Localization for Group of Robots using Matrix Contractors

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Classical contractors
A (classical) contractor associated with a set $X \subset \mathbb{R}^n$ is an operator

$$C : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$$

such that for all boxes $[x] \in \mathbb{IR}^n$:

- **Contraction:** $C([x]) \subset [x]$,
- **Completeness:** $C([x]) \cap X = [x] \cap X$
A matrix contractor of a constraint $\Gamma(A,B,\ldots)$ is an operator

$$C : \mathbb{IR}^{m_a \times n_a} \times \mathbb{IR}^{m_b \times n_b} \times \ldots \rightarrow \mathbb{IR}^{m_a \times n_a} \times \mathbb{IR}^{m_b \times n_b} \times \ldots$$

such that $([A_1],[B_1],\ldots) = C([A],[B],\ldots)$ satisfies:

- **Contraction:** $[A_1] \subset [A]$, $[B_1] \subset [B]$, $\ldots$

- **Completeness:** $\Gamma(A,B,\ldots), A \in [A], B \in [B], \ldots$ implies that $A \in [A_1], B \in [B_1], \ldots$
Example 1. The minimal contractor for the unary constraint "\( S = S^T \)" is

\[
\mathcal{C}([S]) = [S] \cap [S]^T.
\]

where \([S] \in \mathbb{IR}^{n \times n}\). For instance

\[
\mathcal{C}
\begin{pmatrix}
[1,3] & [2,4] \\
[-1,3] & [-1,1]
\end{pmatrix}
= \left( \begin{pmatrix}
[1,3] & [2,4] \\
[-1,3] & [-1,1]
\end{pmatrix} \cap \begin{pmatrix}
[1,3] & [-1,3] \\
[2,4] & [-1,1]
\end{pmatrix}
\right)
= \begin{pmatrix}
[1,3] & [2,3] \\
[2,3] & [-1,1]
\end{pmatrix}.
\]
Example 2. The minimal contractor for the ternary constraint "A + B = C" is

\[
\mathcal{C}_{\text{plus}} \begin{pmatrix} [A] \\ [B] \\ [C] \end{pmatrix} = \begin{pmatrix} [C] - [B] \\ [C] - [A] \\ [A] + [B] \end{pmatrix}.
\]
Example 3. For “$A \cdot B = C$” or for “$R^T = R^{-1}$”, no minimal contractor exists in the literature, to our knowledge.
Example 4. The minimal contractor $C$ associated to “$P \in \mathbb{R}^{m \times m}$ is a positive semi-definite matrix” has been proposed in [3]. For instance:

$$C = \begin{pmatrix}
[-7,3] & [-1,4] & [-5,4] \\
[-1,4] & [-8,3] & [2,9] \\
[0,3] & [-1,2] & [-4,4] \\
[-1,2] & [0.4,3] & [2,5.2] \\
\end{pmatrix}.$$
Interval angles?
The set of angles $\mathbb{A}$ is not a lattice. Thus, we cannot define intervals of angles.
Bisectable abstract domains
Generalize interval algorithms with bisections on a Riemannian manifold $\mathbb{M}$ such as $\mathbb{R}$, $\mathbb{R}^n$, a sphere, the Klein bottle, etc.
Is such a paving always possible? How to define 'intervals' of $\mathbf{M}$?
From the distance $d(a, b)$ between $a$ and $b$, we define the *diameter* $w(X), X \subset M$.  

A *bad* family $\mathbb{IM}$ is a family of subsets of $M$ which satisfies [2]:
1) $\mathbb{IM}$ is a Moore family (containing $\mathbb{M}$), i.e.,

$$\forall i, [a](i) \in \mathbb{IM} \Rightarrow \bigcap_{i} [a](i) \in \mathbb{IM}$$

2) $\mathbb{IM}$ is equipped with a *bisector*, i.e., a function $\beta : \mathbb{IM} \rightarrow \mathbb{IM} \times \mathbb{IM}$ such that $\beta ([x]) = \{[a],[b]\}$:

(i) $[a]$ and $[b]$ do not overlap,

(ii) $[a]$ and $[b]$ cover $[x]$.

(iii) $\beta$ minimizes $\max\{w([a]), w([b])\}$. 
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Embedding
The equation

$$\cos(\alpha) = 1$$

where $\alpha$ is an angle has a unique solution: $\alpha = 2k\pi, \ k \in \mathbb{Z}$. Angles form a Riemannian manifold.
**Embedding.** To avoid the ambiguity, we perform an embedding:

\[
\alpha \mapsto \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}
\]

Or equivalently, using complex numbers:

\[
\alpha \mapsto \cos \alpha + i \cdot \sin \alpha
\]

Now, we introduce a pessimism (*Embedding effect*)
The 'interval angle' \( \left[ \frac{\pi}{2}, \pi \right] \) is represented by the box \([−1,0] + i \cdot [0,1]\).

The sum can be done using the relation:

\[
\begin{align*}
\cos(a + b) &= \cos a \cdot \cos b - \sin a \cdot \sin b \\
\sin(a + b) &= \cos a \cdot \sin b + \sin a \cdot \cos b
\end{align*}
\]

Other operations on angles can also be defined. A classical interval resolution can thus be done.
Example. Solve

\[ a + b = 0, \ a \in \left[ \frac{\pi}{2}, \pi \right], \ b \in \left[ \frac{\pi}{2}, \pi \right] \]

We take \([a] = [b] = \left[ \frac{\pi}{2}, \pi \right].\]
The forward contraction is

\[
\cos(a + b) \in \cos(a) \cdot \cos(b) - \sin(a) \cdot \sin(b)
\]
\[
= \cos([\pi/2, \pi]) \cdot \cos([\pi/2, \pi]) - \sin([\pi/2, \pi]) \cdot \sin([\pi/2, \pi])
\]
\[
= [-1, 0] \cdot [-1, 0] - [0, 1] \cdot [0, 1]
\]

\[
\sin(a + b) \in \cos(a) \cdot \sin(b) + \sin(a) \cdot \cos(b)
\]
\[
= [0, 1] - [0, 1] = [-1, 1]
\]
\[
= [-1, 0] \cdot [0, 1] + [0, 1] \cdot [-1, 0]
\]
\[
= [-1, 0] + [-1, 0] = [-2, 0]
\]

Since 0 \mapsto (1, 0), the backward step yields

\[
\left(\cos[a], \sin[a]\right) = \left(\cos[b], \sin[b]\right) = (-1, 0).
\]
Consider the equivalence relation

\[ \alpha \sim \beta \iff \frac{\beta - \alpha}{2\pi} \in \mathbb{Z} \iff \cos(\alpha - \beta) = 1 \]

The set \( \mathbb{A} \) of all angles is

\[ \mathbb{A} = \mathbb{R} / \sim = \mathbb{R} / 2\pi \mathbb{Z} \cdot \]

Sets \( \mathbb{A}, \mathbb{A} \times \mathbb{A}, \mathbb{A}^{m \times n} \) are Riemannian manifolds.
If $\alpha$ and $\beta$ are angles and if $\rho \in \mathbb{R}$, we can define $\alpha + \beta$, $\alpha - \beta$ and $\rho \cdot \alpha$. 
Matrix contractors
Interval angles?
Localization

Aecs
An arc $\langle \alpha \rangle$ is a connected subset of $\mathbb{A}$. We have $\langle \alpha \rangle = \langle \bar{\alpha}, \tilde{\alpha} \rangle$ with $\bar{\alpha} \in \mathbb{A}$ and $\tilde{\alpha} \in [0, \pi]$. The set of all arcs is denoted by $\mathbb{IA}$. We may define an arc arithmetic. For instance

$$\frac{1}{2} \cdot 2 \cdot \langle 0, \frac{\pi}{2} \rangle = \frac{1}{2} \cdot \langle 0, \pi \rangle = \langle 0, \pi \rangle.$$
IIA is not a Moore family.
The smallest Moore family which contains IIA is the unions of arcs.
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Note. It may be dangerous to deal with union of arcs.

Example of Chabert\cite{1}. With initial domains $[x] = [y] = [1, 9]$, 

$$\begin{cases} 
    y = x \\
    9(x - 5)^2 = 16y 
\end{cases}$$

an explosion of the number of intervals during the propagation occurs.
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Absolute. Estimate the pose of robots $(x_i, y_i, \theta_i)^T, i \in \{1, 2, \ldots \}$.

Relative. Estimate the azimuth, the distances or the bearings. No absolute frame is needed.
Azimuth angles for three robots

Azimuth is measured using a compass and goniometric sensors.
Azimuth contractor
Azimuth matrix is the matrix of azimuth angles $\alpha_{ij}$ between $i^{th}$ robot and $j^{th}$ robot:

$$A = \begin{pmatrix}
0 & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \alpha_{(n-1)n} \\
\alpha_{n1} & \cdots & \alpha_{n(n-1)} & 0
\end{pmatrix}.$$  

By convention, $\alpha_{ii} = 0$.

Note that $A \in (\mathbb{R}^{2\pi\mathbb{Z}})^{n^2}$ which is a Riemannian manifold.
This uncertainty can be represented under the form of an interval matrix.

\[
[A] = \begin{pmatrix}
0 & [\alpha_{12}] & \cdots & [\alpha_{1n}] \\
[\alpha_{21}] & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & [\alpha_{(n-1)n}]
\end{pmatrix},
\]

where \([\alpha_{ij}] \in \mathbb{IA}\). Recall that \(\mathbb{IA}\) is not a Moore family.
For $i, j, k \ i \neq j \neq k$,

\[
\text{azimuth}(A) \Rightarrow \begin{cases} 
\alpha_{ij} - \alpha_{ji} \sim \pi \\
(\alpha_{ij} - \alpha_{ik}) + (\alpha_{jk} - \alpha_{ji}) + (\alpha_{ki} - \alpha_{kj}) \sim \pi \\
(\alpha_{ij} - \alpha_{ki}) \sim (\alpha_{ji} - \alpha_{jk}) + (\alpha_{kj} - \alpha_{ki})
\end{cases}
\]

This can used to build an azimuth contractor $C_{az}([A])$. 
Distance contractor
The distance matrix associated with $n$ points is

$$D = \begin{pmatrix}
0 & d_{12} & \cdots & d_{1n} \\
d_{21} & 0 & \ddots & \vdots \\
\vdots & \vdots & \ddots & d_{(n-1)n} \\
d_{n1} & \cdots & d_{n(n-1)} & 0
\end{pmatrix}$$

To build distance contractor consistent with the constraint "$D$ is a distance matrix", we consider constraints such as:

$$distance(D) \Rightarrow \begin{cases}
\quad d_{ij} = d_{ji} \\
\quad d_{ij} \leq d_{ik} + d_{kj}
\end{cases}$$
Distance-Azimuth contractor
The localization in terms of azimuth angles are done with respect to composition of azimuth $\mathcal{C}_{az}([A])$ and distance $\mathcal{C}_d([D])$ contractors. We can also use mixed constraints such as

$$\text{distaz}(D, A) \Rightarrow \{ \sin(\alpha_{ik} - \alpha_{ij}) \cdot d_{ij} = \sin(\alpha_{ki} - \alpha_{kj}) \cdot d_{kj} \}$$
With five robots
A localization problem with five robots with respect to azimuth and distance is considered.
Take:

\[
[A] = \begin{bmatrix}
0 & [2.2, 1] & -[2.8, 3] & [2.5, 3.3] & -[2.1, 2.2] \\
[5.2, 5.3] & 0 & -[1.9, 2] & [3.1, 3.2] & -[1.6, 1.7] \\
[0.1, 0.2] & [1.2, 1.3] & 0 & -[2.6, 2.7] & -[1.3, 1.5] \\
[5.5, 6.5] & [6.2, 6.3] & -[0.5, 0.6] & 0 & -[0.7, 0.8] \\
[0.9, 0.1] & [1.4, 1.5] & [1.7, 1.8] & [2.2, 2.4] & 0
\end{bmatrix}
\]

\[
[D] = \begin{bmatrix}
\end{bmatrix}
\]
The contracted azimuth matrix is

\[
\begin{pmatrix}
0 & [2.0, 2.1] & -[2.9, 3] & [2.5, 3.3] & -[2.1, 2.2] \\
[5.2, 5.24] & 0 & -[1.9, 2.0] & [3.1, 3.2] & -[1.6, 1.7] \\
[0.1, 0.2] & [1.2, 1.3] & 0 & -[2.6, 2.7] & -[1.3, 1.4] \\
[5.6, 6.4] & [6.2, 6.3] & [0.5, 0.6] & 0 & -[0.7, 0.8] \\
[0.9, 1.0] & [1.4, 1.5] & [1.7, 1.8] & [2.3, 2.4] & 0
\end{pmatrix}.
\]
Pose of five robots with azimuth measurements
The contractions with respect to azimuth and distance are:

\[
[A] = \begin{pmatrix}
0 & [2.0, 2.1] & -[2.94, 3] & [2.6, 2.9] & -[2.1, 2.2] \\
[5.2, 5.24] & 0 & -[1.9, 1.94] & [3.1, 3.2] & -[1.6, 1.7] \\
[0.14, 0.2] & [1.2, 1.3] & 0 & [2.6, 2.7] & -[1.3, 1.4] \\
[5.90, 6.0] & [6.2, 6.3] & [0.5, 0.54] & 0 & -[0.7, 0.8] \\
[0.94, 1.0] & [1.4, 1.5] & [1.7, 1.8] & [2.3, 2.4] & 0 \\
\end{pmatrix}
\]

\[
[D] = \begin{pmatrix}
\end{pmatrix}
\]
Pose of five robots before contraction
One contraction with respect to azimuth
After two contractions with respect to azimuth
Many contractions with respect to azimuth
Many contractions with respect to the distance
Many contractions with respect to both distance and azimuth
With 12 robots
A localization problem with 12 robots with respect to azimuth and distance is considered.
With 12 robots, before contraction
Matrix contractors
Interval angles?
Localization

After contractions
Gilles Chabert.

*Techniques d’intervalles pour la résolution de systèmes d’équations.*


L. Jaulin, B. Desrochers, and D. Massé.

Bisectable Abstract Domains for the resolution of equations involving complex numbers.


L. Jaulin and D. Henrion.

Contracting optimally an interval matrix without losing any positive semi-definite matrix is a tractable problem.

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