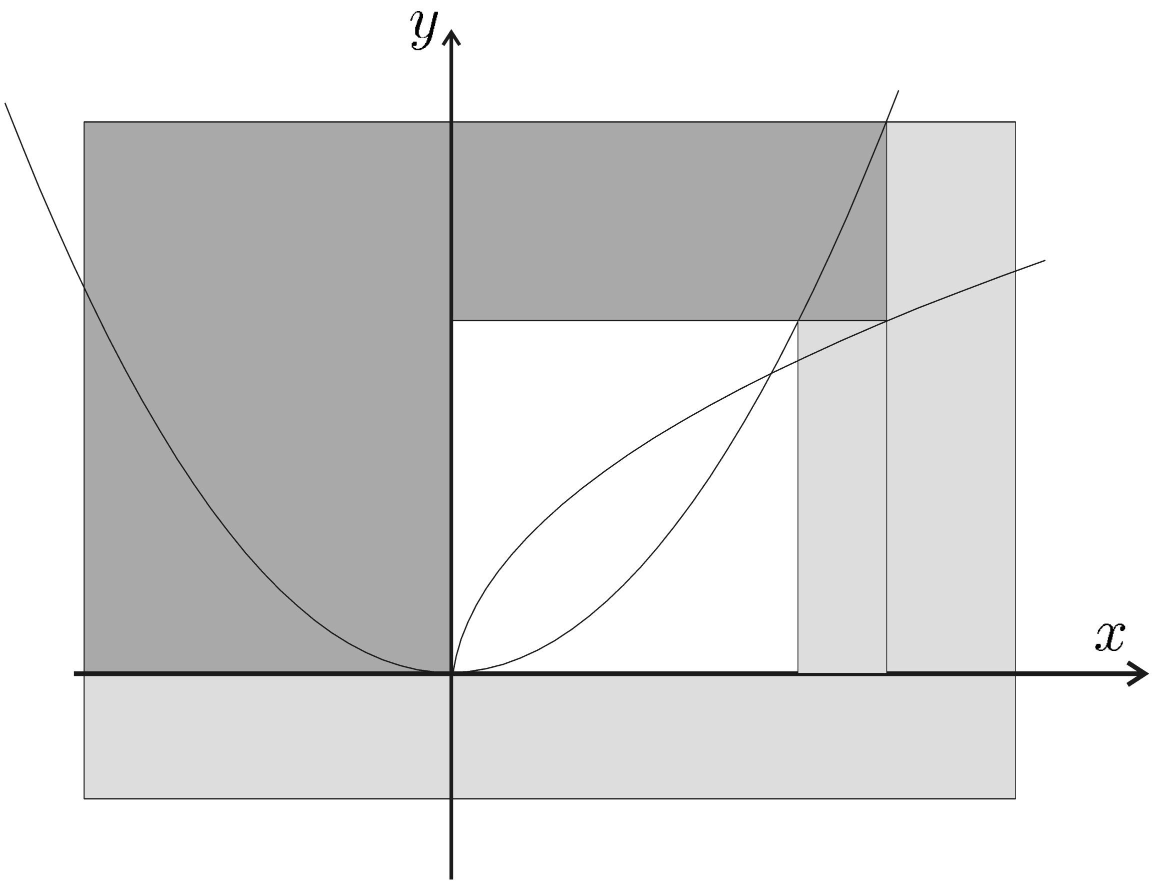
Contractor graph

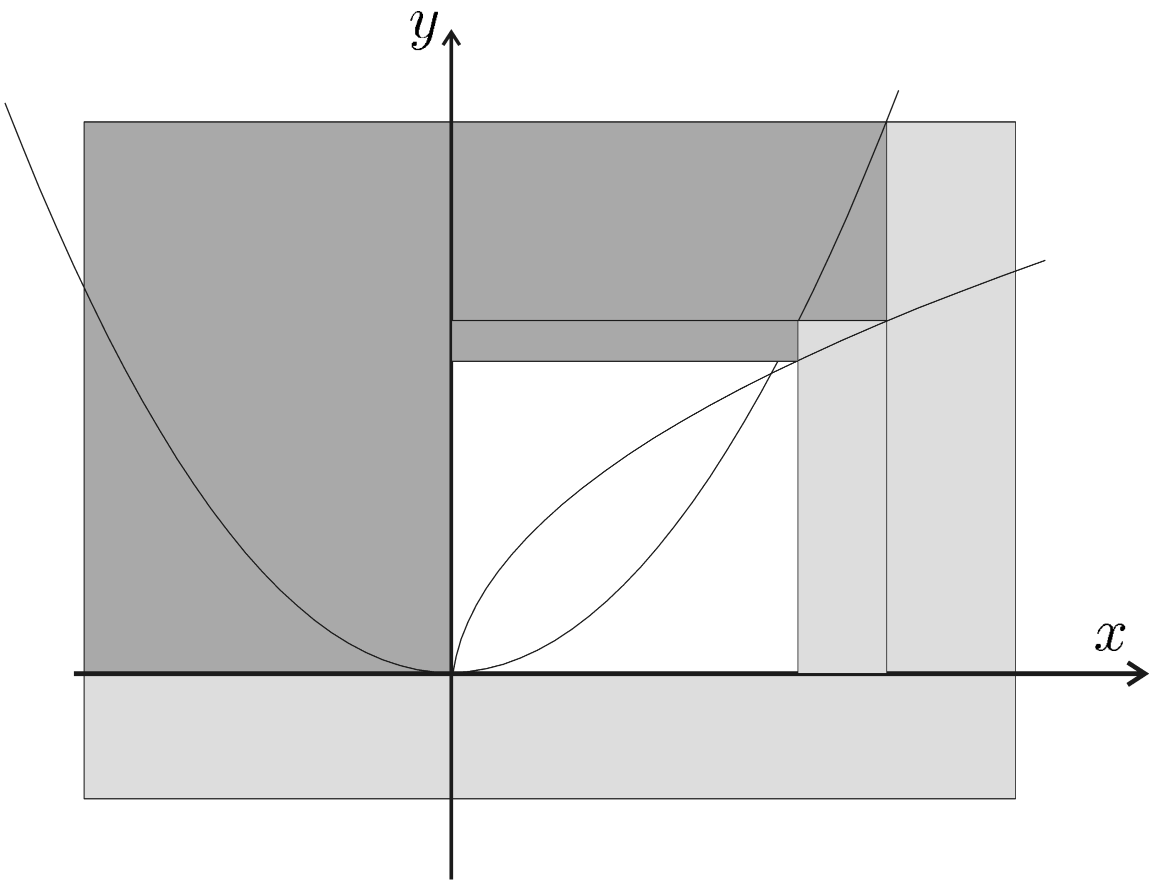

y
x



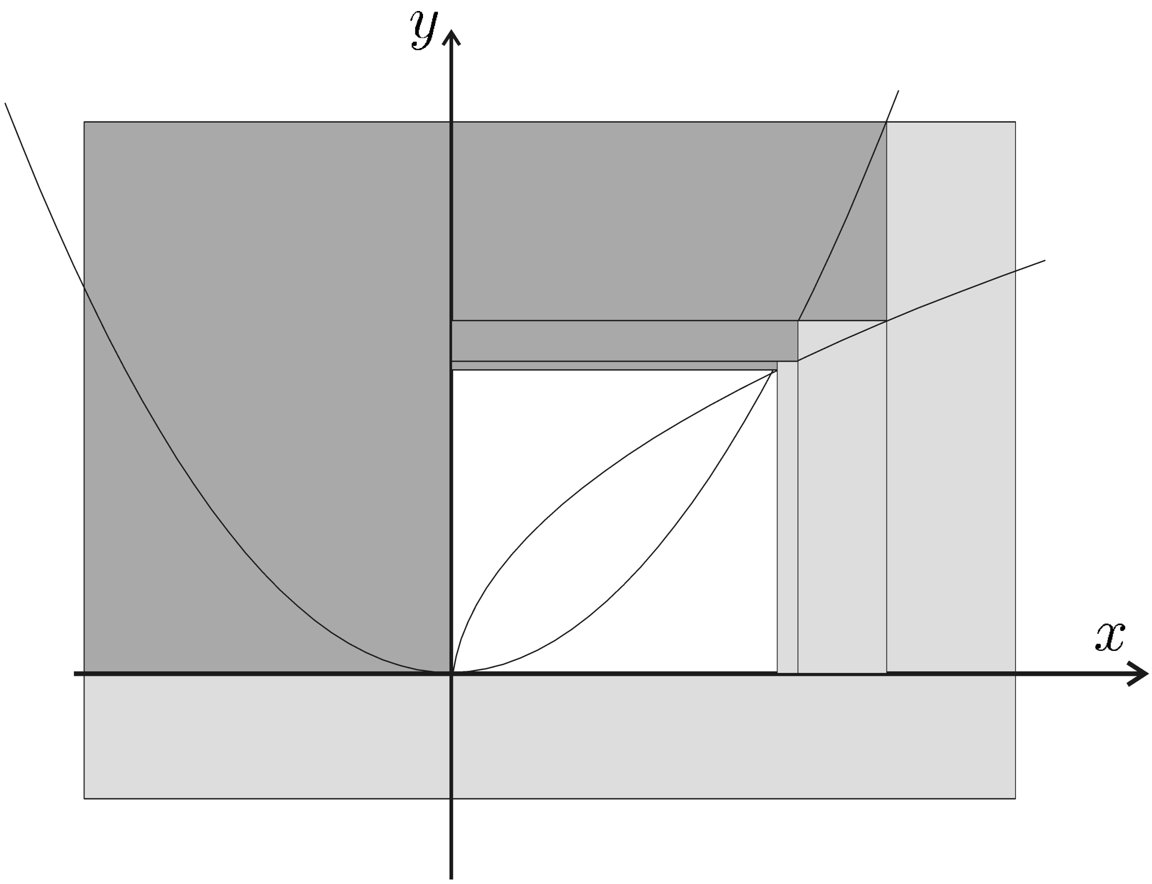
y
x

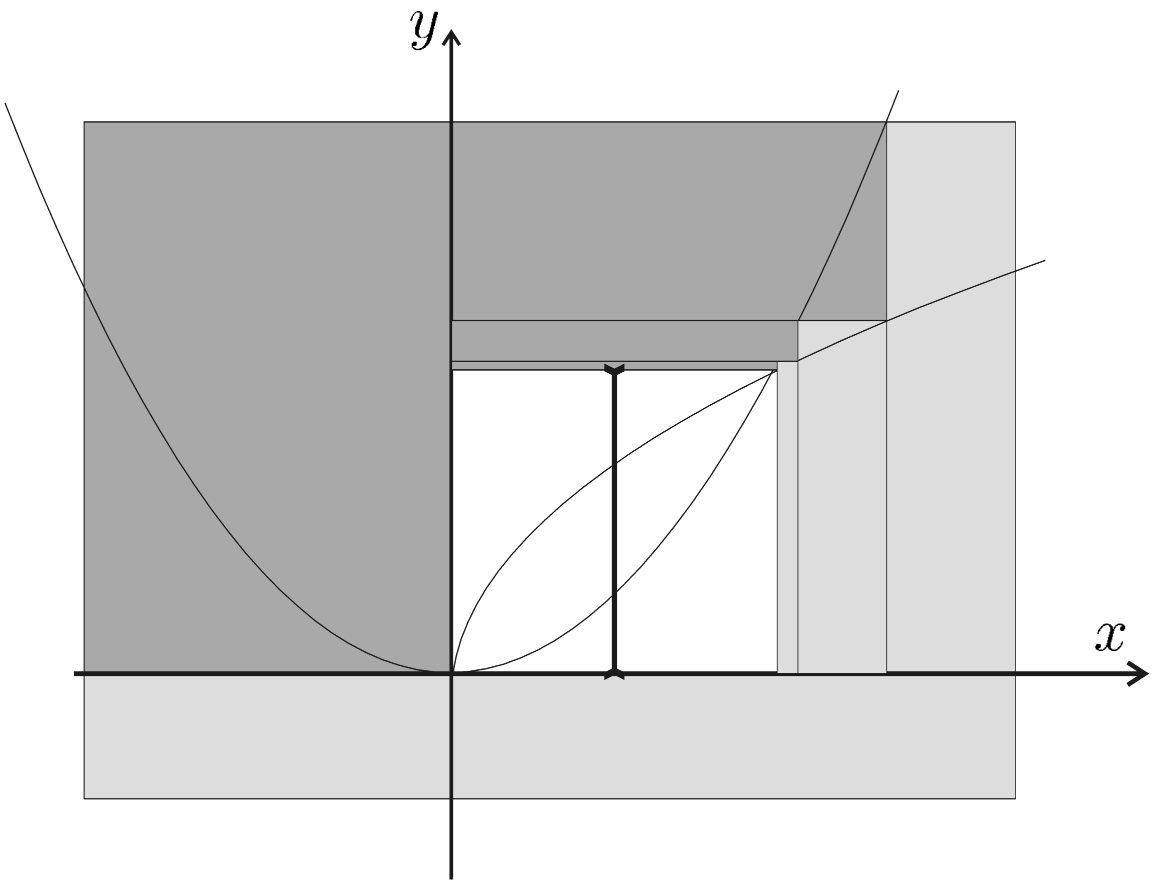

y
x

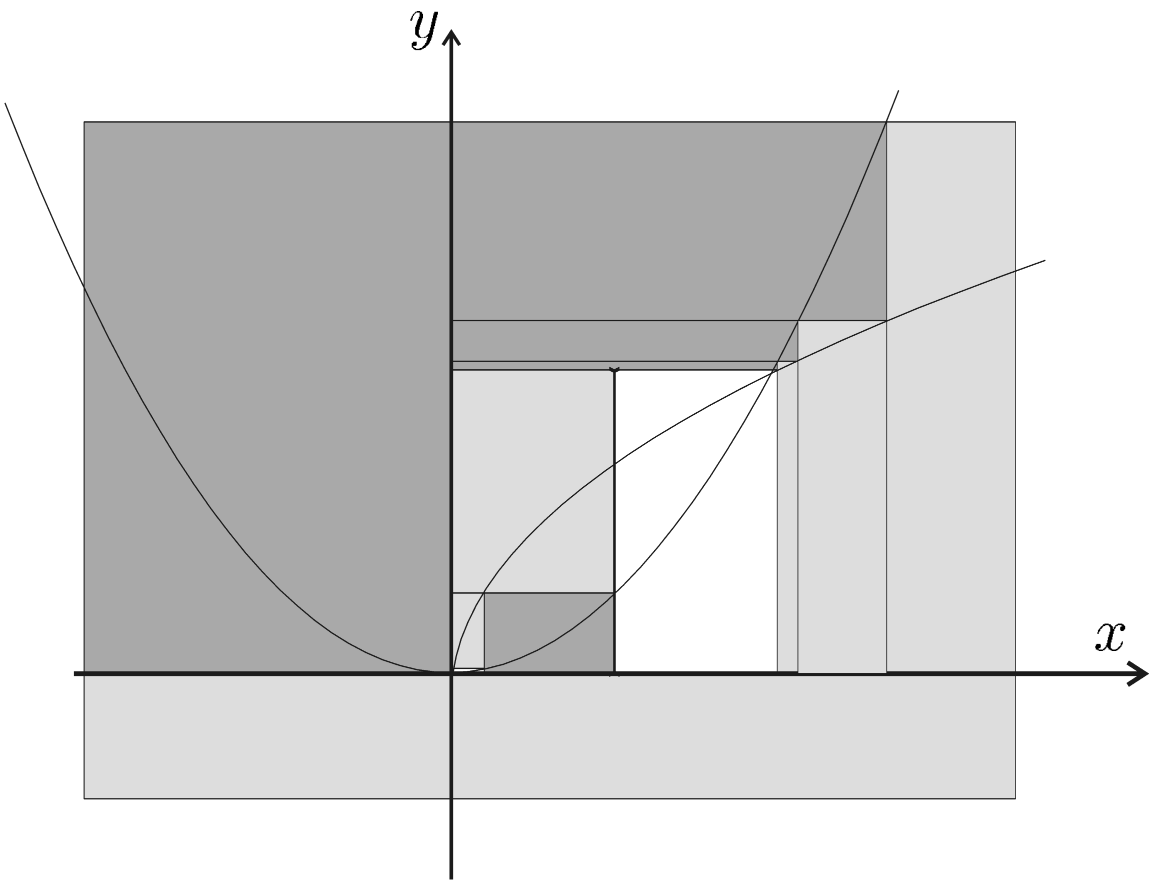

y
x

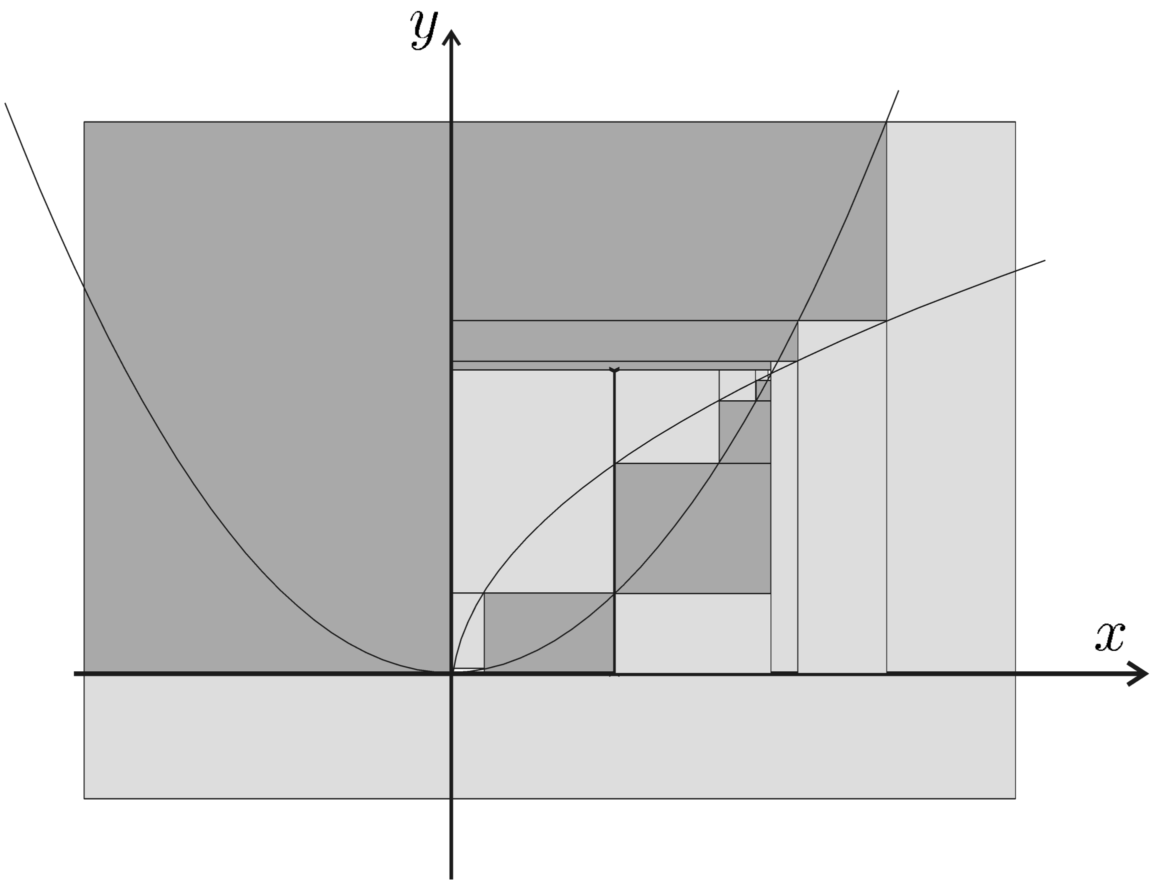


y
x


y
x


y
x


y
x

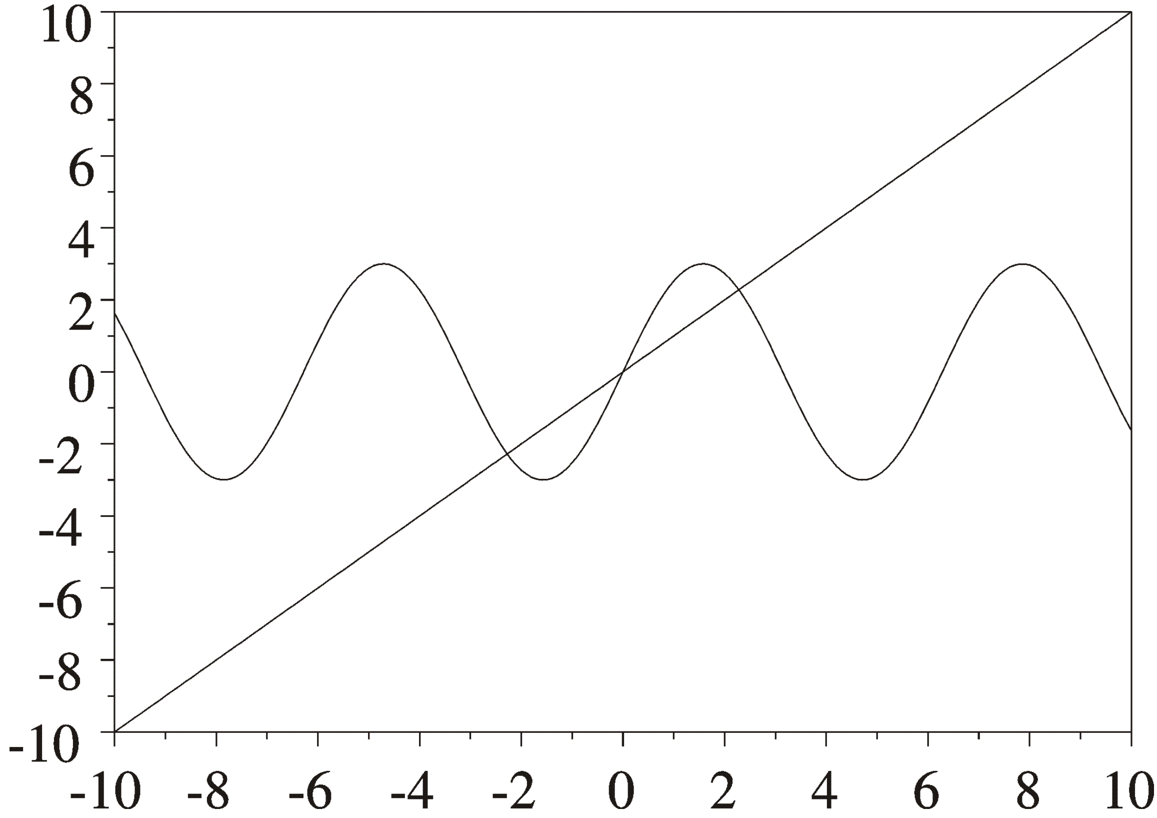

y
x

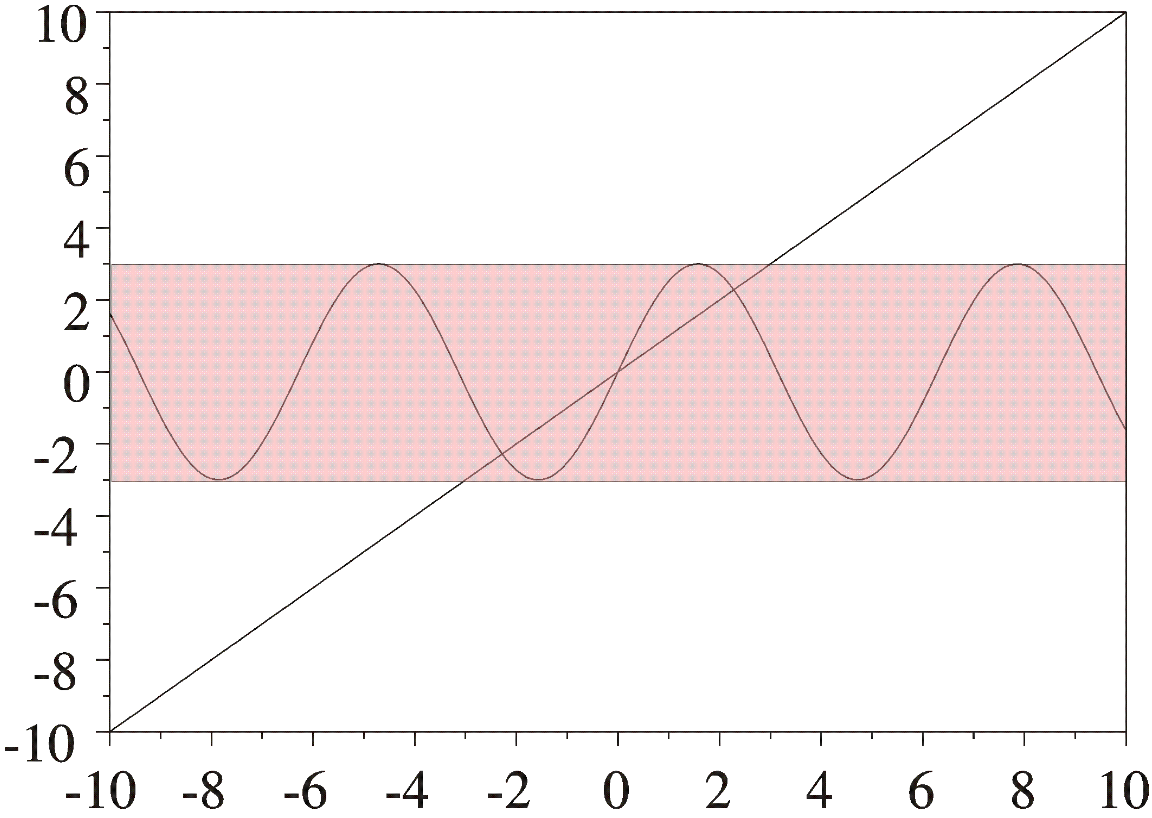
Note that

$$\begin{gathered}\mathcal{C}_1 \text{ is optimal} \\ \mathcal{C}_2 \text{ is optimal} \\ \mathcal{C}_1 \circ \mathcal{C}_2 \text{ is not optimal} \\ (\mathcal{C}_1 \circ \mathcal{C}_2)^\infty \text{ is optimal.}\end{gathered}$$

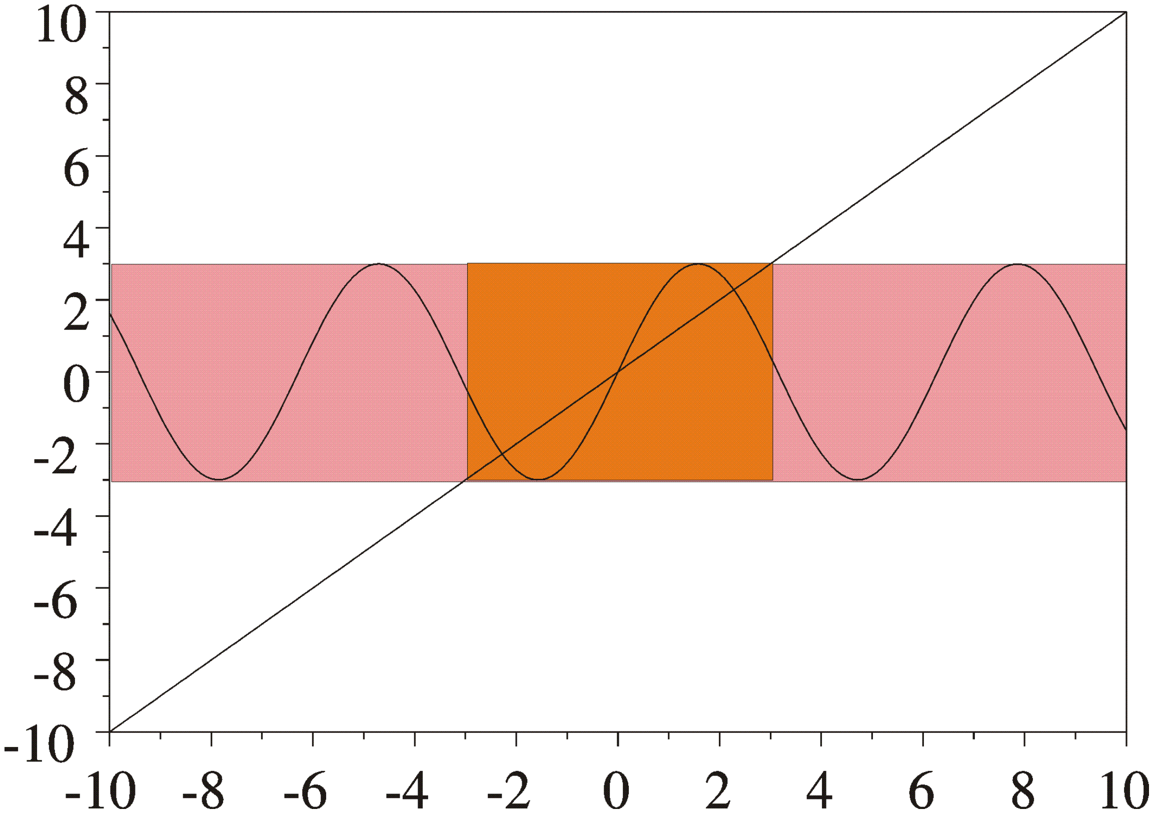
Example 2. Consider the system

$$\left\{\begin{array}{rcl} y &=& 3\sin(x) \\ y &=& x \end{array}\right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10



10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10

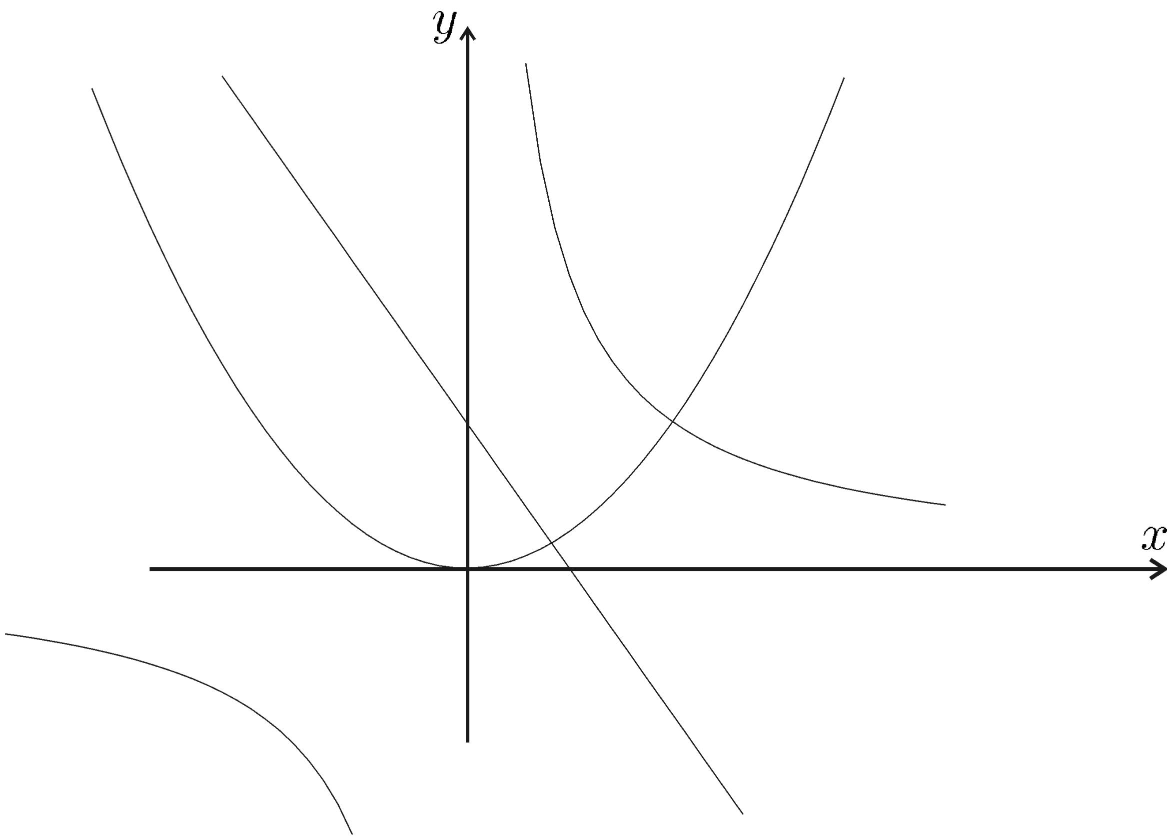


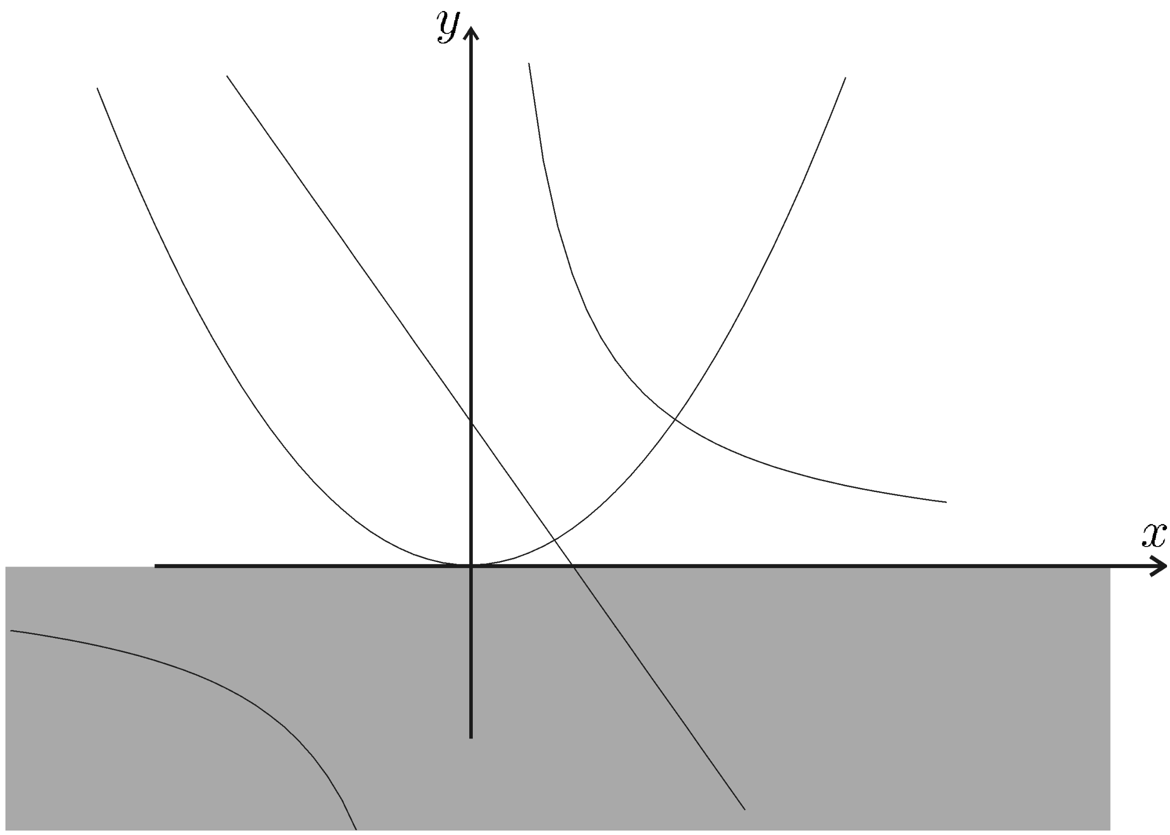
We converge the largest box $[\mathbf{x}]$ such that

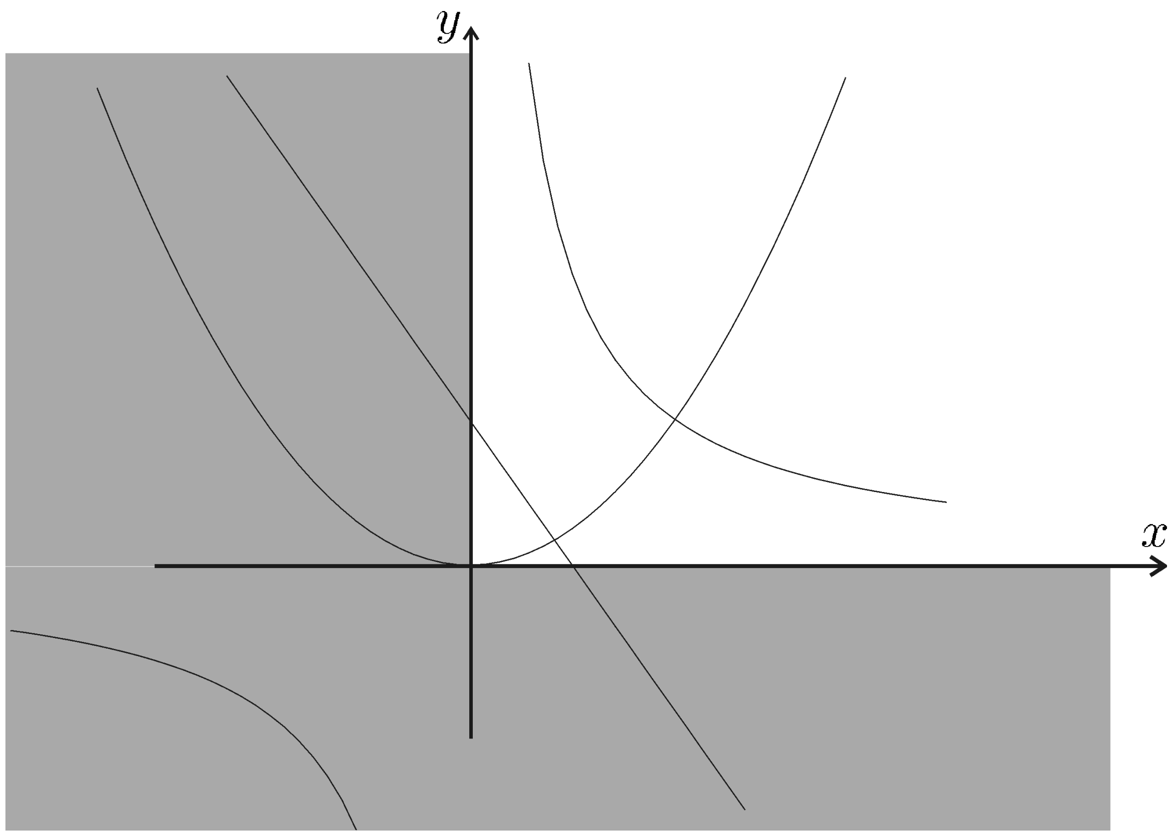
$$\mathcal{C}_1([\mathbf{x}]) = \mathcal{C}_2([\mathbf{x}]) = [\mathbf{x}].$$

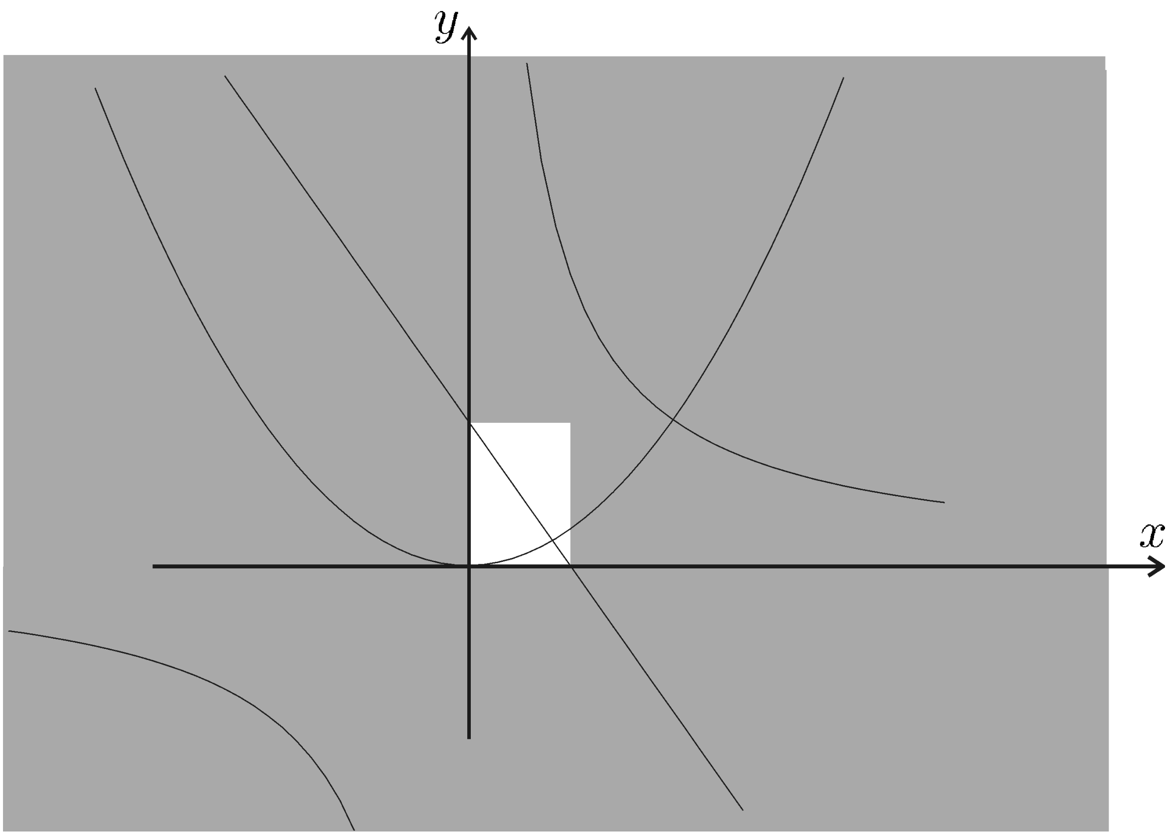
Example 3. Consider the following problem

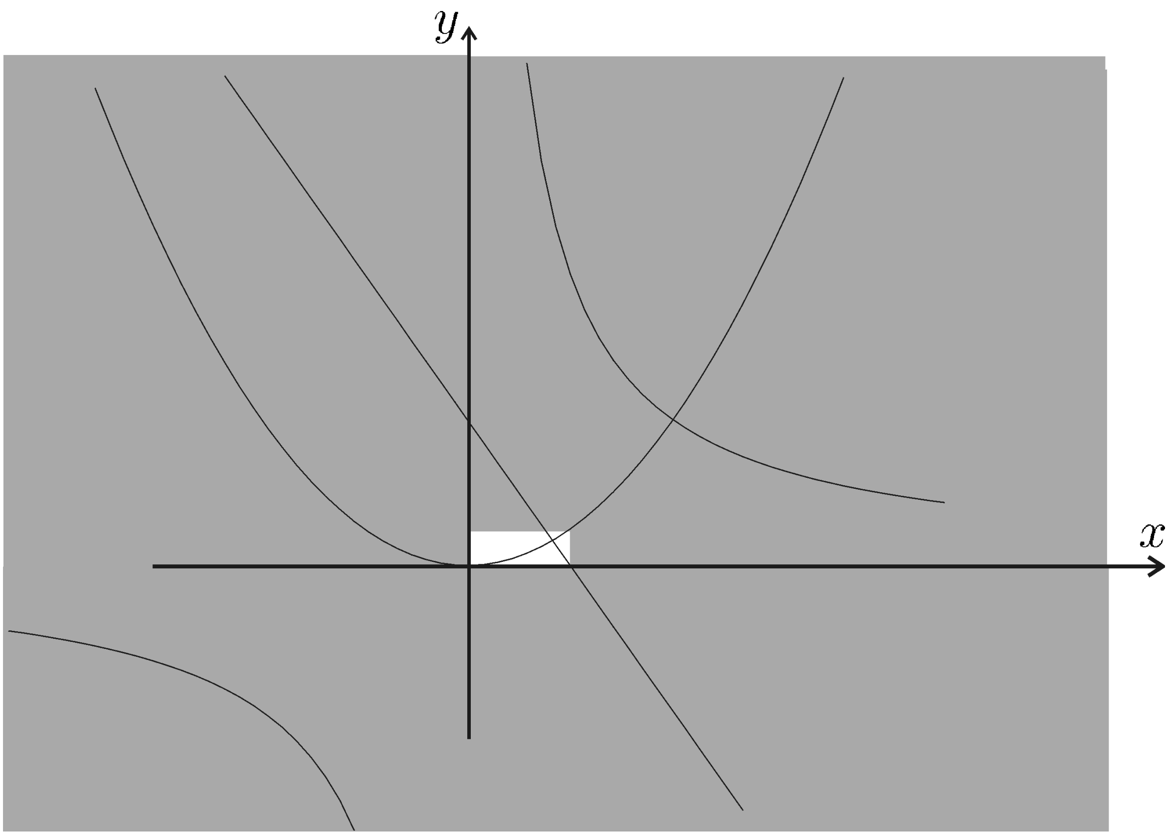
$$\begin{cases} (C_1): & y = x^2 \\ (C_2): & xy = 1 \\ (C_3): & y = -2x + 1 \end{cases}$$


y
x

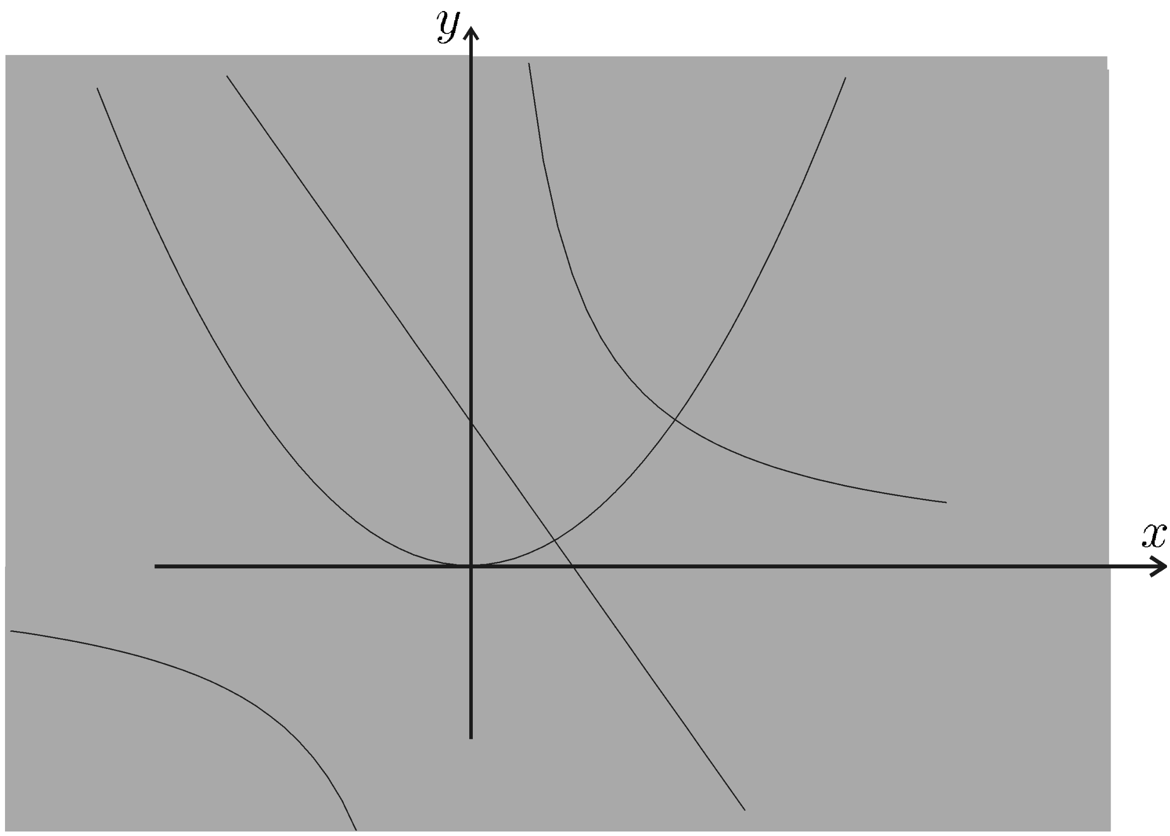

y
x


y
x


y
x



y
x


y
x

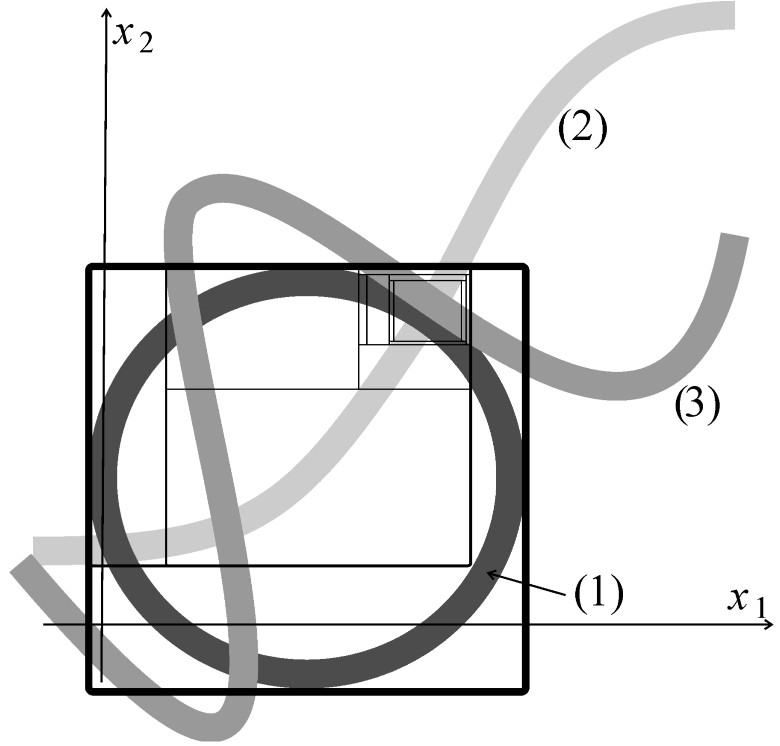
$$
\begin{array}{lll}
(C_1) \Rightarrow & y \in & [-\infty, \infty]^2 \; = \; [0, \infty] \\
(C_2) \Rightarrow & x \in & 1/[0, \infty] \; = [0, \infty] \\
(C_3) \Rightarrow & y \in & [0, \infty] \; \cap \; ((-2)\,.[0, \infty] + 1) \\
 & & = [0, \infty] \; \cap \; ([-\infty, 1]) = [0, 1] \\
 & x \in & [0, \infty] \; \cap \; (-[0, 1]/2 + 1/2) = [0, \frac{1}{2}] \\
(C_1) \Rightarrow & y \in & [0, 1] \cap [0, 1/2]^2 = [0, 1/4] \\
(C_2) \Rightarrow & x \in & [0, 1/2] \; \cap \; 1/[0, 1/4] = \emptyset \\
 & y \in & [0, 1/4] \cap 1/\emptyset = \emptyset
\end{array}
$$

3 Redundant system of equations

Problem

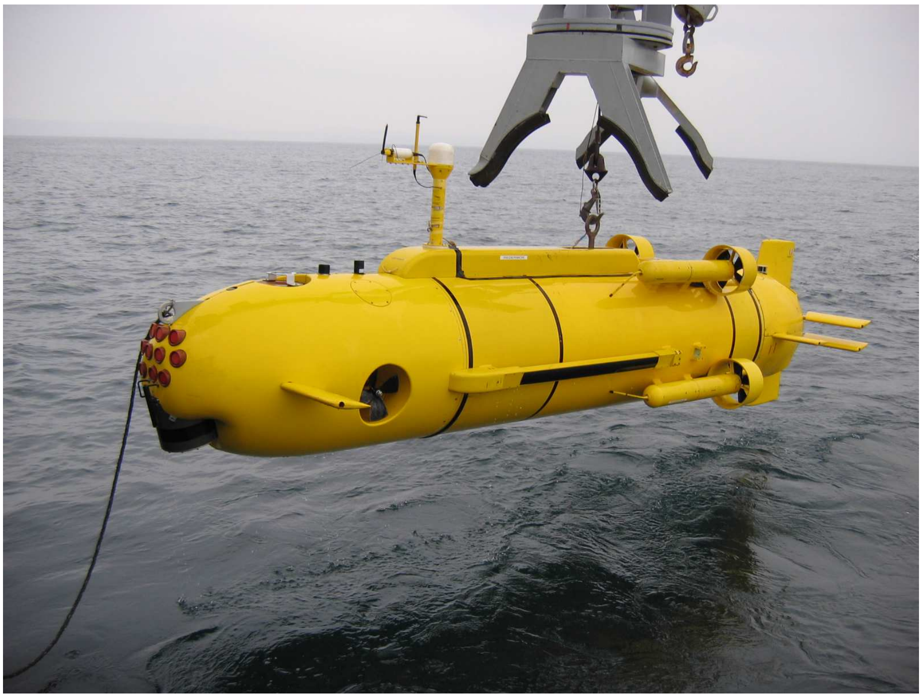
$$\left\{ \begin{array}{c} f_i(\mathbf{x}, \mathbf{y}_i) = 0, \\ \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{y}_i \in [\mathbf{y}_i] \subset \mathbb{R}^{p_i} \\ i \in \{1, \ldots, m\} \end{array} \right. .$$

with $m \gg n \gg 1$.

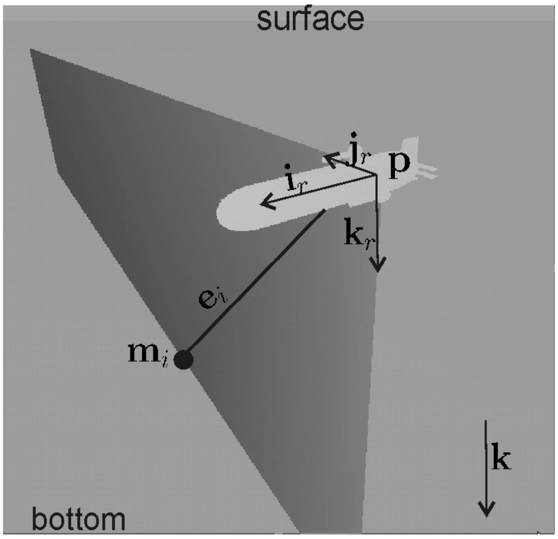


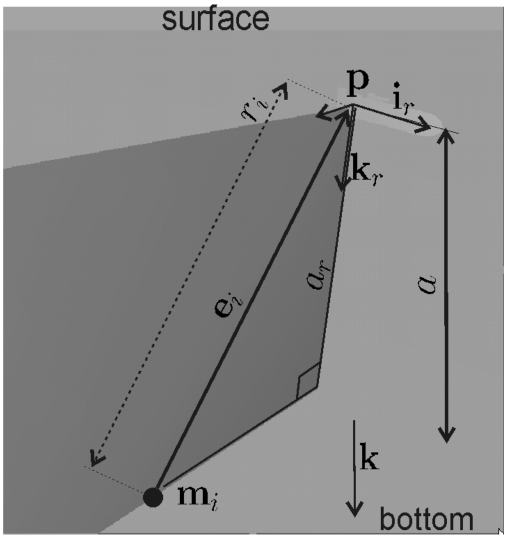
x_2
(2)
(3)
(1)
x_1

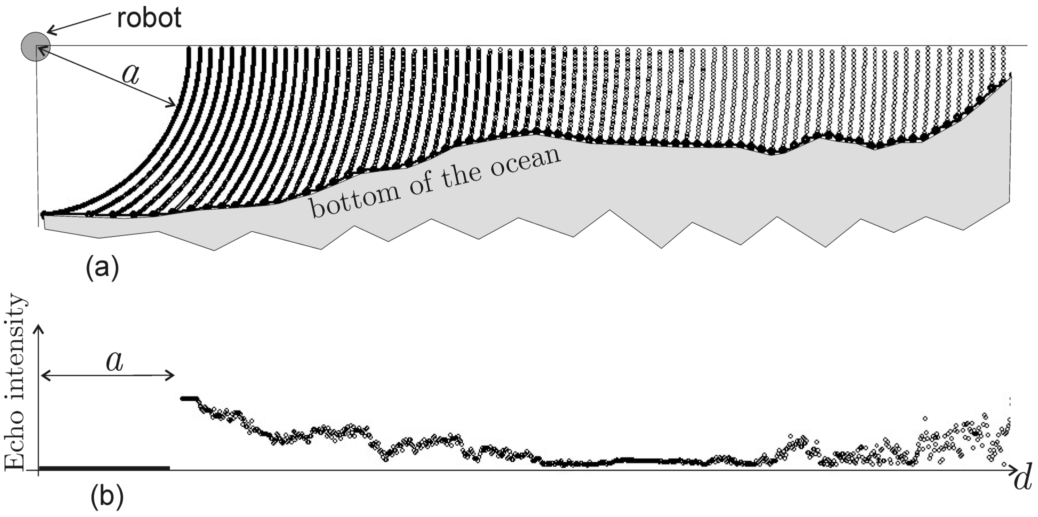
4 SLAM

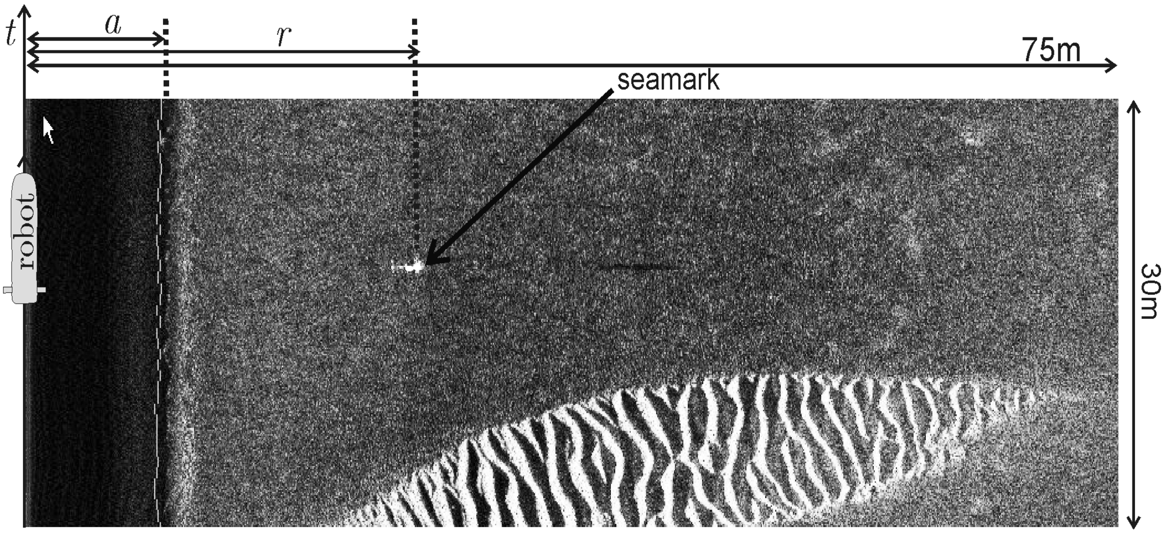


Show the video


surface
$\mathbf{j}_r$
$\mathbf{p}$
$\mathbf{i}_r$
$\mathbf{k}_r$
$\mathbf{e}_i$
$\mathbf{m}_i$
$\mathbf{k}$
bottom


surface
$\mathbf{p}$
$\mathbf{i}_r$
r_i
$\mathbf{k}_r$
a_r
a
$\mathbf{e}_i$
$\mathbf{k}$
$\mathbf{m}_i$
bottom


robot
a
bottom of the ocean
(a)
Echo intensity
a
d
(b)



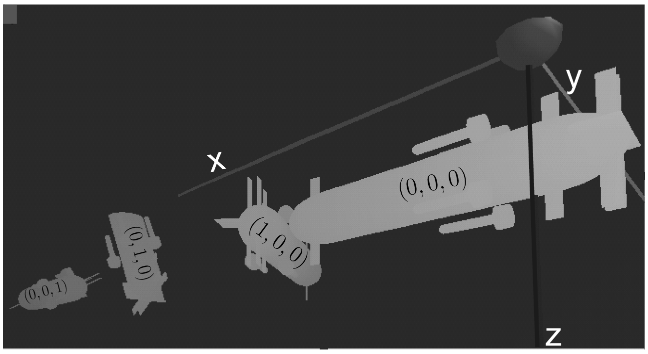
Mine detection with SonarPro

Loch-Doppler returns the speed robot $\mathbf{v}_r$.

$$\mathbf{v}_r \in \tilde{\mathbf{v}}_r + 0.004 * [-1, 1] \cdot \tilde{\mathbf{v}}_r + 0.004 * [-1, 1]$$

Inertial central (Octans III from IXSEA).

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4}.\,[-1,1] \\ 1.75 \times 10^{-4}.\,[-1,1] \\ 5.27 \times 10^{-3}.\,[-1,1] \end{pmatrix}.$$



Six mines have been detected.

i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau(i)$	7054	7092	7374	7748	9038	9688	10024	10817	11172	11232	11279	11688
$\sigma(i)$	1	2	1	0	1	5	4	3	3	4	5	1
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98	37.90	36.71	37.37	31.03	33.51	15.05

4.1 Constraints

$$t \in \{6000.0, 6000.1, 6000.2, \ldots, 11999.4\},$$

$$i \in \{0, 1, \ldots, 11\},$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \cdot \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) \cdot \frac{\pi}{180}\right) & 0 \end{pmatrix} \cdot \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix},$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t)),$$

$$\mathbf{R}_\psi(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{R}_\theta(t) = \begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix},$$

$$\mathbf{R}_\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & -\sin\varphi(t) \\ 0 & \sin\varphi(t) & \cos\varphi(t) \end{pmatrix},$$

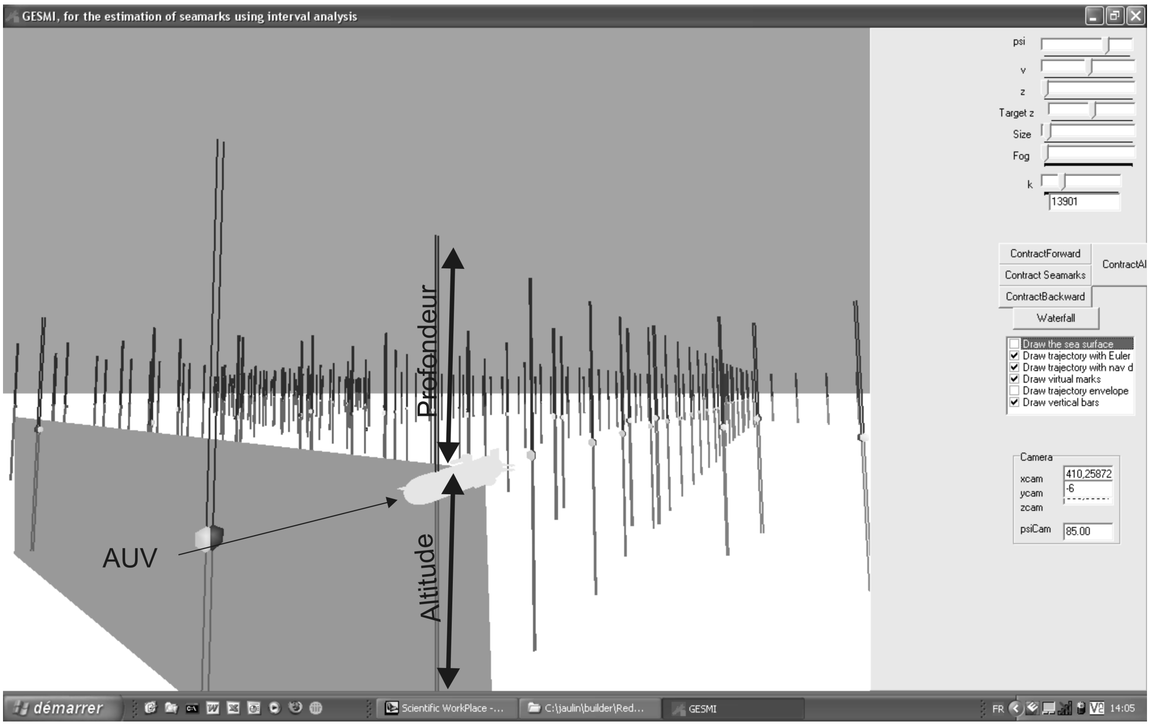
$$\mathbf{R}(t) = \mathbf{R}_\psi(t) \cdot \mathbf{R}_\theta(t) \cdot \mathbf{R}_\varphi(t),$$

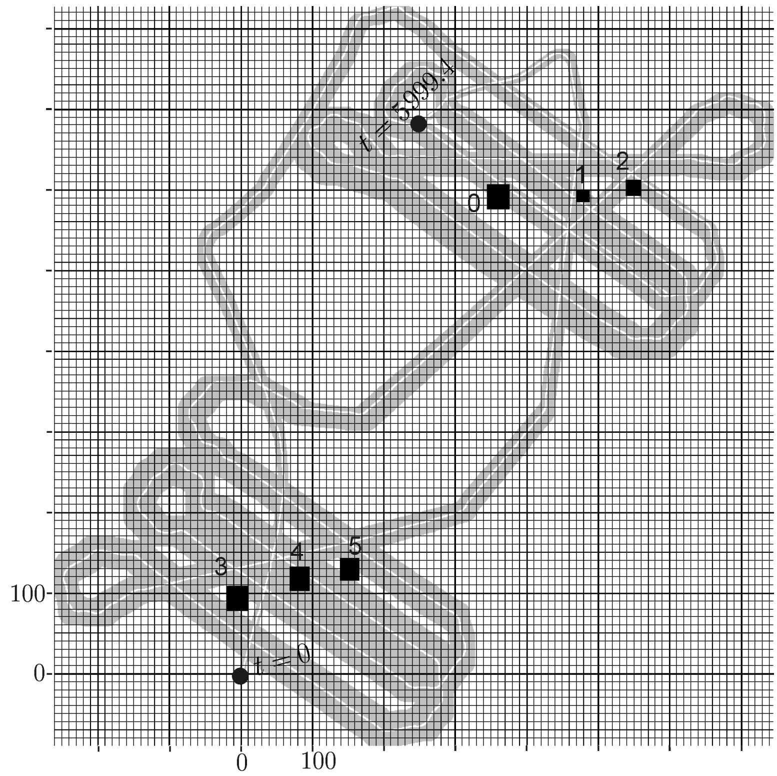
$$\dot{\mathbf{p}}(t) = \mathbf{R}(t) \cdot \mathbf{v}_r(t),$$

$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| \; = r(i),$$

$$\mathbf{R}^\mathsf{T}(\tau(i)) \cdot (\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0, \infty]^{\times 2}.$$

4.2 GESMI


GESMI, for the estimation of seamarks using interval analysis
psi
v
z
Target z
Size
Fog
k
13901
ContractForward
Contract Seamarks
ContractBackward
Waterfall
Draw the sea surface
Draw trajectory with Euler
Draw trajectory with nav d
Draw virtual marks
Draw trajectory envelope
Draw vertical bars
Camera
xcam
ycam
zcam
410,25872
-6
psiCam
85.00
Profondeur
Altitude
AUV
démarrer
Scientific WorkPlace -...
GESMI
FR
14:05



t = 5000.4
0
1
2
3
4
5
t = 0
100
0
0
100

4.3 Quimper language

```
//--------------------------------------------------------------
Constants
N = 59996; // Number of time steps
Variables
  R[N-1][3][3], // rotation matrices
  p[N][3], // positions
  v[N-1][3], // speed vectors
  phi[N-1],theta[N-1],psi[N-1]; // Euler angles
  px[N],py[N]; // for display only
//--------------------------------------------------------------
```

```
function R[3][3]=euler(phi,theta,psi)
  cphi = cos(phi);
  sphi = sin(phi);
  ctheta = cos(theta);
  stheta = sin(theta);
  cpsi = cos(psi);
  spsi = sin(psi);
  R[1][1]=ctheta*cpsi;
  R[1][2]=-cphi*spsi+stheta*cpsi*sphi;
  R[1][3]=spsi*sphi+stheta*cpsi*cphi;
  R[2][1]=ctheta*spsi;
  R[2][2]=cpsi*cphi+stheta*spsi*sphi;
  R[2][3]=-cpsi*sphi+stheta*cphi*spsi;
  R[3][1]=-stheta;
  R[3][2]=ctheta*sphi;
  R[3][3]=ctheta*cphi;
end
```

```
contractor-list rotation
  for k=1:N-1;
    R[k]=euler(phi[k],theta[k],psi[k]);
  end
end
//---------------------------------
contractor-list statequ
  for k=1:N-1;
    p[k+1]=p[k]+0.1*R[k]*v[k];
  end
end
//---------------------------------
contractor init
  inter k=1:N-1;
    rotation(k)
  end
end
```

```
contractor fwd
  inter k=1:N-1;
    statequ(k)
  end
end
//-------------------------------
contractor bwd
  inter k=1:N-1;
    statequ(N-k)
  end
end
```

```
main
  p[1] :=read("gps_init.dat");
  v :=read("Quimper_v.dat");
  phi :=read("Quimper_phi.dat");
  theta :=read("Quimper_theta.dat");
  psi :=read("Quimper_psi.dat");
  init;
  fwd;
  bwd;
  column(p,px,1);
  column(p,py,2);
  print("--- Robot positions: ---");
  newplot("gesmi.dat");
  plot(px,py,color(rgb(1,1,1),rgb(0,0,0)));
end
```

5 Sailboat robotics

5.1 Vaimos

Vaimos (IFREMER and ENSTA)

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right).$$

With the controller $\mathbf{u} = \mathbf{g}\left(\mathbf{x}\right)$, the robot satisfies an equation of the form

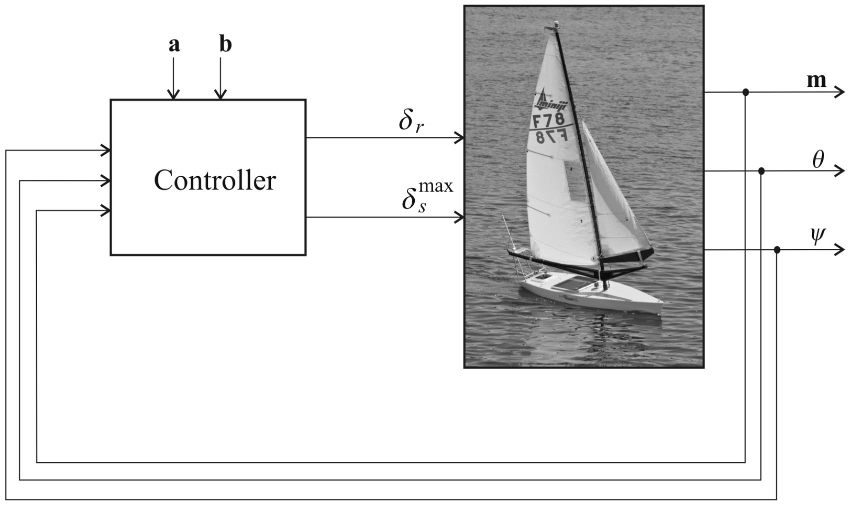
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right).$$

With all uncertainties, the robot satisfies.

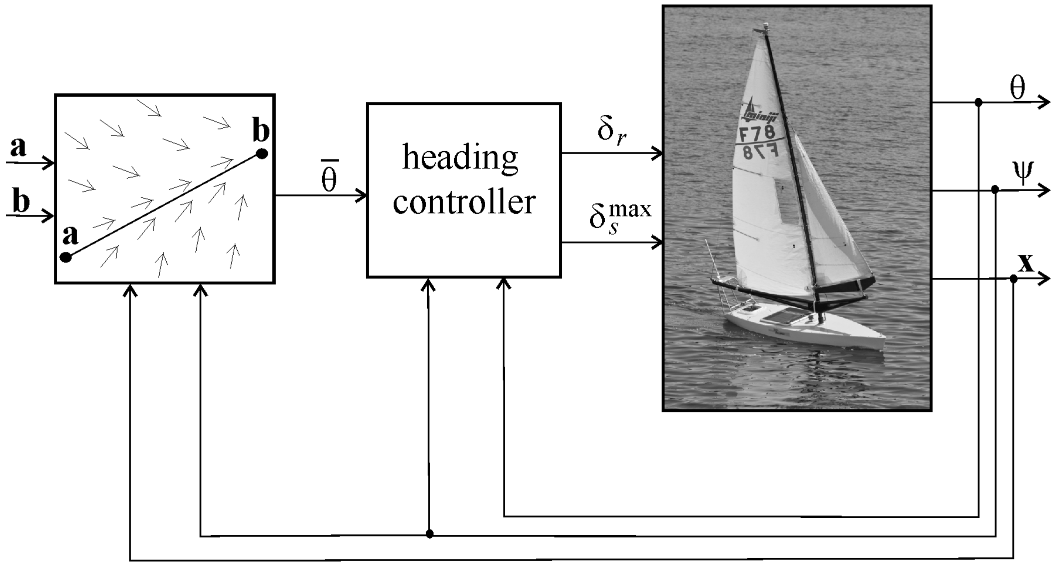
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

5.2 Line following



Controller of a sailboat robot



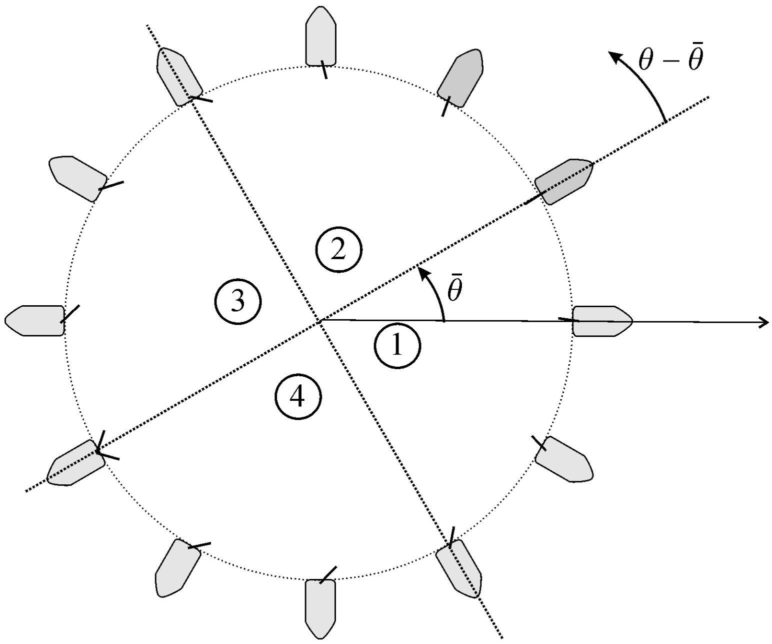
a
b
a
b
θ̄
heading
controller
δ_r
δ_s^max
F78
θ
ψ
x

Heading controller

$$\left\{\begin{array}{lcl} \delta_r & = & \left\{\begin{array}{ll} \delta_r^{\max}.\sin\left(\theta-\bar{\theta}\right) & \text{if } \cos\left(\theta-\bar{\theta}\right)\geq 0 \\ \delta_r^{\max}.\mathrm{sign}\left(\sin\left(\theta-\bar{\theta}\right)\right) & \text{otherwise} \end{array}\right. \\ \delta_s^{\max} & = & \frac{\pi}{2}.\left(\frac{\cos(\psi-\bar{\theta})+1}{2}\right)^q. \end{array}\right.$$

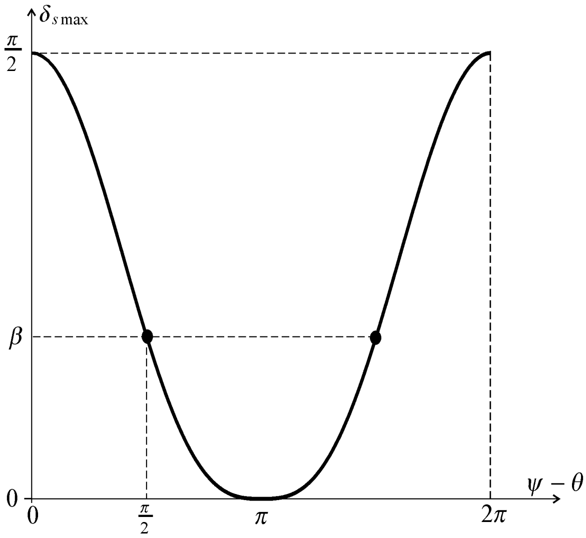
Rudder

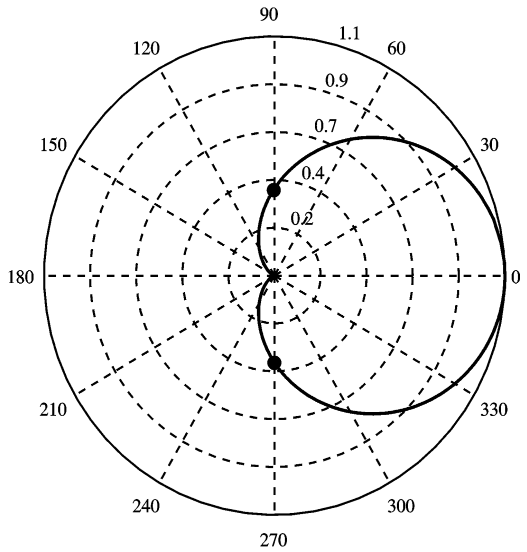
$$\delta_r = \begin{cases} \delta_r^{\max}.\sin\left(\theta - \bar{\theta}\right) & \text{if } \cos\left(\theta - \bar{\theta}\right) \geq 0 \\ \delta_r^{\max}.\mathrm{sign}\left(\sin\left(\theta - \bar{\theta}\right)\right) & \text{otherwise} \end{cases}$$



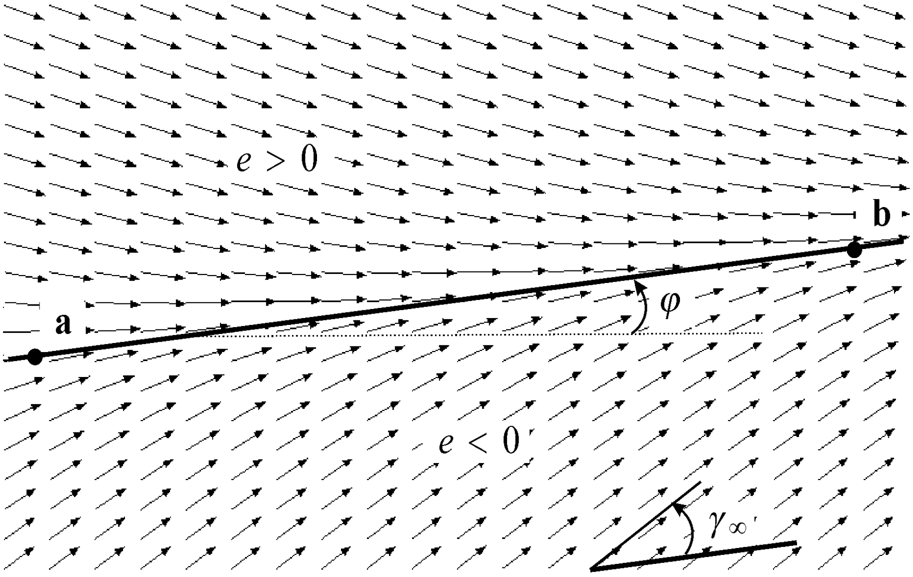
Sail

$$\delta_s^{\max} = \frac{\pi}{2} \cdot \left(\frac{\cos\left(\psi - \bar{\theta}\right) + 1}{2} \right)^q \text{ with } q = \frac{\log\left(\frac{\pi}{2\beta}\right)}{\log(2)}$$



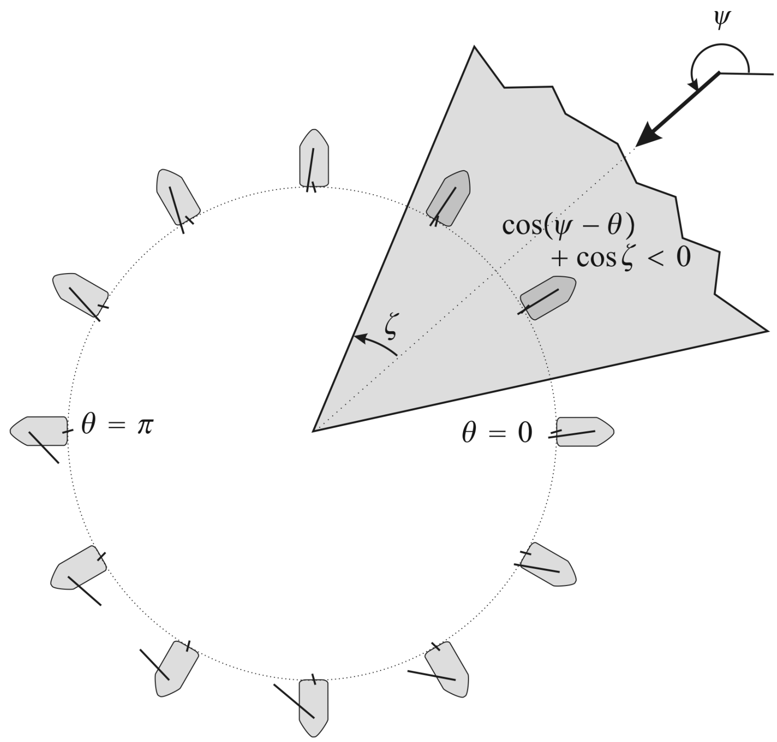


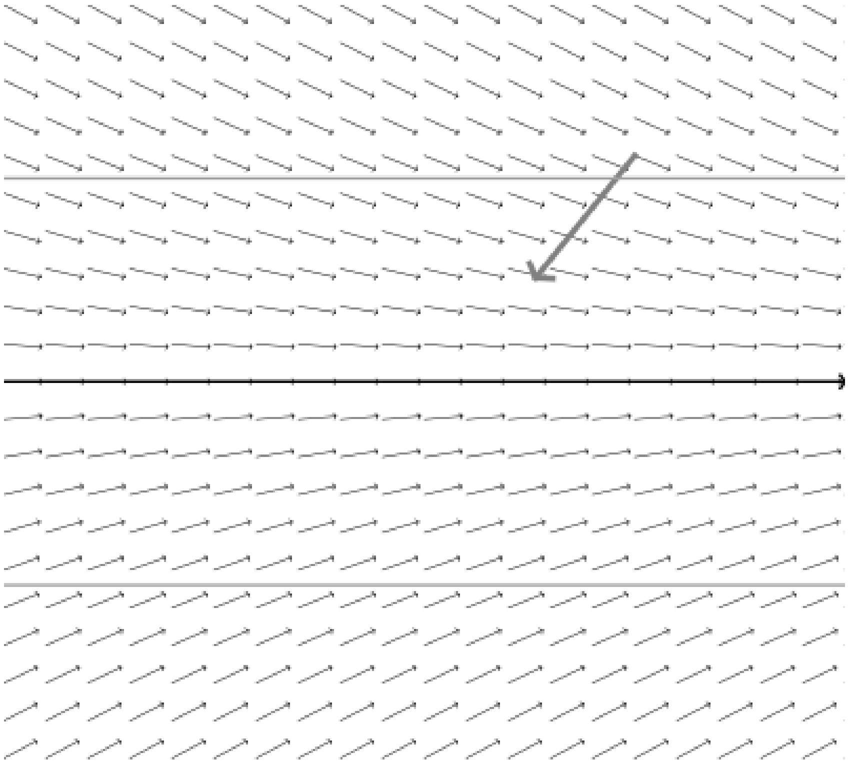
5.3 Vector field



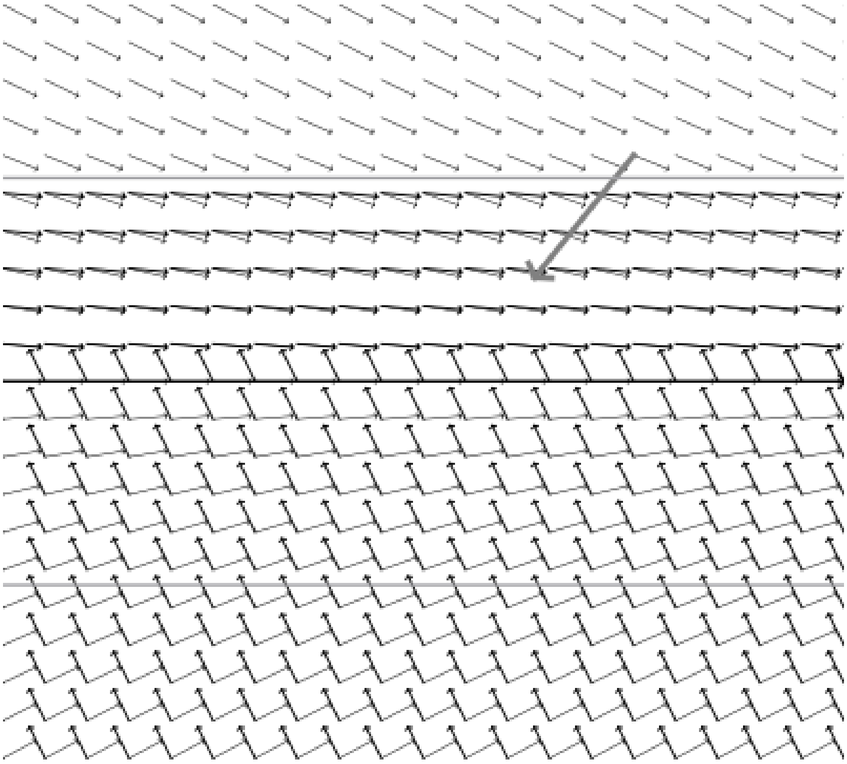
Nominal vector field: $\theta^* = \varphi - \frac{2.\gamma_\infty}{\pi}.\text{atan}\left(\frac{e}{r}\right).$

A course θ^* may be unfeasible





$$\theta^* = -\frac{2.\gamma_\infty}{\pi}.\mathsf{atan}\left(\frac{e}{r}\right)$$



Keep close hauled strategy.

http://youtu.be/pHteidmZpnY

5.4 Controller

Controller $\bar{\theta}\left(\mathbf{m},\mathbf{a},\mathbf{b},\psi,\gamma_{\infty},r,\zeta\right)$

$e = \det\left(\frac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}, \mathbf{m}-\mathbf{a}\right)$

$\varphi = \text{atan2}(\mathbf{b}-\mathbf{a})$

$\theta^* = \varphi - \frac{2.\gamma_{\infty}}{\pi}.\text{atan}\left(\frac{e}{r}\right)$

if $\cos\left(\psi-\theta^*\right)+\cos\zeta<0$

  or $(|e|<r$ and $(\cos(\psi-\varphi)+\cos(\zeta)<0))$

    then $\bar{\theta}=\pi+\psi-\zeta.\text{sign}(e)$;

    else $\bar{\theta}=\theta^*$;

end

Without hysteresis

Controller in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$; out: $\delta_r, \delta_s^{\max}$; inout: q

$e = \det\left(\frac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}, \mathbf{m}-\mathbf{a}\right)$

if $|e| > \frac{r}{2}$ then $q = \text{sign}(e)$

$\varphi = \text{atan2}(\mathbf{b}-\mathbf{a})$

$\theta^* = \varphi - \frac{2.\gamma_\infty}{\pi}.\text{atan}\left(\frac{e}{r}\right)$

if $\cos(\psi - \theta^*) + \cos\zeta < 0$

  or $(|e| < r$ and $(\cos(\psi - \varphi) + \cos\zeta < 0))$

   then $\bar{\theta} = \pi + \psi - q.\zeta.$

   else $\bar{\theta} = \theta^*$

end

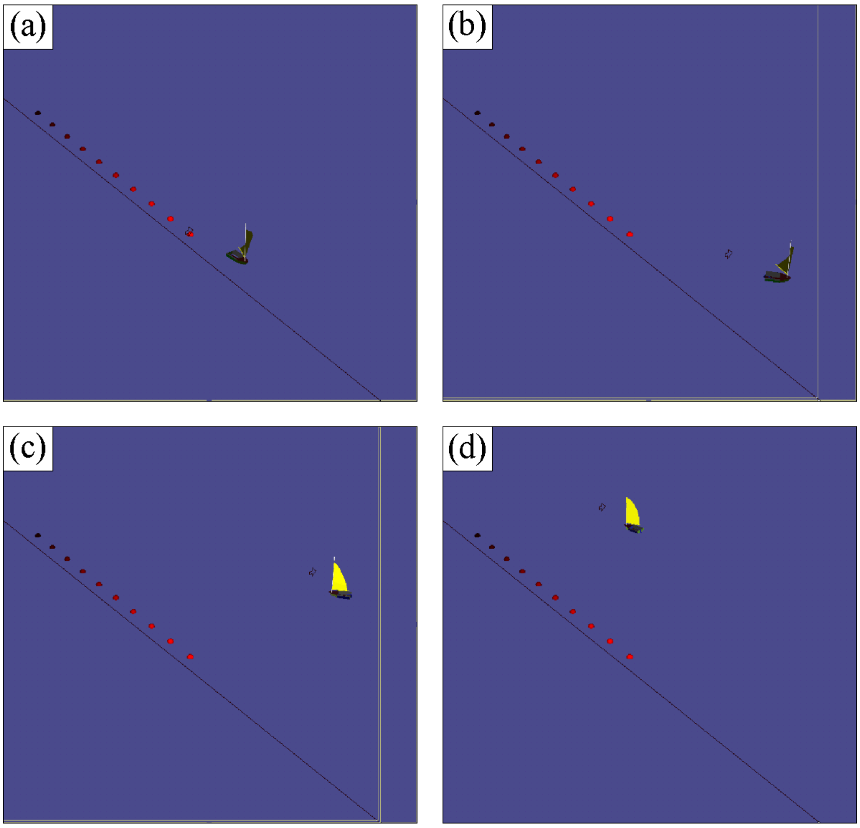
if $\cos\left(\theta - \bar{\theta}\right) \geq 0$ then $\delta_r = \delta_r^{\max}.\sin\left(\theta - \bar{\theta}\right)$

else $\delta_r = \delta_r^{\max}.\text{sign}\left(\sin\left(\theta - \bar{\theta}\right)\right)$

$\delta_s^{\max} = \frac{\pi}{2}.\left(\frac{\cos(\psi - \bar{\theta}) + 1}{2}\right)^q.$

With hysteresis

5.5 Validation by simulation


(a)
(b)
(c)
(d)

5.6 Theoretical validation

When the wind is known, the sailboat with the heading controller is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

The system

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right)$$

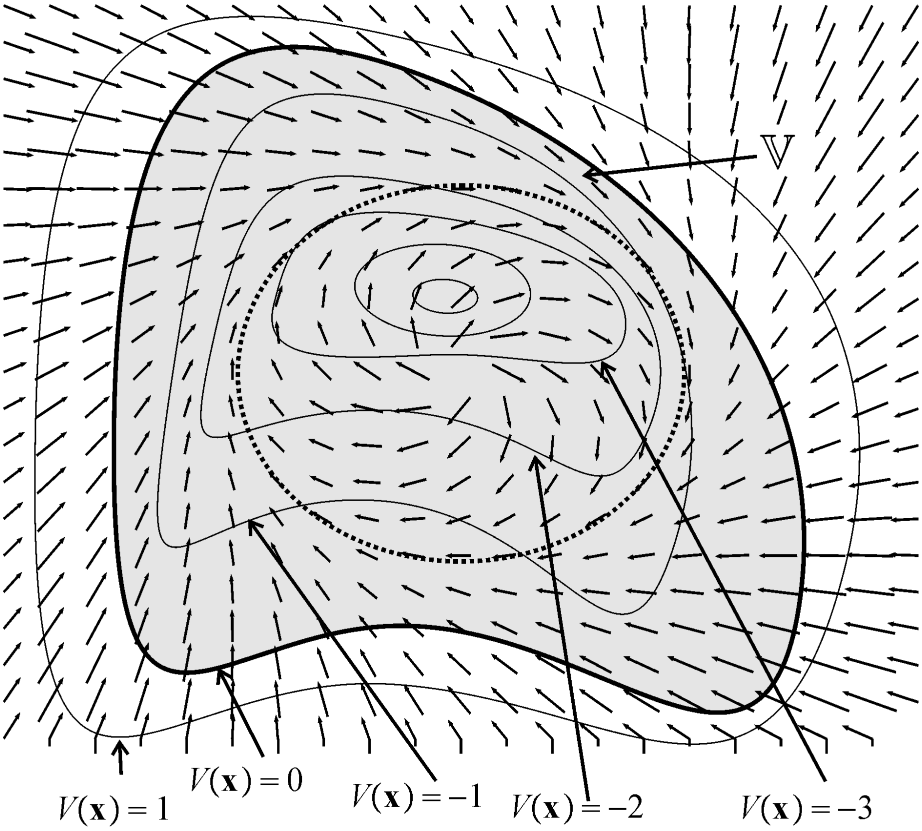
is Lyapunov-stable (1892) is there exists $V\left(\mathbf{x}\right) \geq 0$ such that

$$\begin{aligned} \dot{V}\left(\mathbf{x}\right) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}. \\ V\left(\mathbf{x}\right) &= 0 \text{ iff } \mathbf{x} = \mathbf{0} \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$. The system is V-stable* if

$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\right).$$

*Jaulin, Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE TRO.



𝕍
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$
\begin{aligned}
& \left(V\left(\mathbf{x}\right) \geq 0 \;\Rightarrow\; \dot{V}\left(\mathbf{x}\right) < 0 \right) \\
\Leftrightarrow \;& \left(V\left(\mathbf{x}\right) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) < 0 \right) \\
\Leftrightarrow \;& \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow \;& \neg \left(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) \geq 0 \text{ and } V(\mathbf{x}) \geq 0 \right)
\end{aligned}
$$

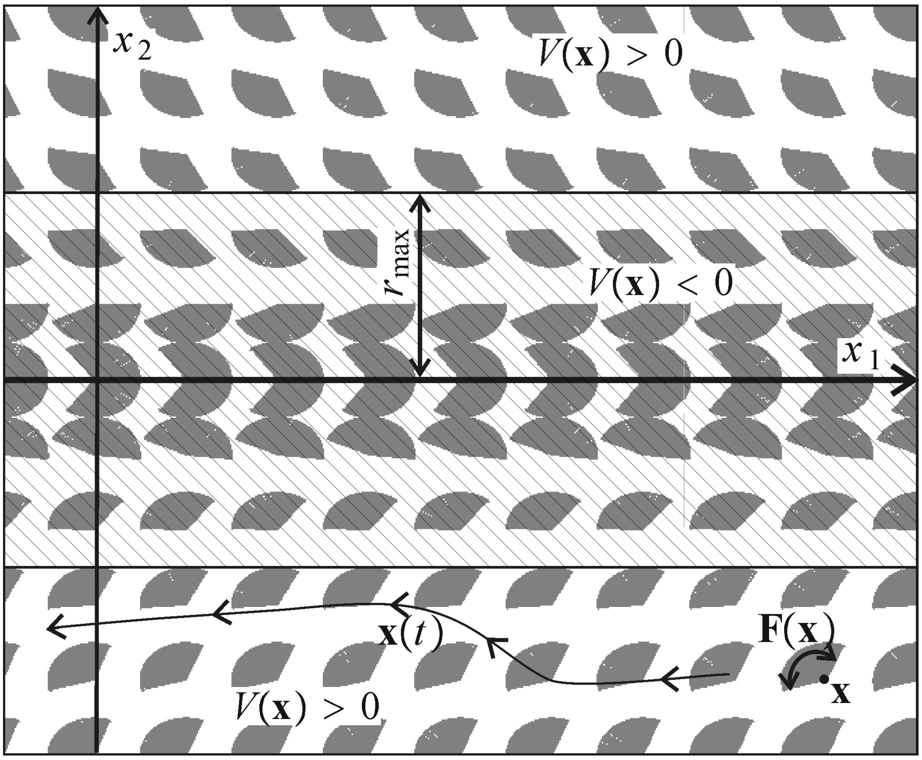
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V-stability.

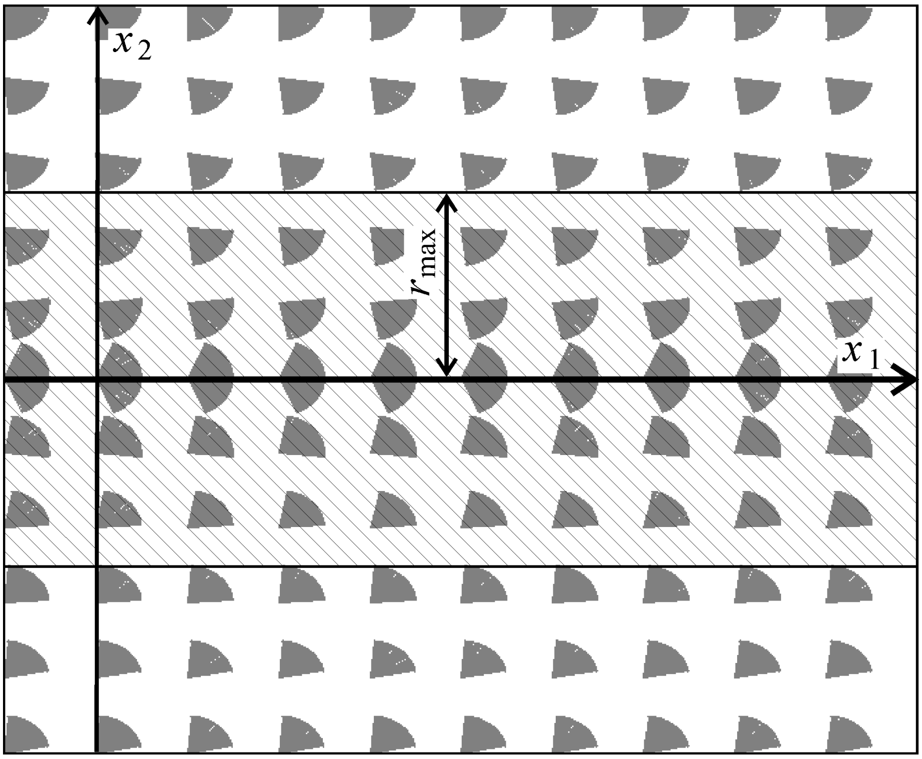
Theorem. We have

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a} \geq 0 \\ \quad \mathbf{a} \in \mathbf{F}(\mathbf{x}) \\ \quad V(\mathbf{x}) \geq 0 \end{array} \right. \quad \text{inconsistent} \;\; \Leftrightarrow \;\; \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}) \;\text{ is } V\text{-stable}$$



Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\max}^2.$$

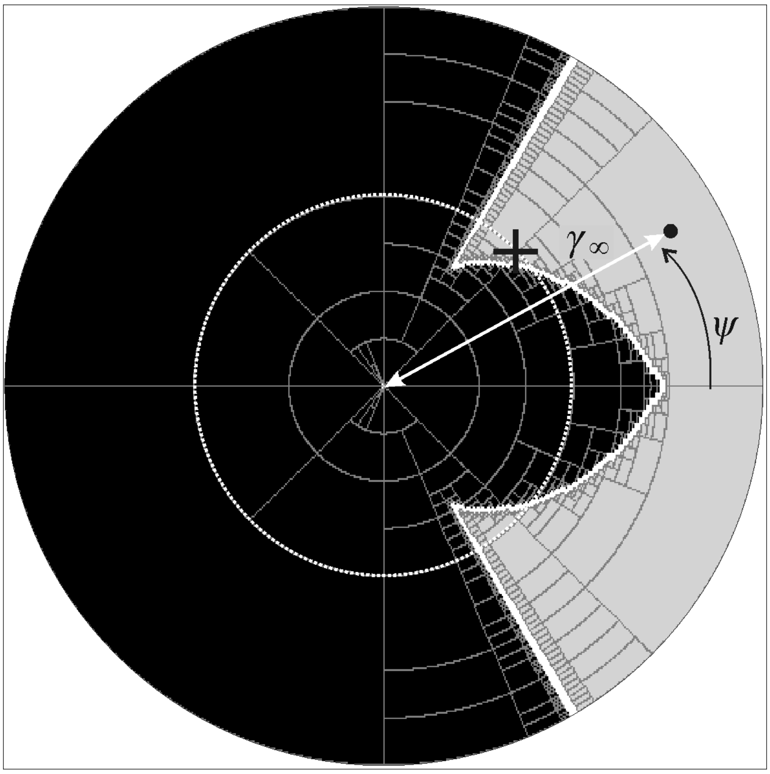

x 2
r max
x 1

5.7 Parametric case

Consider the differential inclusion

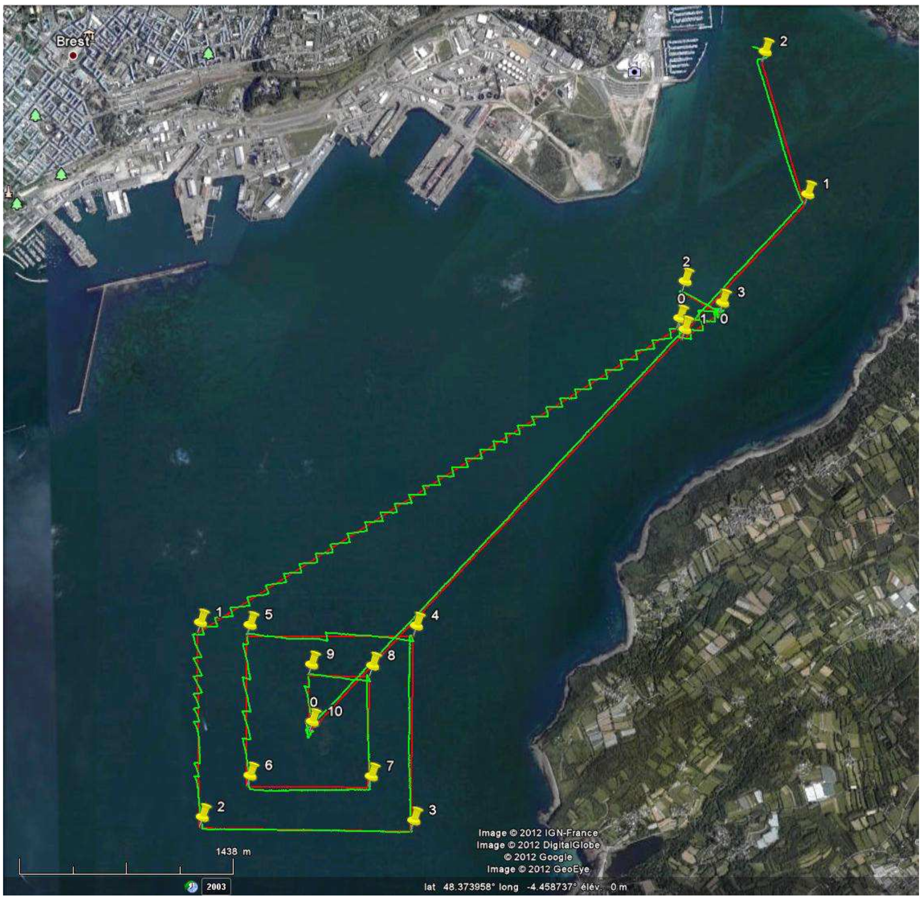
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, \mathbf{p}).$$

We characterize the set $\mathbb{P}$ of all $\mathbf{p}$ such that the system is V-stable.


+
γ∞
ψ

5.8 Experimental validation

Brest


Brest
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
1438 m
2003
lat 48.373958° long -4.458737° élév. 0 m

Brest-Douarnenez. January 17, 2012, 8am


Brest
PRO
BR D98 505
YAMAHA

miniji
F78

Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Plougastel-Daoulas
Le Conquet
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google
Start
vaimos.txt -...
Logs
sftp://root...
Command Pr...
Command Pr...
data
Interface po...
VCtrl (Runni...
VAIMOS aut...
Google Earth
FR
98%
01:42

Middle of Atlantic ocean

350 km made by Vaimos in 53h, September 6-9, 2012.

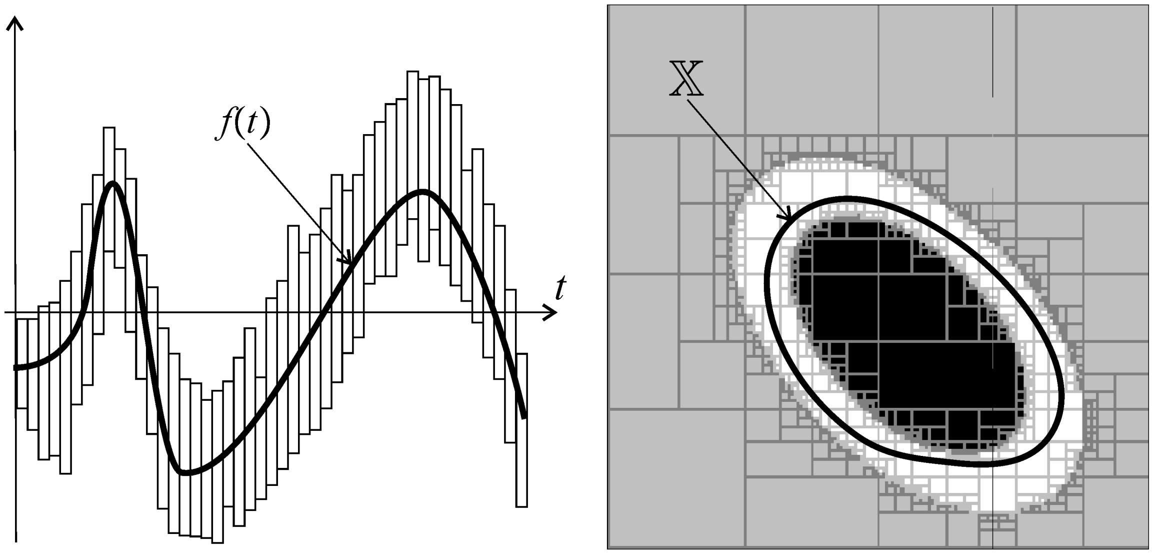
Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

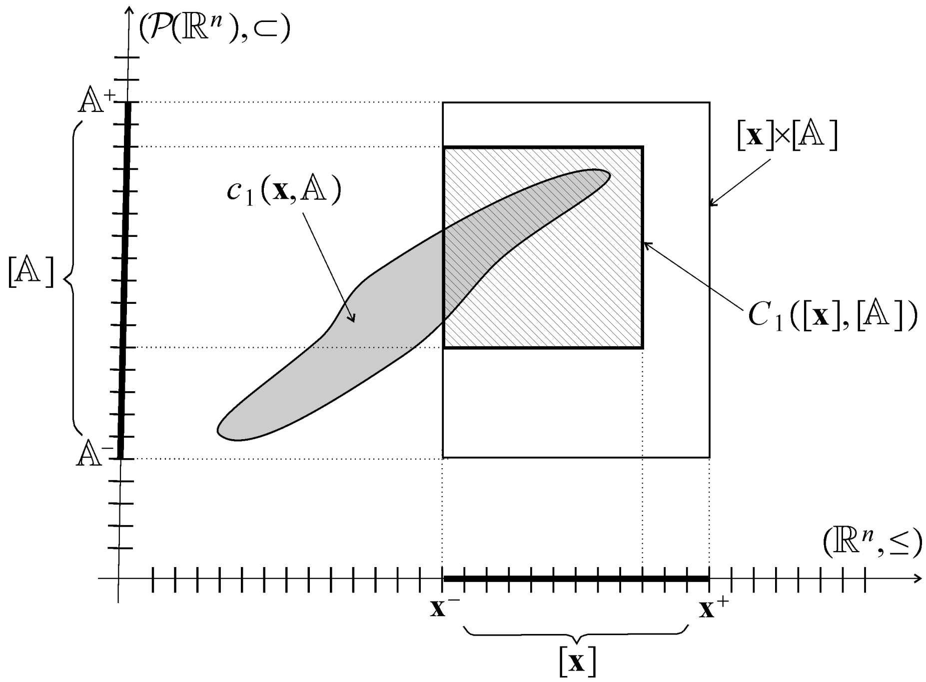
Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.

6 Perspectives



An interval function (or tube) and a set interval



$(\mathcal{P}(\mathbb{R}^n),\subset)$
$\mathbb{A}^+$
$[\mathbf{x}]\times[\mathbb{A}]$
$c_1(\mathbf{x},\mathbb{A})$
$[\mathbb{A}]$
$C_1([\mathbf{x}],[\mathbb{A}])$
$\mathbb{A}^-$
$(\mathbb{R}^n,\leq)$
$\mathbf{x}^-$
$\mathbf{x}^+$
$[\mathbf{x}]$