Reachability analysis of an underwater robot with ballast

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1. Reachability

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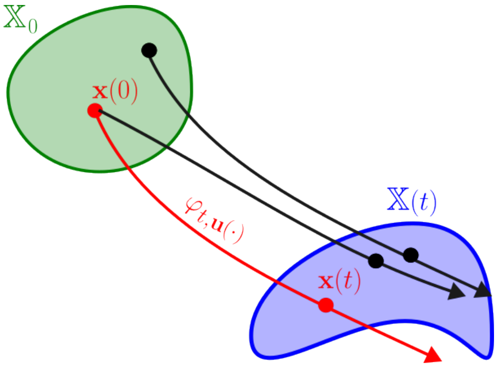
We have

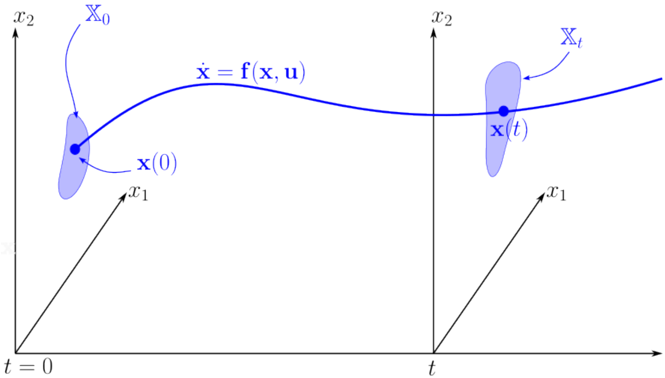
- a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- an uncertain input $\mathbf{u}(t) \in [\mathbf{u}]$
- an initial state vector $\mathbf{x}(0) \in \mathbb{X}_0$

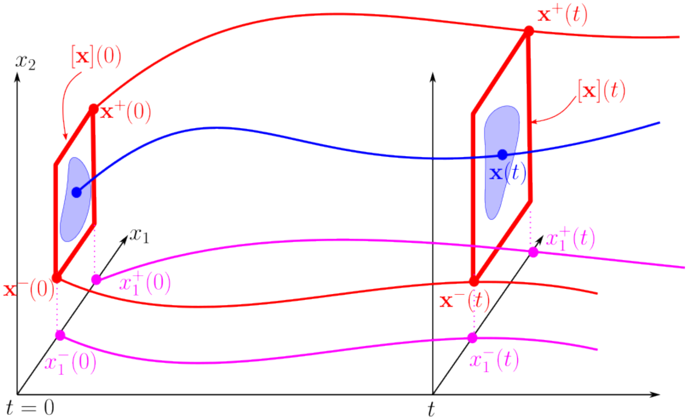
The *reach set is*

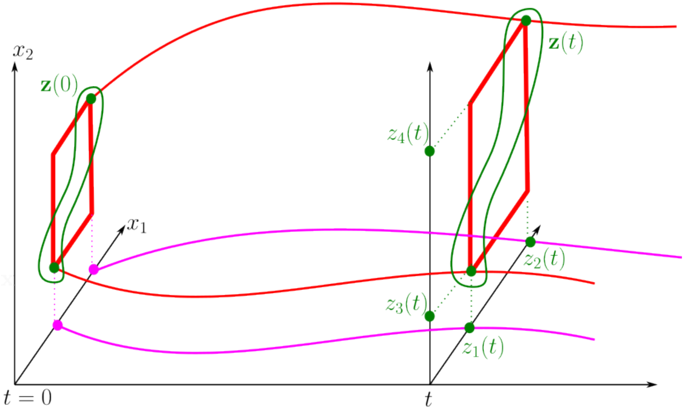
$$\mathbb{X}(t) = \left\{ \mathbf{a} | \, \exists \mathbf{x}(0) \in \mathbb{X}_0, \exists \mathbf{u}(\cdot) \in [\mathbf{u}], \mathbf{a} = \varphi_{t,\mathbf{u}(\cdot)}(\mathbf{x}(0)) \right\}$$

where $\varphi_{t,\mathbf{u}(\cdot)}$ is the flow.


$\mathbb{X}_0$
$\mathbf{x}(0)$
$\varphi_{t,\mathbf{u}(\cdot)}$
$\mathbb{X}(t)$
$\mathbf{x}(t)$


x_2
$\mathbb{X}_0$
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
x_2
$\mathbb{X}_t$
$\mathbf{x}(t)$
$\mathbf{x}(0)$
x_1
x_1
$t = 0$
t


$\mathbf{x}^+(t)$
x_2
$[\mathbf{x}](0)$
$\mathbf{x}^+(0)$
$[\mathbf{x}](t)$
$\mathbf{x}(t)$
x_1
$x_1^+(t)$
$\mathbf{x}^-(0)$
$x_1^+(0)$
$\mathbf{x}^-(t)$
$x_1^-(0)$
$x_1^-(t)$
$t = 0$
t


x_2
$\mathbf{z}(0)$
x_1
$\mathbf{z}(t)$
$z_4(t)$
$z_2(t)$
$z_3(t)$
$z_1(t)$
$t=0$
t

Define

$$\mathbf{z} = (x_1^-, x_1^+, \ldots, x_n^-, x_n^+).$$

Our contribution is to define an ODE

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z})$$

such that

$$\mathbb{X}_t \subset [\mathbf{x}](t) = \underbrace{[x_1^-(t), x_1^+(t)] \times \cdots \times [x_n^-(t), x_n^+(t)]}_{\simeq \mathbf{z}(t)}.$$

2. Interval analysis

Surface of a sphere s of radious $r \in [2,3]$:

$$s = 4\pi r^2$$

Thus

$$s \in 4 * [3.1, 3.2] * [2,3]^2$$

```
from codac import *
PI=Interval([3.1,3.2])
R=Interval([2,3])
S=4*PI*sqr(R)


-->  S=[49.6, 115.2]
```

Proposition. Consider the system

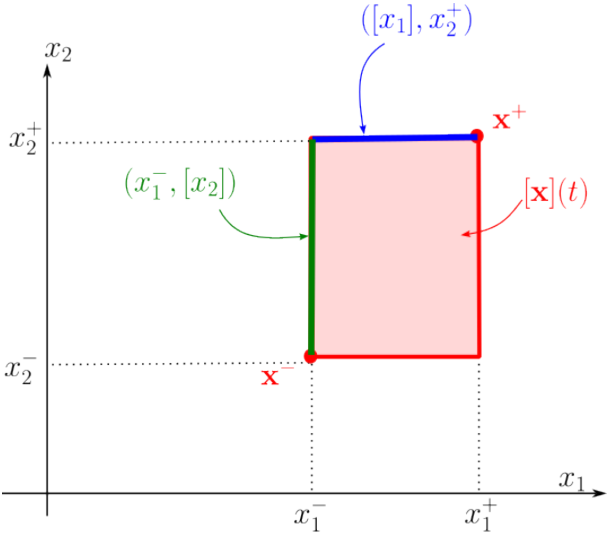
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

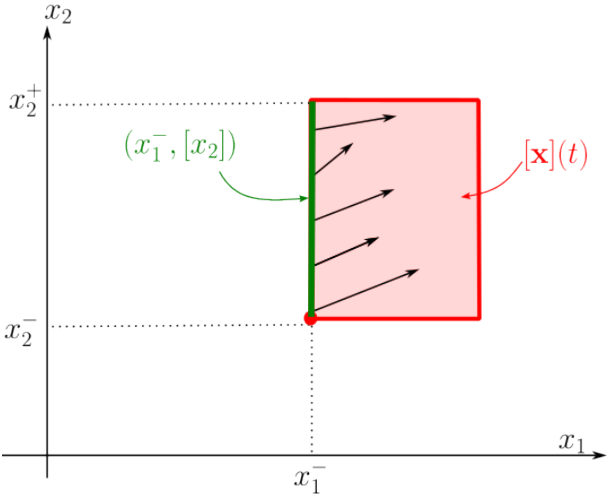
or equivalently

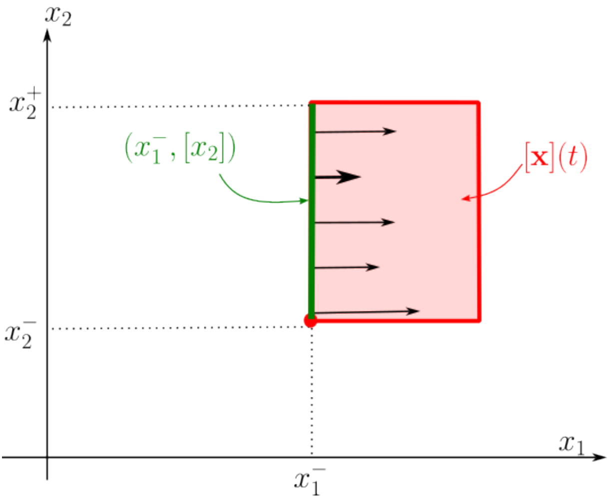
$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2, \dots, x_n, \mathbf{u}) \\ \vdots & & \vdots \\ \dot{x}_n & = & f_n(x_1, x_2, \dots, x_n, \mathbf{u}) \end{array}$$

An interval enclosure is

$$\underbrace{\begin{pmatrix} \dot{x}_1^- \\ \dot{x}_1^+ \\ \dot{x}_2^- \\ \dot{x}_2^+ \\ \vdots \\ \dot{x}_n^- \\ \dot{x}_n^+ \end{pmatrix}}_{=\dot{\mathbf{z}}} = \underbrace{\begin{pmatrix} \text{lb}(& [f_1](x_1^-, [x_2], [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ \text{ub}(& [f_1](x_1^+, [x_2], [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ \text{lb}(& [f_2]([x_1], x_2^-, [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ \text{ub}(& [f_2]([x_1], x_2^+, [x_3], \dots, [x_n], [\mathbf{u}]) &) \\ & \vdots & \\ \text{lb}(& [f_n]([x_1], [x_2], [x_3], \dots, x_n^-, [\mathbf{u}]) &) \\ \text{ub}(& [f_n]([x_1], [x_2], [x_3], \dots, x_n^+, [\mathbf{u}]) &) \end{pmatrix}}_{=\mathbf{g}(\mathbf{z},[\mathbf{u}])}$$

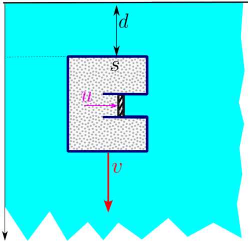

$([x_1], x_2^+)$
x_2
x_2^+
$\mathbf{x}^+$
$(x_1^-, [x_2])$
$[\mathbf{x}](t)$
x_2^-
$\mathbf{x}^-$
x_1
x_1^-
x_1^+


x_2
x_2^+
x_2^-
$(x_1^-, [x_2])$
$[\mathbf{x}](t)$
x_1
x_1^-



The left face should not go faster than the bold arrow

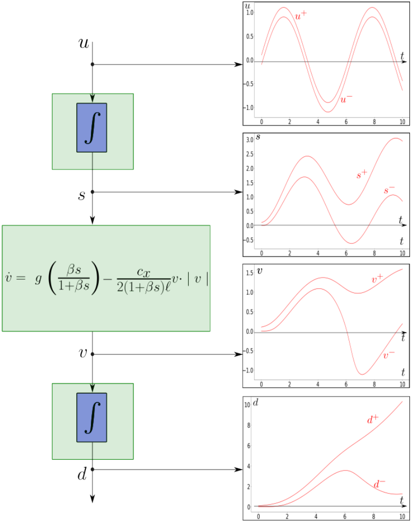
3. Robot with ballast



$$\begin{cases} \dot{s} = u \\ \dot{v} = \frac{s}{1+s} - \frac{1}{1+s} v \cdot |v| \\ \dot{d} = v \end{cases}$$

$$
\begin{aligned}
\dot{s}^- &= u^- \\
\dot{s}^+ &= u^+ \\
\dot{v}^- &= \mathsf{lb}\left(\frac{[s]}{1+[s]} - \frac{1}{1+[s]}\cdot v^- \cdot \mid v^- \mid\right) \\
\dot{v}^+ &= \mathsf{ub}\left(\frac{[s]}{1+[s]} - \frac{1}{1+[s]}\cdot v^+ \cdot v^+ \mid\right) \\
\dot{d}^- &= v^- \\
\dot{d}^+ &= v^+
\end{aligned}
$$

$$\underbrace{\begin{pmatrix} \dot{s}^- \\ \dot{s}^+ \\ \dot{v}^- \\ \dot{v}^+ \\ \dot{d}^- \\ \dot{d}^+ \end{pmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{pmatrix} u^- \\ u^+ \\ \frac{s^-}{1+s^-} - \max\begin{pmatrix} \frac{v^-\cdot|v^-|}{1+s^-} \\ \frac{v^-\cdot|v^-|}{1+s^+} \end{pmatrix} \\ \frac{s^+}{1+s^+} - \min\begin{pmatrix} \frac{v^+\cdot|v^+|}{1+s^-} \\ \frac{v^+\cdot|v^+|}{1+s^+} \end{pmatrix} \\ v^- \\ v^+ \end{pmatrix}}_{\mathbf{g}(\mathbf{z})}$$


u
s
v
d
$\dot{v} = g\left(\frac{\beta s}{1+\beta s}\right) - \frac{cx}{2(1+\beta s)\ell} v \cdot | v |$
u^+
u^-
s^+
s^-
v^+
v^-
d^+
d^-
t

References

1. Monotone systems [8]
2. Interval analysis [4][3]
3. Tubes: interval tube arithmetic [2][7][1][6]
4. Ballast robot [5]

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