Characterization of trajectories using constraint programming and abstract interpretation

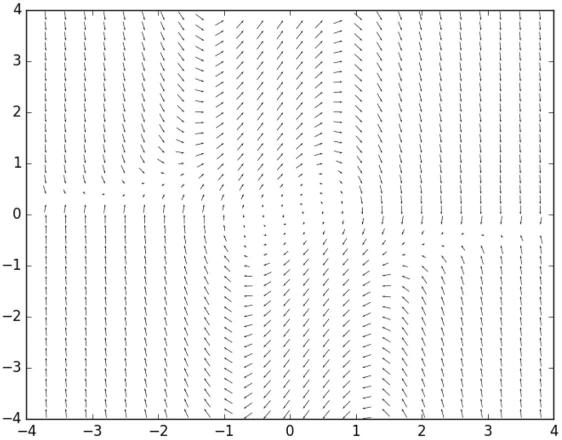
T. Le Mézo, **L. Jaulin**, B. Zerr

Capture set

We consider a state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

Example: The Van der Pol system

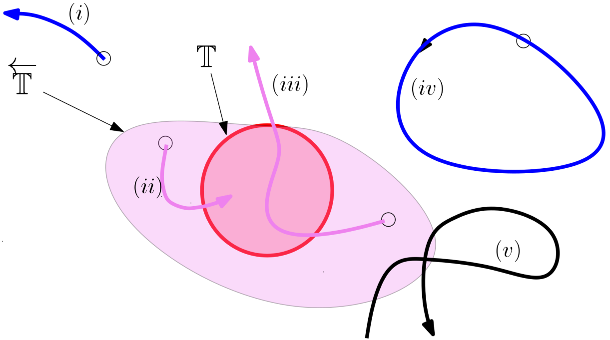
$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$


4
3
2
1
0
−1
−2
−3
−4
−4
−3
−2
−1
0
1
2
3
4

Let φ be the flow map.
The *capture* set of the *target* $\mathbb{T} \subset \mathbb{R}^n$ is:

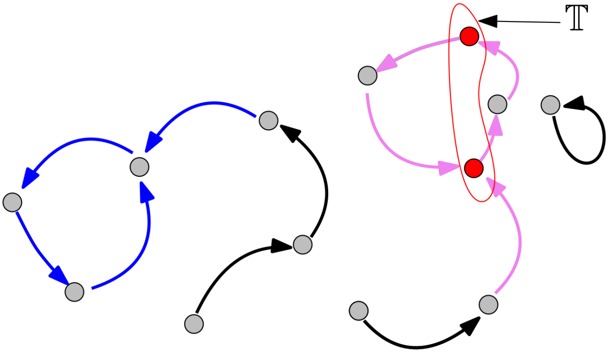
$$\overleftarrow{\mathbb{T}} = \{\mathbf{x}_0 \mid \exists t \geq 0, \varphi(t, \mathbf{x}_0) \in \mathbb{T}\}.$$

To each state, we associate a path.

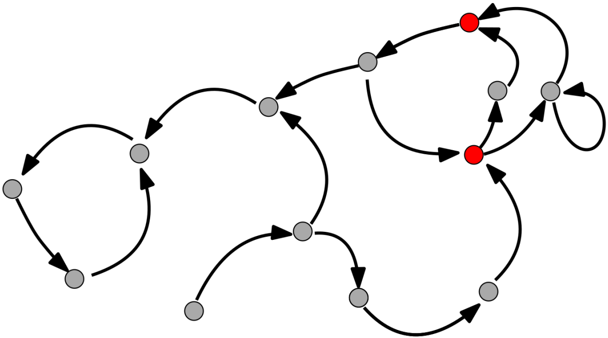


Graph analogy

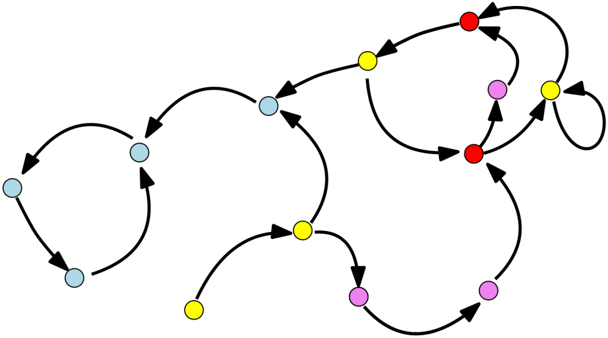
A deterministic graph $\mathscr{G}_1$ with a target $\mathbb{T}$ (red), a dead path (blue).



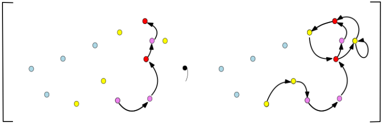
It can be approximated by a non deterministic graph $\mathscr{G}_2$:



Using a backward method, we compute an interval containing $\overleftarrow{\mathbb{T}}$.



Which corresponds to an interval of graphs:



Our new approach: bracket $\overleftarrow{\mathbb{T}}$, we search for paths not for states.

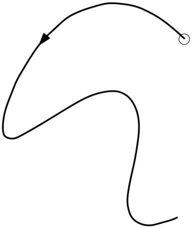
Maze

An ***interval*** is *a domain* which encloses a real number.

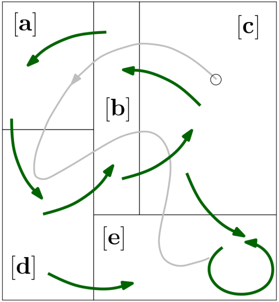
A ***polygon*** is *a domain* which encloses a vector of $\mathbb{R}^n$.

A ***maze*** is *a domain* which encloses a path.

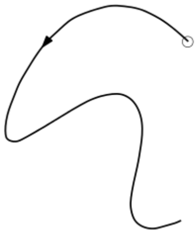
A maze is a set of paths.



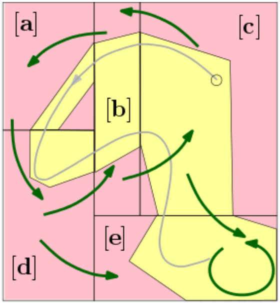
$\in$



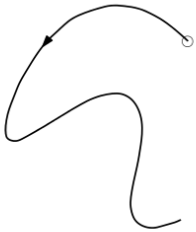
Mazes can be made more accurate by adding polygones.



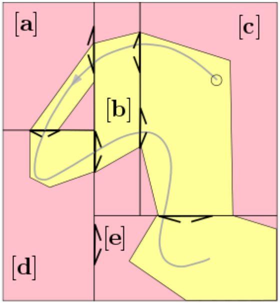
$\in$



Or using doors instead of a graph



$\in$



Here, a **maze** $\mathscr{L}$ is composed of

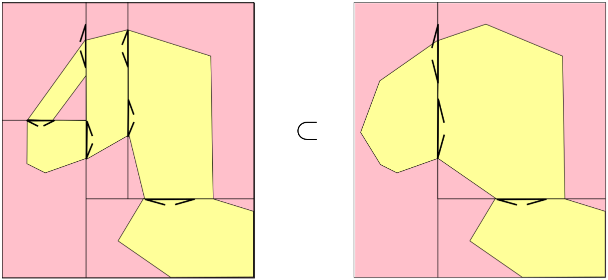
- A paving $\mathscr{P}$
- A polygon for each box of $\mathscr{P}$
- Doors between adjacent boxes

The set of mazes forms a lattice with respect to $\subset$.

$\mathscr{L}_a \subset \mathscr{L}_b$ means :

- the boxes of $\mathscr{L}_a$ are subboxes of the boxes of $\mathscr{L}_b$.
- The polygones of $\mathscr{L}_a$ are included in those of $\mathscr{L}_b$
- The doors of $\mathscr{L}_a$ are thinner than those of $\mathscr{L}_b$.

The left maze contains less paths than the right maze.



Note that yellow polygons are convex.

Inner approximation of $\overleftarrow{\mathbb{T}}$

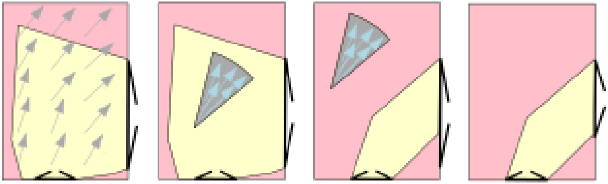
Main idea: Compute an outer approximation of the complementary of $\overleftarrow{\mathbb{T}}$:

$$\overline{\overleftarrow{\mathbb{T}}} = \{\mathbf{x}_0 \mid \forall t \geq 0, \varphi(t, \mathbf{x}_0) \notin \mathbb{T}\}$$

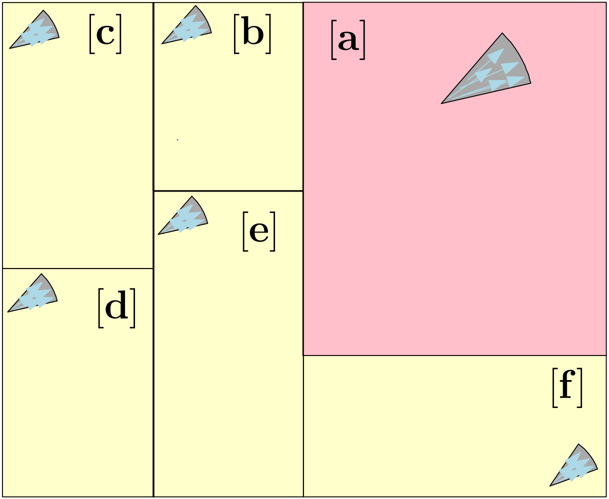
Thus, we search for a path that never reach $\mathbb{T}$.

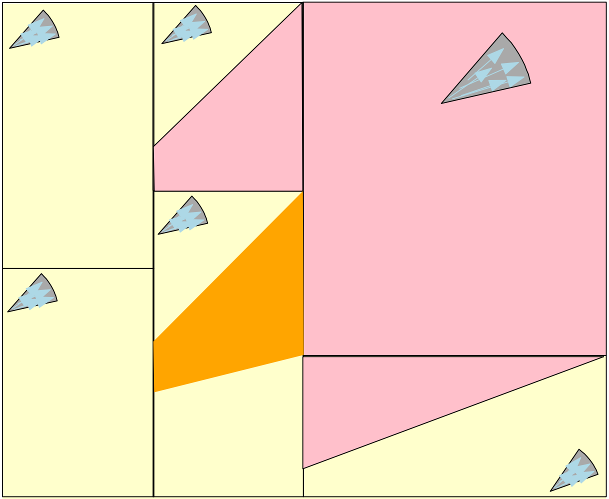
Target contractor. If a box [x] of $\mathscr{P}$ is included in $\mathbb{T}$ then remove [x] and close all doors entering in [x] .

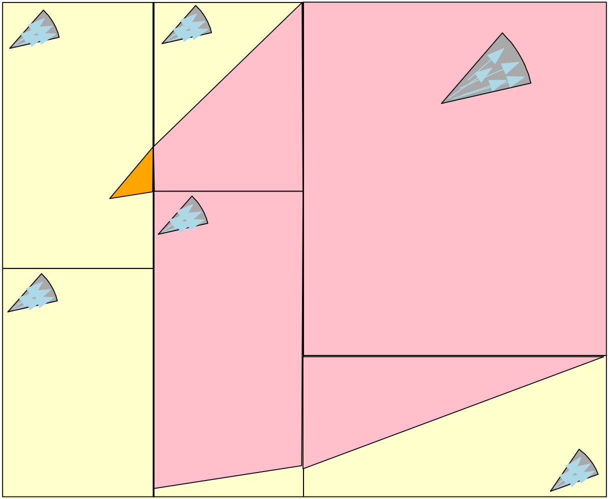
Flow contractor. For each box $[\mathbf{x}]$ of $\mathscr{P}$, we contract the polygon using the constraint $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

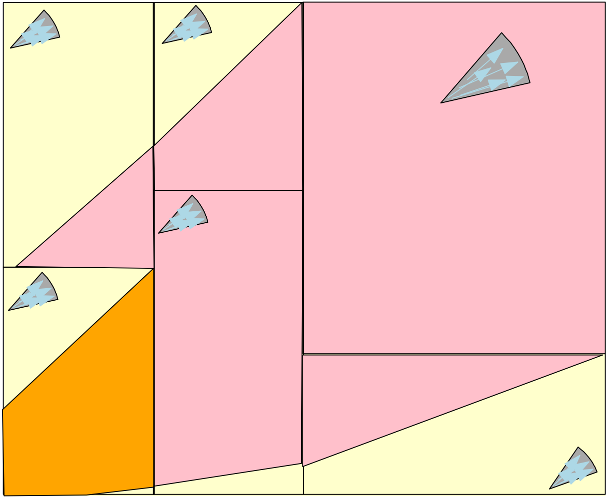


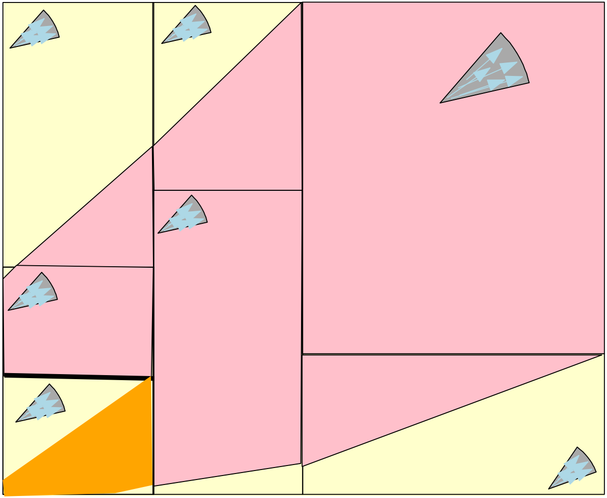
Inner propagation

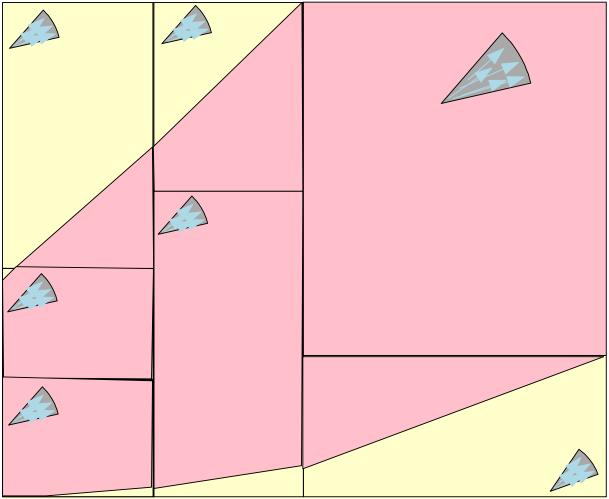

[c]
[b]
[a]
[e]
[d]
[f]

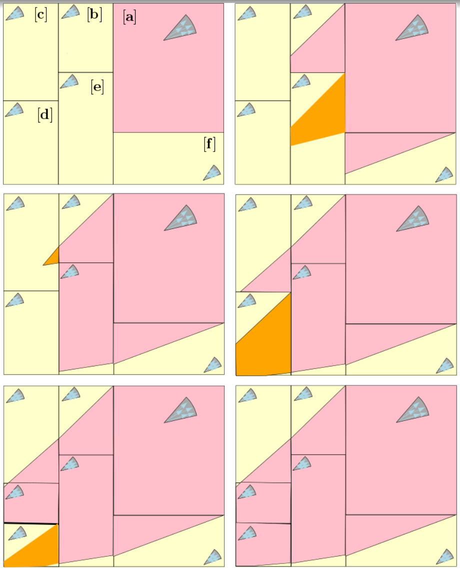




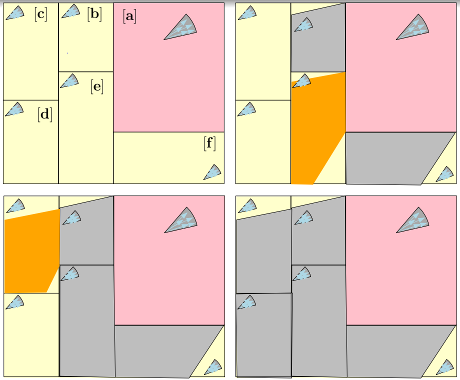







[c]
[b]
[a]
[e]
[d]
[f]

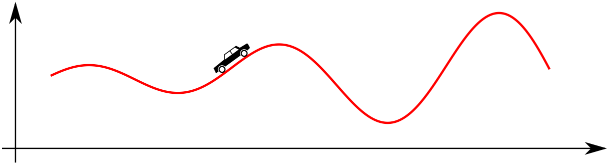
Outer propagation

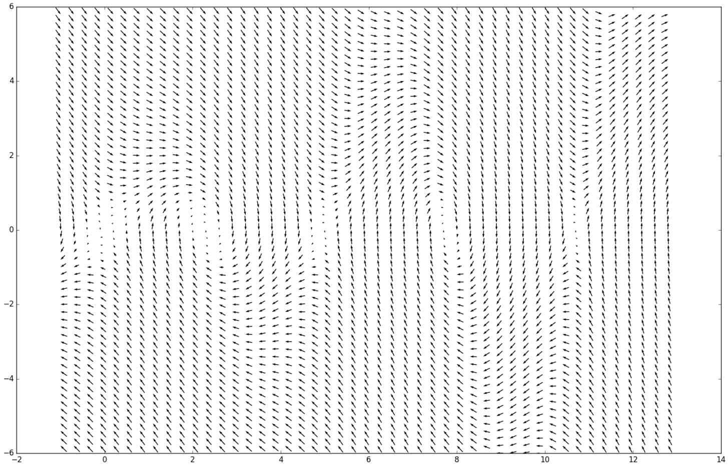


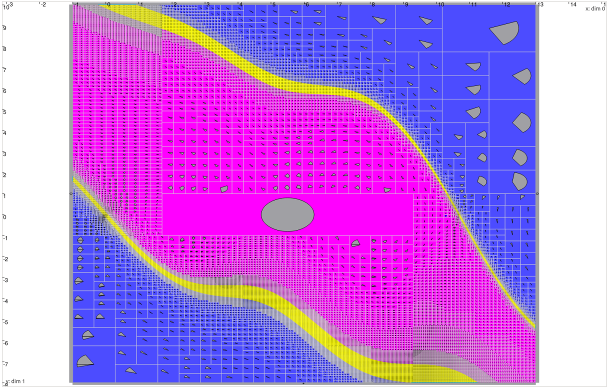
An interpretation can be given only when the fixed point is reached.

Car on the hill

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 9.81\sin\left(\frac{11}{24}\cdot\sin x_1 + 0.6\cdot\sin(1.1\cdot x_1)\right) - 0.7\cdot x_2 \end{cases}$$



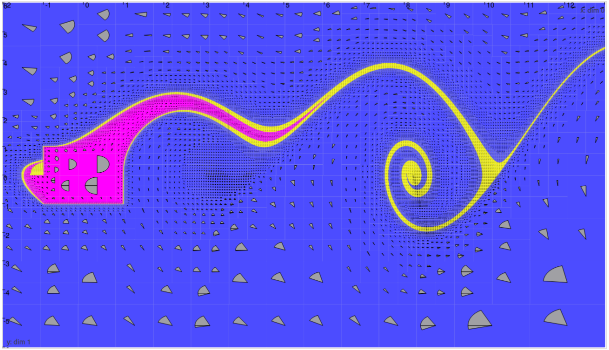

6
4
2
0
−2
−4
−6
−2
0
2
4
6
8
10
12
14



Research box $\mathbb{X}_0 = [-1, 13] \times [-10, 10]$

Blue: $\mathbb{T}_{out} = \overline{\mathbb{X}_0}$; Red: $\mathbb{T}_{in} = [2, 9] \times [-1, 1]$

Combined with an outer propagation

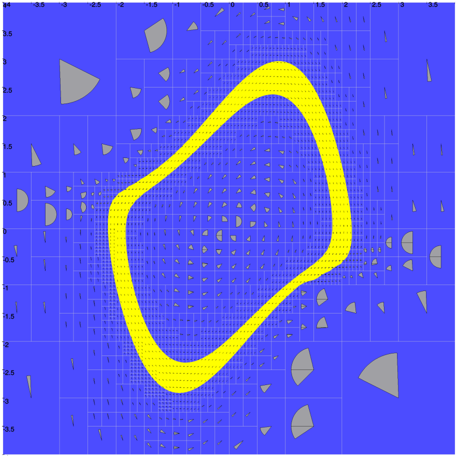


Van der Pol system

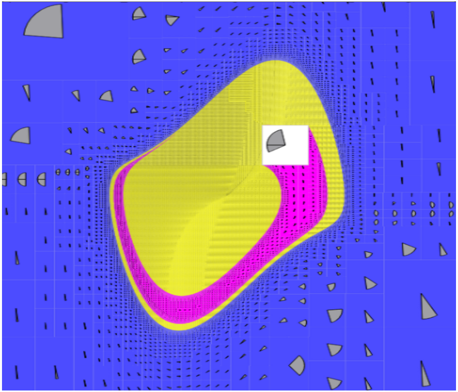
Consider the system

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$

and the box $\mathbb{X}_0 = [-4, 4] \times [-4, 4]$.

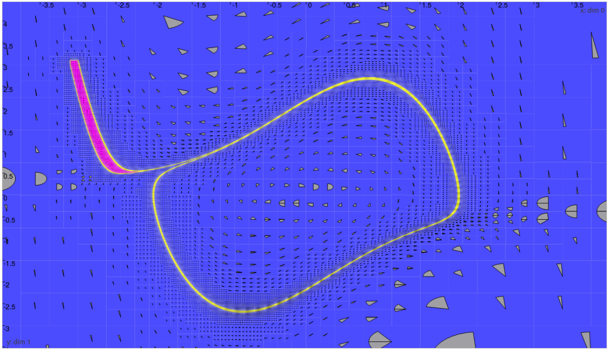


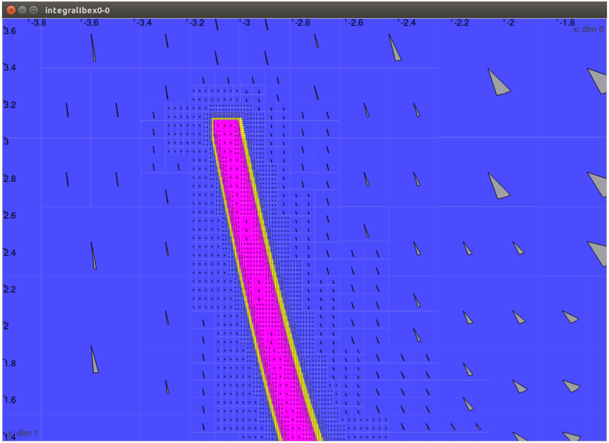
$$\mathbf{f} \to -\mathbf{f}\ ;\ \mathbb{T} = \overline{\mathbb{X}_0} \cup [-0.1, 0.1]^2.$$



$$\mathbf{f} \rightarrow -\mathbf{f} \ ; \ \mathbb{T}_{out} = \overline{\mathbb{X}_0} \ ; \ \mathbb{T}_{in} = [0.5, 1]^2.$$

Combined with an outer propagation




integralIbex0-0
-3.8
-3.6
-3.4
-3.2
-3
-2.8
-2.6
-2.4
-2.2
-2
-1.8
x: dim 0
3.6
3.4
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
y: dim 1