Localization for Group of Robots using Matrix Contractors

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Classical contractors

A (classical) contractor associated with a set $\mathbb{X} \subset \mathbb{R}^n$ is an operator

$$\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$$

such that for all boxes $[\mathbf{x}] \in \mathbb{IR}^n$:

- Contraction: $\mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}]$,
- Completeness: $\mathscr{C}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X}$

Matrix contractors

A matrix contractor of a constraint $\Gamma(\mathbf{A},\mathbf{B},\dots)$ is an operator

$$\mathscr{C}:\mathbb{IR}^{m_a\times n_a}\times\mathbb{IR}^{m_b\times n_b}\times\dots\to\mathbb{IR}^{m_a\times n_a}\times\mathbb{IR}^{m_b\times n_b}\times\dots$$

such that $([\mathbf{A}_1],[\mathbf{B}_1],\dots)=\mathscr{C}([\mathbf{A}],[\mathbf{B}],\dots)$ satisfies :

- Contraction: $[\mathbf{A}_1]\subset[\mathbf{A}]$, $[\mathbf{B}_1]\subset[\mathbf{B}]$,...
- Completeness: $\Gamma(\mathbf{A},\mathbf{B},\dots),\mathbf{A}\in[\mathbf{A}],\ \mathbf{B}\in[\mathbf{B}]$,... implies that $\mathbf{A}\in[\mathbf{A}_1],\ \mathbf{B}\in[\mathbf{B}_1],\dots$

Example 1. The minimal contractor for the unary constraint "$\mathbf{S} = \mathbf{S}^T$" is

$$\mathscr{C}([\mathbf{S}]) = [\mathbf{S}] \cap [\mathbf{S}]^{\mathrm{T}}.$$

where $[\mathbf{S}] \in \mathbb{IR}^{n \times n}$. For instance

$$\begin{aligned} \mathscr{C}\begin{pmatrix} [1,3] & [2,4] \\ [-1,3] & [-1,1] \end{pmatrix} &= \begin{pmatrix} [1,3] & [2,4] \\ [-1,3] & [-1,1] \end{pmatrix} \cap \begin{pmatrix} [1,3] & [-1,3] \\ [2,4] & [-1,1] \end{pmatrix} \\ &= \begin{pmatrix} [1,3] & [2,3] \\ [2,3] & [-1,1] \end{pmatrix}. \end{aligned}$$

Example 2. The minimal contractor for the ternary constraint "$\mathbf{A}+\mathbf{B}=\mathbf{C}$" is

$$\mathscr{C}_{plus}\begin{pmatrix}[\mathbf{A}]\\ [\mathbf{B}]\\ [\mathbf{C}]\end{pmatrix}=\begin{pmatrix}[\mathbf{C}]-[\mathbf{B}]\\ [\mathbf{C}]-[\mathbf{A}]\\ [\mathbf{A}]+[\mathbf{B}]\end{pmatrix}.$$

Example 3. For "$\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$" or for "$\mathbf{R}^T = \mathbf{R}^{-1}$", no minimal contractor exists in the literature, to our knowledge.

Example 4. The minimal contractor $\mathscr{C}$ associated to "$\mathbf{P} \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix" has been proposed in [3]. For instance:

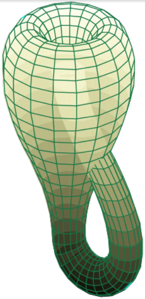
$$\mathscr{C}\left(\begin{pmatrix} [-7,3] & [-1,4] & [-5,4] \\ [-1,4] & [-8,3] & [2,9] \\ [-5,4] & [2,9] & [4,9] \end{pmatrix}\right)$$
$$= \begin{pmatrix} [0,3] & [-1,2] & [-4,4] \\ [-1,2] & [0.4,3] & [2,5.2] \\ [-4,4] & [2,5.2] & [4,9] \end{pmatrix}$$

Interval angles?

The set of angles $\mathbb{A}$ is not a lattice. Thus, we cannot define intervals of angles.

Bisectable abstract domains

Generalize interval algorithms with bisections on a Riemannian manifold $\mathbb{M}$ such a $\mathbb{R}$, $\mathbb{R}^n$, a sphere, the Klein bottle, etc.



Is such a paving always possible? How to define 'intervals' of $\mathbb{M}$?

From the distance $d(a,b)$ between a and b, we define the *diameter* $w(\mathbb{X})$, $\mathbb{X} \subset \mathbb{M}$.

A *bad* family $\mathbb{IM}$ is a family of subsets of $\mathbb{M}$ which satisfies [2]:

1) $\mathbb{IM}$ is a Moore family (containing $\mathbb{M}$), *i.e.*,

$$\forall i, [a](i) \in \mathbb{IM} \Rightarrow \bigcap_i [a](i) \in \mathbb{IM}$$

2) $\mathbb{IM}$ is equipped with a *bisector*, *i.e.*, a function $\beta : \mathbb{IM} \to \mathbb{IM} \times \mathbb{IM}$ such that $\beta([x]) = \{[a], [b]\}$:
(i) $[a]$ and $[b]$ do not overlap,
(ii) $[a]$ and $[b]$ cover $[x]$
(iii) β minimizes $\max\{w([a]), w([b])\}$.

E. H. Moore

Eliakim Hastings Moore

Born	January 26, 1862 Marietta, Ohio, U.S.
Died	December 30, 1932 (aged 70) Chicago, Illinois, U.S.
Nationality	American
Fields	Mathematics
Institutions	University of Chicago 1892–31 Yale University 1887–89

Embedding

The equation

$$cos\left(\alpha\right) = 1$$

where α is an angle has a unique solution: $\alpha = 2k\pi,\ k \in \mathbb{Z}$.
Angles form a Riemannian manifold.

Embedding. To avoid the ambiguity, we perform an embedding:

$$\alpha \mapsto \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Or equivalenltly, using complex numbers:

$$\alpha \mapsto \cos\alpha + i \cdot \sin\alpha$$

Now, we introduce a pessimism (*Embedding effect*)

The 'interval angle' $[\frac{\pi}{2}, \pi]$ is represented by the box $[-1, 0] + i \cdot [0, 1]$.
The sum can be done using the relation:

$$\begin{aligned} \cos(a+b) &= \cos a \cdot \cos b - \sin a \cdot \sin b \\ \sin(a+b) &= \cos a \cdot \sin b + \sin a \cdot \cos b \end{aligned}$$

Other operations on angles can also be defined.
A classical interval resolution can thus be done.

Example. Solve

$$a + b = 0, a \in [\frac{\pi}{2}, \pi], b \in [\frac{\pi}{2}, \pi]$$

We take $[a] = [b] = [\frac{\pi}{2}, \pi]$.

The forward contraction is

$$\begin{array}{rcl} \cos(a+b) & \in & \cos[a]\cdot\cos[b]-\sin[a]\cdot\sin[b] \\ & = & \cos\left([\frac{\pi}{2},\pi]\right)\cdot\cos\left([\frac{\pi}{2},\pi]\right)-\sin\left([\frac{\pi}{2},\pi]\right)\cdot\sin\left([\frac{\pi}{2},\pi]\right) \\ & = & [-1,0]\cdot[-1,0]-[0,1]\cdot[0,1] \end{array}$$

$$\begin{array}{rcl} \sin(a+b) & \in & \cos[a]\cdot\sin[b]+\sin[a]\cdot\cos[b] \\ & = & [0,1]-[0,1]=[-1,1] \\ & = & [-1,0]\cdot[0,1]+[0,1]\cdot[-1,0] \\ & = & [-1,0]+[-1,0]=[-2,0] \end{array}$$

Since $0 \mapsto (1,0)$, the backward step yields

$$(\cos[a],\sin[a])=(\cos[b],\sin[b])=(-1,0).$$

Consider the equivalence relation

$$\alpha \sim \beta \Leftrightarrow \frac{\beta - \alpha}{2\pi} \in \mathbb{Z} \Leftrightarrow \cos(\alpha - \beta) = 1$$

The set $\mathbb{A}$ of all angles is

$$\mathbb{A} = \frac{\mathbb{R}}{\sim} = \frac{\mathbb{R}}{2\pi\mathbb{Z}}.$$

Sets $\mathbb{A}, \mathbb{A} \times \mathbb{A}, \mathbb{A}^{m \times n}$ are Riemannian manifolds.

If α and β are angles and if $\rho \in \mathbb{R}$, we can define $\alpha + \beta$, $\alpha - \beta$ and $\rho \cdot \alpha$.

Arcs

An *arc* $\langle\alpha\rangle$ is a connected subset of $\mathbb{A}$. We have $\langle\alpha\rangle = \langle\overline{\alpha},\widetilde{\alpha}\rangle$ with $\overline{\alpha}\in\mathbb{A}$ and $\widetilde{\alpha}\in[0,\pi]$.

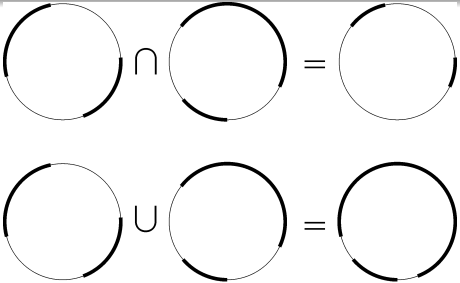
The set of all arcs is denoted by $\mathbb{IA}$.

We may define an arc arithmetic. For instance

$$\frac{1}{2}\cdot 2\cdot\left\langle 0,\frac{\pi}{2}\right\rangle = \frac{1}{2}\cdot\langle 0,\pi\rangle = \langle 0,\pi\rangle\,.$$

$\mathbb{IA}$ is not a Moore family.

The smallest Moore family which contains $\mathbb{IA}$ is the unions of arcs.



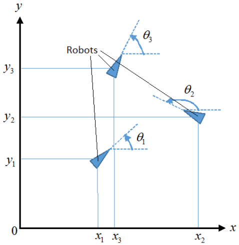
Note. It may be dangerous to deal with union of arcs.
Example of Chabert[1]. With initial domains $[x] = [y] = [1, 9]$,

$$\begin{cases} y = x \\ 9\left(x-5\right)^2 = 16y \end{cases}$$

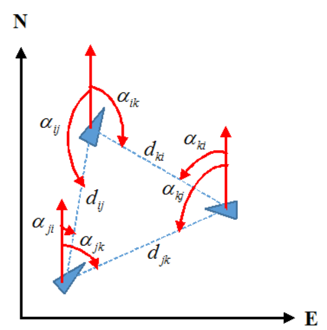
an explosion of the number of intervals during the propagation occurs.

Localization

Absolute. Estimate the pose of robots $(x_i, y_i, \theta_i)^T, i \in \{1, 2, \dots\}$.



Relative. Estimate the azimuth, the distances or the bearings. No absolute frame is needed.



Azimuth angles for three robots

Azimuth is measured using a compass and goniometric sensors.

Azimuth contractor

Azimuth matrix is the matrix of azimuth angles α_{ij} between i^{th} robot and j^{th} robot:

$$\mathbf{A} = \begin{pmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{(n-1)\,n} \\ \alpha_{n1} & \cdots & \alpha_{n\,(n-1)} & 0 \end{pmatrix}.$$

By convention, $\alpha_{ii} = 0$.
Note that $\mathbf{A} \in \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)^{n^2}$ which is a Riemannian manifold.

This uncertainty can be represented under the form of an interval matrix.

$$[\mathbf{A}] = \begin{pmatrix} 0 & [\alpha_{12}] & \cdots & [\alpha_{1n}] \\ [\alpha_{21}] & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \left[\alpha_{(n-1)\,n)}\right] \\ [\alpha_{n1}] & \cdots & \left[\alpha_{n\,(n-1)}\right] & 0 \end{pmatrix}.$$

where $[\alpha_{ij}] \in \mathbb{IA}$. Recall that $\mathbb{IA}$ is not a Moore family.

For $i, j, k \ i \neq j \neq k$,

$$azimuth(A) \Rightarrow \begin{cases} \alpha_{ij} - \alpha_{ji} \sim \pi \\ (\alpha_{ij} - \alpha_{ik}) + (\alpha_{jk} - \alpha_{ji}) + (\alpha_{ki} - \alpha_{kj}) \sim \pi \\ (\alpha_{ij} - \alpha_{ki}) \sim (\alpha_{ji} - \alpha_{jk}) + (\alpha_{kj} - \alpha_{ki}) \end{cases}$$

This can used to build an azimuth contractor $\mathscr{C}_{az}([\mathbf{A}])$.

Distance contractor

The distance matrix associated with n points is

$$\mathbf{D} = \begin{pmatrix} 0 & d_{12} & \cdots & d_{1n} \\ d_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_{(n-1)\,n} \\ d_{n1} & \cdots & d_{n\,(n-1)} & 0 \end{pmatrix}$$

To build distance contractor consistent with the constraint "**D** is a distance matrix", we consider constraints such as:

$$distance(\mathbf{D}) \Rightarrow \begin{cases} d_{ij} = d_{ji} \\ d_{ij} \leq d_{ik} + d_{kj} \end{cases}$$

Distance-Azimuth contractor

The localization in terms of azimuth angles are done with respect to composition of azimuth $\mathscr{C}_{az}([\mathbf{A}])$ and distance $\mathscr{C}_{d}([\mathbf{D}])$ contractors. We can also use mixed constraints such as

$$distaz(\mathbf{D},\mathbf{A}) \Rightarrow \{\sin(\alpha_{ik}-\alpha_{ij})\cdot d_{ij} = \sin(\alpha_{ki}-\alpha_{kj})\cdot d_{kj}$$

With five robots

A localization problem with five robots with respect to azimuth and distance is considered.

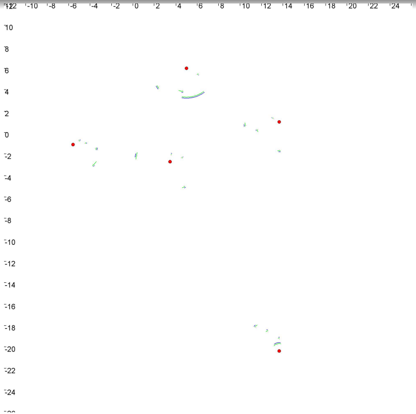
Take:

$$[\mathbf{A}] = \begin{pmatrix} 0 & [2,2.1] & -[2.8,3] & [2.5,3.3] & -[2.1,2.2] \\ [5.2,5.3] & 0 & -[1.9,2] & [3.1,3.2] & -[1.6,1.7] \\ [0.1,0.2] & [1.2,1.3] & 0 & -[2.6,2.7] & -[1.3,1.5] \\ [5.5,6.5] & [6.2,6.3] & -[0.5,0.6] & 0 & -[0.7,0.8] \\ [0.9,0.1] & [1.4,1.5] & [1.7,1.8] & [2.2,2.4] & 0 \end{pmatrix}$$

$$[\mathbf{D}] = \begin{pmatrix} 0 & [9.9,10] & [8,10] & [22,31] & [10,13] \\ [9.8,10] & 0 & [9,12] & [19,22] & [15,20] \\ [7,9] & [8,11] & 0 & [19,21] & [7,10] \\ [23,32] & [20,22] & [20,21] & 0 & [26,28] \\ [10,15] & [15,22] & [8,10] & [25,30] & 0 \end{pmatrix}$$

The contracted azimuth matrix is

$$\begin{pmatrix} 0 & [2.0, 2.1] & -[2.9, 3] & [2.5, 3.3] & -[2.1, 2.2] \\ [5.2, 5.24] & 0 & -[1.9, 2.0] & [3.1, 3.2] & -[1.6, 1.7] \\ [0.1, 0.2] & [1.2, 1.3] & 0 & -[2.6, 2.7] & -[1.3, 1.4] \\ [5.6, 6.4] & [6.2, 6.3] & [0.5, 0.6] & 0 & -[0.7, 0.8] \\ [0.9, 1.0] & [1.4, 1.5] & [1.7, 1.8] & [2.3, 2.4] & 0 \end{pmatrix}.$$

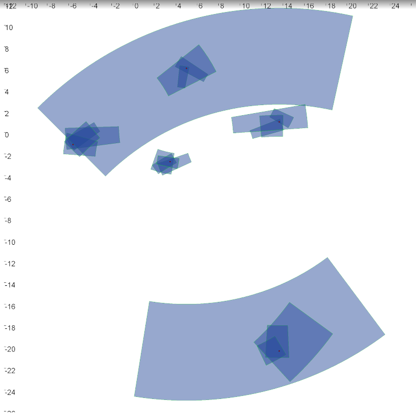


Pose of five robots with azimuth measurements

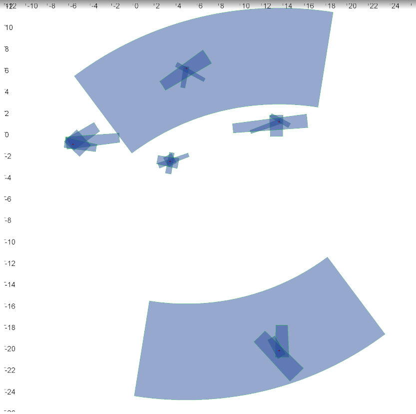
The contractions with respect to azimuth and distance are:

$$[\mathbf{A}] = \begin{pmatrix} 0 & [2.0,2.1] & -[2.94,3] & [2.6,2.9] & -[2.1,2.2] \\ [5.2,5.24] & 0 & -[1.9,1.94] & [3.1,3.2] & -[1.6,1.7] \\ [0.14,0.2] & [1.2,1.3] & 0 & [2.6,2.7] & -[1.3,1.4] \\ [5.90,6.0] & [6.2,6.3] & [0.5,0.54] & 0 & -[0.7,0.8] \\ [0.94,1.0] & [1.4,1.5] & [1.7,1.8] & [2.3,2.4] & 0 \end{pmatrix}$$

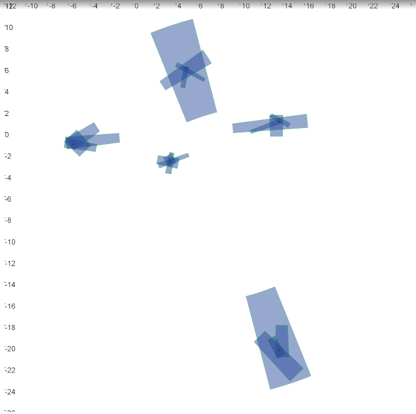
$$[\mathbf{D}] = \begin{pmatrix} 0 & [9.9,10.1] & [8.1,9] & [25,30.6] & [10.6,13] \\ [9.9,10.1] & 0 & [10.3,11] & [20.4,22] & [16.7,20] \\ [8.1,9] & [10.3,11] & 0 & [20,21] & [8,10] \\ [25.1,30.6] & [20.4,22] & [20,21] & 0 & [26,28] \\ [10.6,13] & [16.7,20] & [8,10] & [26,28] & 0 \end{pmatrix}$$



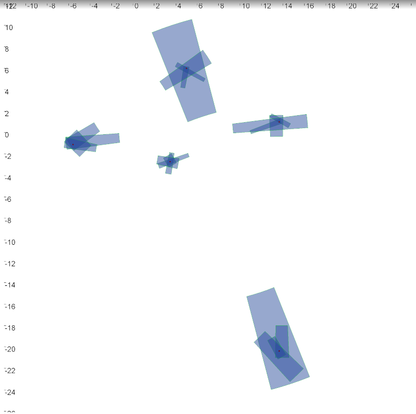
Pose of five robots before contraction



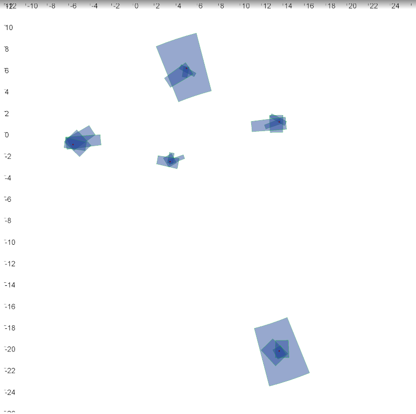
One contraction with respect to azimuth



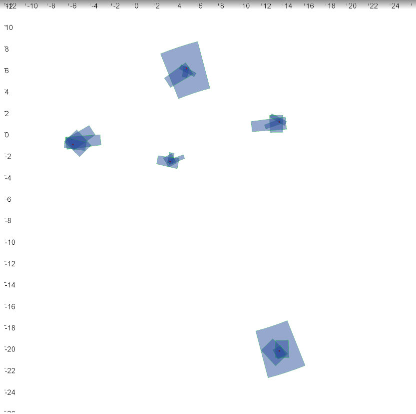
After two contractions with respect to azimuth



Many contractions with respect to azimuth



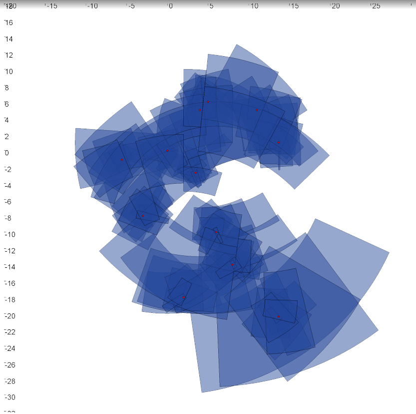
Many contractions with respect to the distance



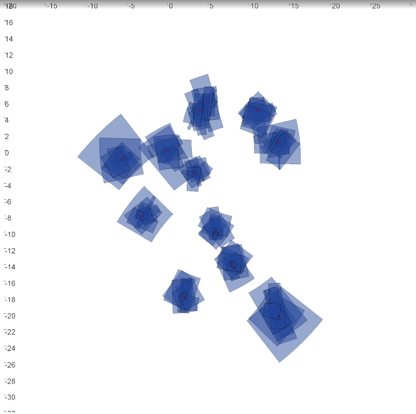
Many contractions with respect to both distance and azimuth

With 12 robots

A localization problem with 12 robots with respect to azimuth and distance is considered.



With 12 robots, before contraction



After contractions

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