Lie symmetries applied to interval integration

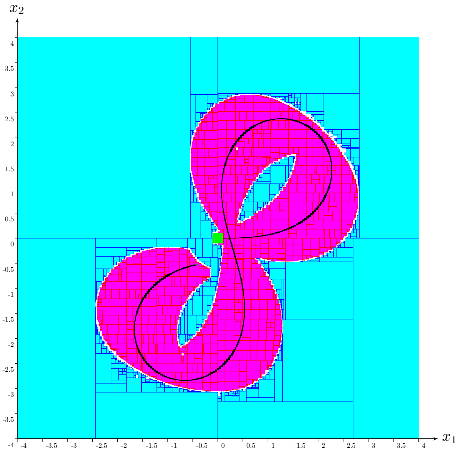
J. Damers, L. Jaulin, S. Rohou

Réunion MRIS, Palaiseau
2019, November 8

Consider the system

$$\begin{cases} \dot{x}_1 = u_1 \cdot \cos x_3 \\ \dot{x}_2 = u_1 \cdot \sin x_3 \\ \dot{x}_3 = u_2 \end{cases}$$

where u_1, u_2 is time dependent.



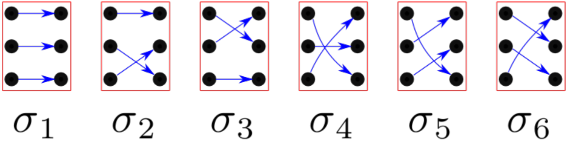
$$\operatorname{Proj}_{(x_1,x_2)} \bigcup_{t\in[0,14]} \mathbb{X}_t$$

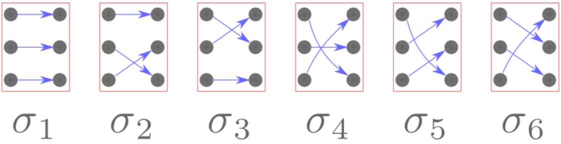
Group action

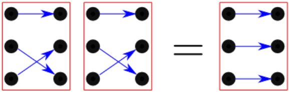
Define by $\mathbb{A} = \{a, b, c\}$.
The *symmetric group* is the set of all permutations in $\mathbb{X}$ is

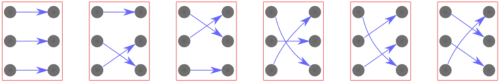
$$\begin{aligned} S_3 &= \{\sigma_1, \ldots, \sigma_6\} \\ &= \{abc \to abc, abc \to acb, abc \to bac, \\ &\quad abc \to cba, abc \to bca, abc \to cab\} \end{aligned}$$

It is a group with respect to $\circ$.

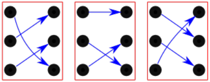
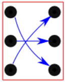

σ_1 σ_2 σ_3 σ_4 σ_5 σ_6



$\sigma_2 \circ \sigma_2 =$  $= \sigma_1$



σ_1 σ_2 σ_3 σ_4 σ_5 σ_6

$\sigma_6 \circ \sigma_2 \circ \sigma_6^{-1} =$  $=$  $= \sigma_4$

σ_6^{-1} σ_2 σ_6

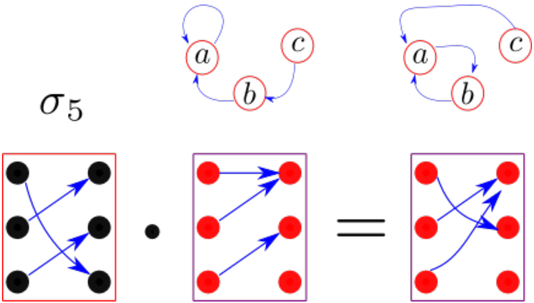
Action. Define by $\mathbb{F}$ the set of applications from $\mathbb{A}$ to $\mathbb{A}$.
For instance

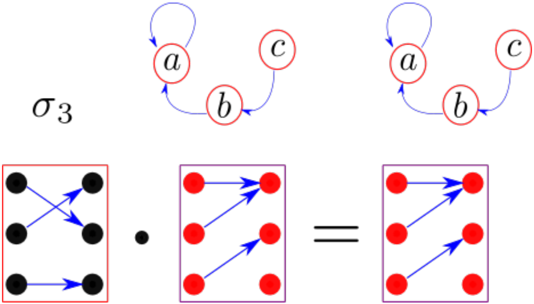
$$f_{aab} = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} a \\ a \\ b \end{pmatrix} \right) \in \mathbb{F}$$

Given $\sigma \in S_3$, we define the *action* of σ on f as

$$\sigma \bullet f = f \circ \sigma.$$

σ_5




σ_3
a
b
c
a
b
c

$(S_3, \circ, \bullet)$ is a *group action* since

- $(S_3, \circ)$ is a group
- $\forall f \in \mathbb{F}, \sigma_1 \bullet f = f$ (identity)
- $(\sigma_i \circ \sigma_j) \bullet f = \sigma_i \circ (\sigma_j \bullet f)$ (compatibility)

Stabilizer. For f in $\mathbb{F}$, the stabilizer group (or symmetry group) of G with respect to f is

$$G_f = Sym(f) = \{\sigma \in S_3 | \sigma \bullet f = f\}.$$

In our example we can check that

$$G_{f_{aab}} = \{\sigma_1, \sigma_3, \sigma_5\}$$

Differential group action

Action.

Define by $\mathbb{F}$ the set of all state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$

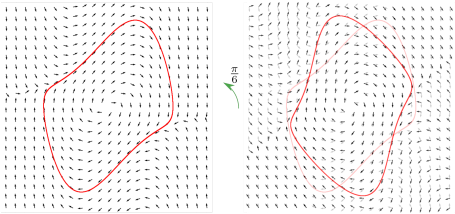
If $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, we define

$$\mathbf{g} \bullet \mathbf{f} = \left(\frac{d\mathbf{g}}{d\mathbf{x}} \circ \mathbf{g}^{-1} \right) \cdot \left(\mathbf{f} \circ \mathbf{g}^{-1} \right).$$

Proposition. If $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y} = \mathbf{g}(\mathbf{x})$, we have $\dot{\mathbf{y}} = \mathbf{g} \bullet \mathbf{f}(\mathbf{y})$.

Example. If $\mathbf{g}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, we get

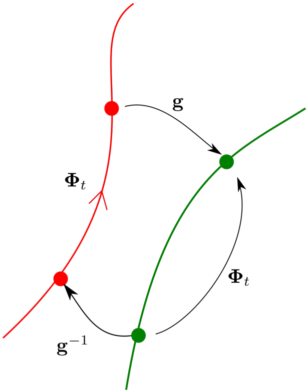
$$(\mathbf{g} \bullet \mathbf{f})(\mathbf{x}) = \underbrace{\left(\frac{d\mathbf{g}}{d\mathbf{x}}(\mathbf{g}^{-1}(\mathbf{x}))\right)}_{\mathbf{A}} \cdot (\mathbf{f}(\mathbf{g}^{-1}(\mathbf{x}))$$
$$= \mathbf{A} \cdot \mathbf{f}(\mathbf{A}^{-1} \cdot \mathbf{x}).$$

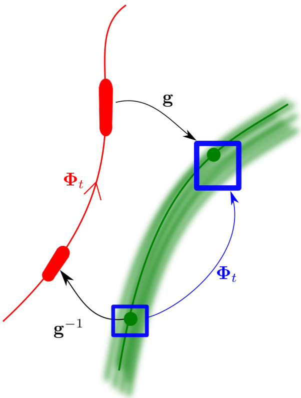


A transformation $\mathbf{g}$ is a *stabilizer* of $\mathbf{f}$ if $\mathbf{g} \bullet \mathbf{f} = \mathbf{f}$.
Equivalently, $\mathbf{g} \in Sym(\mathbf{f})$.

Proposition. Define $\mathbf{\Phi}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ as the flow associated to $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$. We have:

$$\mathbf{g}\bullet\mathbf{f}=\mathbf{f}\Leftrightarrow\mathbf{\Phi}_t=\mathbf{g}\circ\mathbf{\Phi}_t\circ\mathbf{g}^{-1}.$$

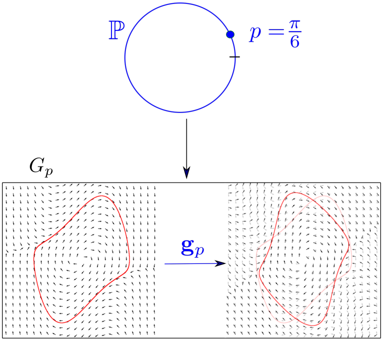



g
Φ_t
Φ_t
g^{-1}

Lie group of symmetries

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and a manifold $\mathbb{P}$.
A Lie group $G_{\mathbf{p}}$ of symmetries is a family of transformations $\mathbf{g}_{\mathbf{p}}$, $\mathbf{p} \in \mathbb{P}$ such that

- $(G_{\mathbf{p}}, \circ)$ is a Lie group
- $\forall \mathbf{p} \in \mathbb{P}, \mathbf{g}_{\mathbf{p}} \bullet \mathbf{f} = \mathbf{f}$.



Here, $(G_\mathbf{p}, \circ)$ is a Lie group but $\mathbf{g_p} \bullet \mathbf{f} \neq \mathbf{f}$

Transport function.

Given a Lie group of symmetries $G_{\mathbf{p}}$.

A *transport function* $\mathbf{h}(\mathbf{x},\mathbf{a})$ returns $\mathbf{p}$ so that $\mathbf{g_p}(\mathbf{a}) = \mathbf{x}$:

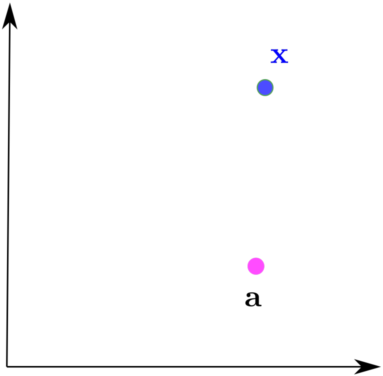
$$\mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a})}(\mathbf{a}) = \mathbf{x}$$

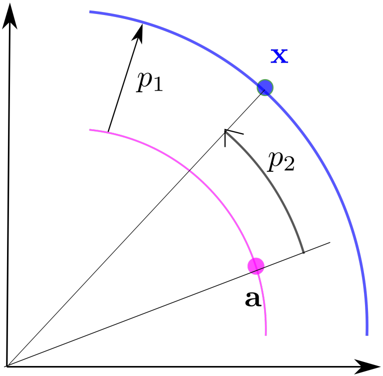
Example. Consider the Lie group of symmetries:

$$G_{\mathbf{p}} = \left\{ \mathbf{g}_{\mathbf{p}} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to p_1 \cdot \begin{pmatrix} \cos p_2 & -\sin p_2 \\ \sin p_2 & \cos p_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

To get the transport function $\mathbf{h}(\mathbf{x}, \mathbf{a})$, we use the equivalence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{p_1 \cdot \begin{pmatrix} \cos p_2 & -\sin p_2 \\ \sin p_2 & \cos p_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\mathbf{g}_{\mathbf{p}}(\mathbf{a})} \Leftrightarrow \mathbf{p} = \mathbf{h}(\mathbf{x}, \mathbf{a})$$


x
a



$$\mathbf{p} = \mathbf{h}(\mathbf{x}, \mathbf{a}) \text{ and } \mathbf{g_p}(\mathbf{a}) = \mathbf{x}$$

Interval integration

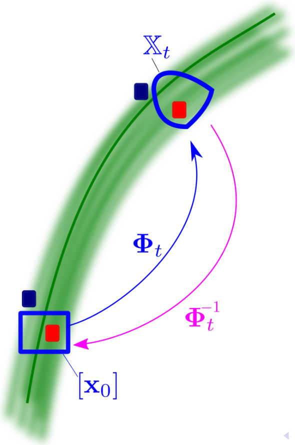
Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x}_0 \in [\mathbf{x}_0]$.
$\mathbf{\Phi}_t$ is the flow associated to $\mathbf{f}$

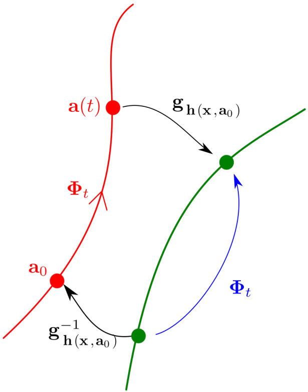
The set of all states $\mathbf{x}$ at time t consistent with the initial box $[\mathbf{x}_0]$ is [5]

$$\mathbb{X}_t = \mathbf{\Phi}_{-t}^{-1}([\mathbf{x}_0])$$

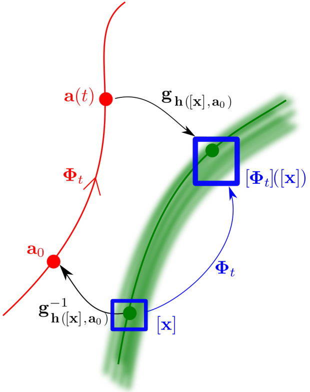


Theorem. We have one reference $\mathbf{a}(t) = \mathbf{\Phi}_t(\mathbf{a}_0)$.
If $\mathbf{h}(\mathbf{x},\mathbf{a})$ is a transport function for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then

$$\mathbf{\Phi}_t(\mathbf{x}) = \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t).$$



An inclusion function for $\mathbf{\Phi}_t(\mathbf{x})$ is thus $\mathbf{\Phi}_t([\mathbf{x}]) = \mathbf{g}_{\mathbf{h}([\mathbf{x}],\mathbf{a}_0)} \circ \mathbf{a}(t)$.



Method [2]

- Enclose [1] [7], a reference $\mathbf{a}(t)$ in a thin tube $[\mathbf{a}(t)]$.
- Find a Lie group of symmetries $G_{\mathbf{p}}$.
- Give an expression for the transport function $\mathbf{h}(\mathbf{x},\mathbf{a})$.
- Solve the set inversion problem.

Using separators [6], we can compute

$$\bigcup_{t\in\mathbb{T}} \mathbb{X}_t = \bigcup_{t\in\mathbb{T}} \mathbf{\Phi}_{-t}^{-1}([\mathbf{x}_0])$$

with $\mathbf{\Phi}_t(\mathbf{x}) = \mathbf{g}_{\mathbf{h}(\mathbf{x},\mathbf{a}_0)} \circ \mathbf{a}(t)$ and $\mathbb{T}$ is either

- a discrete set $\mathbb{T} = \{1, \dots, m\}$;
- an interval $\mathbb{T} = [0, t_{\max}]$.

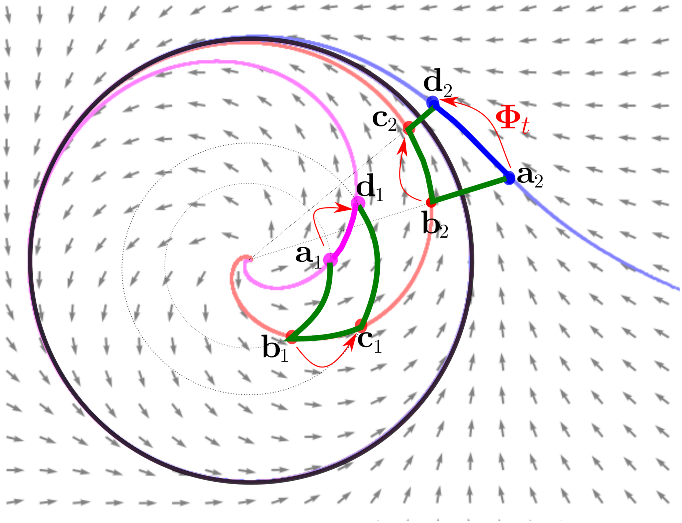
Hydon system

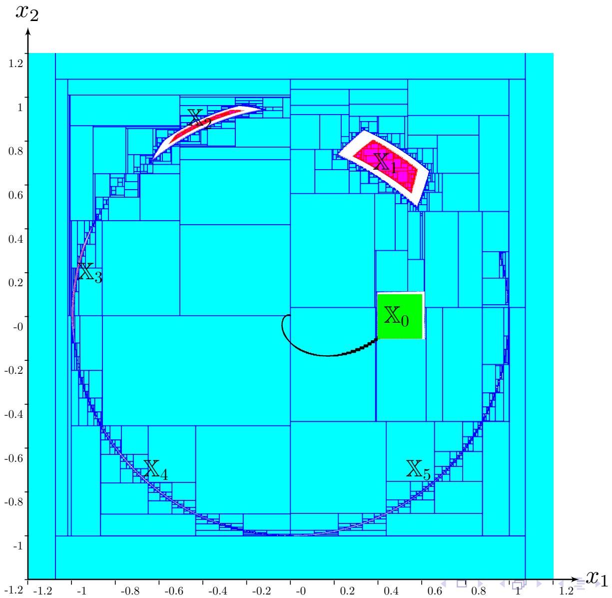
Example. Consider the system [3]

$$\begin{cases} \dot{x}_1 &= -x_1^3 - x_1 x_2^2 + x_1 - x_2 \\ \dot{x}_2 &= -x_2^3 - x_1^2 x_2 + x_1 + x_2 \end{cases}$$

It has the following symmetry

$$\mathbf{g_p}(\mathbf{x}) = \frac{1}{\sqrt{p_2 + \left(x_1^2 + x_2^2\right)(1 - p_2)}} \cdot \begin{pmatrix} \cos p_1 & -\sin p_1 \\ \sin p_1 & \cos p_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$


$\mathbf{d}_2$
$\mathbf{c}_2$
Φ_t
$\mathbf{a}_2$
$\mathbf{d}_1$
$\mathbf{b}_2$
$\mathbf{a}_1$
$\mathbf{b}_1$
$\mathbf{c}_1$


x_2
x_1
$\mathbb{X}_0$
$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
1.2
1
0.8
0.6
0.4
0.2
-0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.2
-1
-0.8
-0.6
-0.4
-0.2
-0
0.2
0.4
0.6
0.8
1
1.2

Dead reckoning

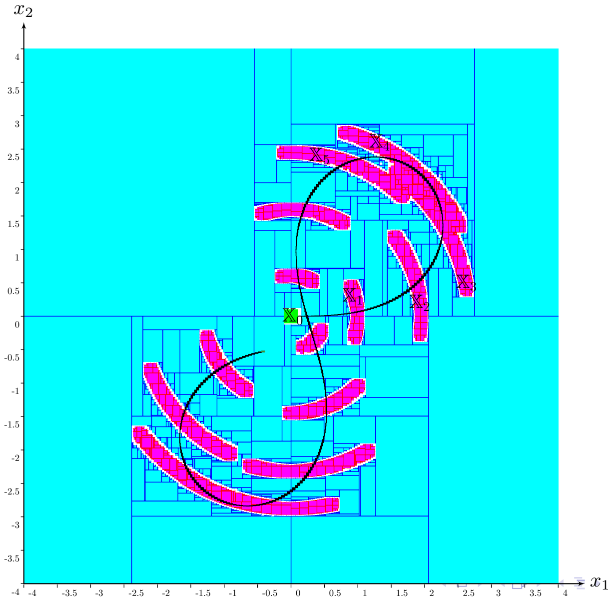
Consider the system [4]

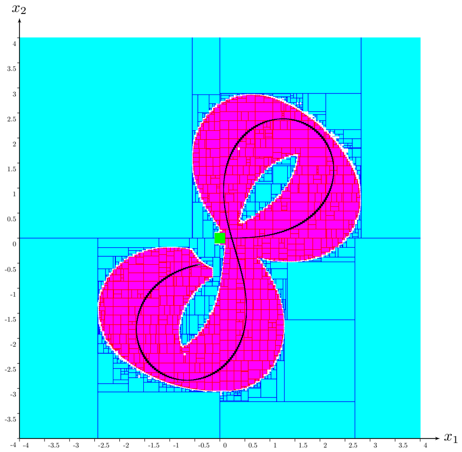
$$\begin{cases} \dot{x}_1 & = & u_1 \cdot \cos x_3 \\ \dot{x}_2 & = & u_1 \cdot \sin x_3 \\ \dot{x}_3 & = & u_2 \end{cases}$$

To avoid the time dependence in $\mathbf{u}$, we rewrite the system into

$$\begin{cases} \dot{x}_1 & = & u_1(x_4) \cdot \cos x_3 \\ \dot{x}_2 & = & u_1(x_4) \cdot \sin x_3 \\ \dot{x}_3 & = & u_2(x_4) \\ \dot{x}_4 & = & 1 \end{cases}$$

where x_4 is the clock variable.


x_2
x_1
$\mathbb{X}_0$
$\mathbb{X}_1$
$\mathbb{X}_2$
$\mathbb{X}_3$
$\mathbb{X}_4$
$\mathbb{X}_5$
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4



$$\underset{(x_1,x_2)}{\text{Proj}} \bigcup_{t\in[0,14]} \mathbb{X}_t$$

A. Chapoutot, J. Alexandre dit Sandretto, and O. Mullier.
Dynibex, available at ,
http://perso.ensta-paristech.fr/c̃hapoutot/dynibex/.

2015.

J. Damers, L. Jaulin, and S. Rohou.
Guaranteed interval integration for large initial boxes.
In *Book of Abstracts of the 12th Summer Workshop on Interval Methods (SWIM 2019)*, July 2019.

P. E. Hydon.
Symmetry methods for differential equations, A beginner's guide.
Cambridge Texts in Applied Mathematics, Cambridge, 2000.

L. Jaulin.
Mobile Robotics.

ISTE editions, 2015.

L. Jaulin.
Inner and outer set-membership state estimation.
Reliable Computing, 22:47–55, 2016.

L. Jaulin and B. Desrochers.
Introduction to the algebra of separators with application to path planning.
Engineering Applications of Artificial Intelligence, 33:141–147, 2014.

S. Rohou, L. Jaulin, M. Mihaylova, F. Le Bars, and S. Veres.
Guaranteed Computation of Robots Trajectories.
Robotics and Autonomous Systems, 93:76–84, 2017.