Chain of set inversion problems; Application to reachability analysis

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Backward reach set

Given $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$ and $\mathbb{Y}$. Set

$$\varphi^k = \underbrace{\mathbf{f} \circ \mathbf{f} \circ \cdots \circ \mathbf{f}}_{k \text{ times}},$$

For a given $\bar{k}$, compute

$$\mathbb{X}_0 = (\varphi^{\bar{k}})^{-1}(\mathbb{Y}).$$

Equivalently, solve the chain

$$\mathbb{X}_0 = \mathbf{f}^{-1}(\mathbb{X}_1), \mathbb{X}_1 = \mathbf{f}^{-1}(\mathbb{X}_2), \dots, \mathbb{X}_{\bar{k}} = \mathbb{Y}$$

Robust case

With an input vector $\mathbf{u}(k) \in [\mathbf{u}]$:

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)).$$

The initial feasible set $\mathbb{X}_0$ becomes also uncertain: it depends on $\mathbf{u}(k)$.

Define φ^k as:

$$\begin{array}{rcl}\varphi^1(\mathbf{x}(0),\mathbf{u}(0)) & = & \mathbf{f}(\mathbf{x}(0),\mathbf{u}(0)) \\ \varphi^{k+1}(\mathbf{x}(0),\mathbf{u}(0:k)) & = & \mathbf{f}(\varphi^k(\mathbf{x}(0),\mathbf{u}(0:k-1)),\mathbf{u}(k))\end{array}$$

Define

$$\begin{array}{rcl}\mathbb{X}_0^{\subset} & = & \{\mathbf{x}_0 | \forall \mathbf{u}(0:\bar{k}-1) \in [\mathbf{u}]^{\bar{k}}, \varphi^{\bar{k}}(\mathbf{x}(0),\mathbf{u}(0:\bar{k}-1)) \in \mathbb{Y}\} \\ \mathbb{X}_0^{\supset} & = & \{\mathbf{x}_0 | \exists \mathbf{u}(0:\bar{k}-1) \in [\mathbf{u}]^{\bar{k}}, \varphi^{\bar{k}}(\mathbf{x}(0),\mathbf{u}(0:\bar{k}-1)) \in \mathbb{Y}\},\end{array}$$

we have

$$\mathbb{X}_0^{\subset} \subset \mathbb{X}_0 \subset \mathbb{X}_0^{\supset}.$$

The sets $\mathbb{X}_0^{\subset}$ and $\mathbb{X}_0^{\supset}$ are the *minimal* and the *maximal backward reach set.*

Thick set inversion

Set inversion problem

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

If $\mathbf{f}_\mathbf{p}$ depends on a parameter vector $\mathbf{p} \in \mathbb{R}^q$.

$$\mathbb{X}^{\subset} \subset \mathbb{X}(\mathbf{p}) \subset \mathbb{X}^{\supset},$$

where

$$\begin{aligned} \mathbb{X}^{\subset} &= \bigcap_{\mathbf{p} \in [\mathbf{p}]} \mathbf{f}_\mathbf{p}^{-1}(\mathbb{Y}) = \{\mathbf{x} | \forall \mathbf{p} \in [\mathbf{p}], \mathbf{f}_\mathbf{p}(\mathbf{x}) \in \mathbb{Y}\} \\ \mathbb{X}^{\supset} &= \bigcup_{\mathbf{p} \in [\mathbf{p}]} \mathbf{f}_\mathbf{p}^{-1}(\mathbb{Y}) = \{\mathbf{x} | \exists \mathbf{p} \in [\mathbf{p}], \mathbf{f}_\mathbf{p}(\mathbf{x}) \in \mathbb{Y}\}. \end{aligned}$$

The pair $[\![\mathbb{X}]\!]=[\mathbb{X}^{\subset},\mathbb{X}^{\supset}]$ is a *thick set*. It partitions $\mathbb{R}^n$ into three zones: the clear zone $\mathbb{X}^{\subset}$, the penumbra $\mathbb{X}^{\partial}=\mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$ and the dark zone $\mathbb{R}^n\backslash\mathbb{X}^{\supset}$.

Notation. The *thick set inversion problem* is denoted by

$$[\![\mathbb{X}]\!] = \mathbf{f}_{[\mathbf{p}]}^{-1}(\mathbb{Y}).$$

One box in the penumbra. Consider the thick set inversion problem $[\![\mathbb{X}]\!] = \mathbf{f}_{[\mathbf{p}]}^{-1}([\mathbf{y}])$ where

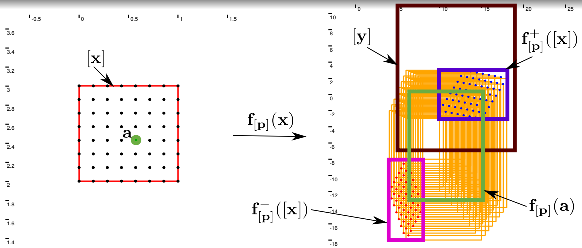
$$\mathbf{f}_{\mathbf{p}}(\mathbf{x}) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Take $[\mathbf{p}] = [2,3] \times [2,5] \times [4,5] \times [-6,-1]$, $[\mathbf{y}] = [5,19] \times [-7,11]$ and $\mathbf{x} \in [\mathbf{x}] = [0,1] \times [2,3]$.

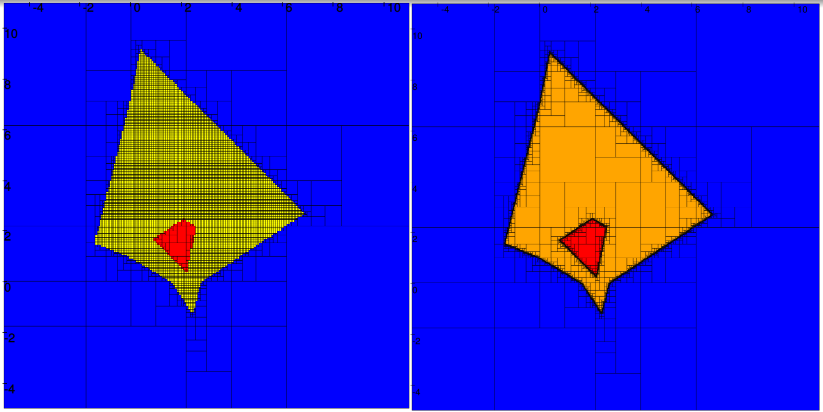
$$\mathbf{f}^{-}(\mathbf{x},[\mathbf{p}]) \in \left[\mathbf{f}^{-}_{[\mathbf{p}]}\right]([\mathbf{x}]) = \begin{pmatrix} 2 \cdot [0,1] + 2 \cdot [2,3] \\ 4 \cdot [0,1] - 6 \cdot [2,3] \end{pmatrix} = \begin{pmatrix} [4,8] \\ [-18,-8] \end{pmatrix}$$

and

$$\mathbf{f}^{+}(\mathbf{x},[\mathbf{p}]) \in \left[\mathbf{f}^{+}_{[\mathbf{p}]}\right]([\mathbf{x}]) = \begin{pmatrix} 3 \cdot [0,1] + 5 \cdot [2,3] \\ 5 \cdot [0,1] - 1 \cdot [2,3] \end{pmatrix} = \begin{pmatrix} [10,18] \\ [-3,3] \end{pmatrix}.$$



We conclude that [x] is inside the penumbra


-4
-2
0
2
4
6
8
10
10
8
6
4
2
0
-2
-4

Links and chains

Links

If, for a box $[\mathbf{p}] \subset \mathbb{R}^q$, the set

$$\mathbf{f}_{[\mathbf{p}]}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, [\mathbf{p}]) = \{\mathbf{y} \in \mathbb{R}^p \,|\, \exists \mathbf{p} \in [\mathbf{p}]\,,\ \mathbf{y} = \mathbf{f}_{\mathbf{p}}(\mathbf{x})\}$$

is a box, then $\mathbf{f}$ is said to be a *link*.
Links will be composed later to build a *chain*.

Due to their specific box-shaped structure, link can be inverted with respect to $\mathbf{x}$ without bisections on the $\mathbf{p}$-space.

Example. The function

$$f(\mathbf{x}, p) = 20e^{-x_1 p} - 8e^{-x_2 p}.$$

is a scalar link function.

Proposition. Consider ℓ scalar link functions, $f_i(\mathbf{x},\mathbf{p}_i)$, where vectors $\{\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_\ell\}$ are vector components of the vector $\mathbf{p}$. The function

$$\mathbf{f}(\mathbf{x},\mathbf{p}) = \begin{pmatrix} f_1(\mathbf{x},\mathbf{p}_1) \\ \vdots \\ f_\ell(\mathbf{x},\mathbf{p}_\ell) \end{pmatrix}$$

is a link.

Example. The function

$$\mathbf{f}(\mathbf{x},\mathbf{p}) = \begin{pmatrix} p_1 x_1 x_2 + p_1 p_2 sin(x_2) \\ p_3 x_1 + p_3 p_4 cos(x_1) x_2 \end{pmatrix}$$

is a link.

Consequence.

If $\mathbf{f_p}(\mathbf{x})$ is a link, for all $\mathbf{x}$, the set $\mathbf{f_{[p]}}(\mathbf{x})$ is a box.
If $\mathbf{x} \in [\mathbf{x}]$, the two bounds of the box $\mathbf{f_{[p]}}(\mathbf{x})$ are included inside the boxes $\left[\mathbf{f}^-_{\mathbf{[p]}}\right]([\mathbf{x}])$ and $\left[\mathbf{f}^+_{\mathbf{[p]}}\right]([\mathbf{x}])$.

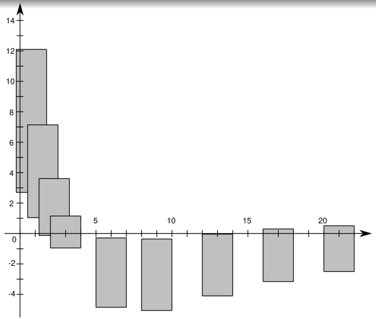
Parameter estimation with uncertain time measurement

Example. Let us estimate the parameters q_1 and q_2 of the model

$$y(\mathbf{q},t) = 20e^{-q_1 t} - 8e^{-q_2 t}.$$

We assume that 10 measurements y_i have been collected at time t_i.

i	$[t_i]$	$[y_i]$
1	[0.25, 1.25]	[2.7,12.1]
2	[1, 2]	[1.04,7.14]
3	[1.75, 2.75]	[-0.13,3.61]
4	[2.5, 3.5]	[-0.95,1.15]
5	[5.5, 6.5]	[-4.85,-0.29]
6	[8.5, 9.5]	[-5.06,-0.36]
7	[12.5, 13.5]	[-4.1,-0.04]
8	[16.5, 17.5]	[-3.16,0.3]
9	[20.5, 21.5]	[-2.5,0.51]
10	[24.5, 25.5]	[-2,0.6]



Data $([t_i], [y_i])$

We are interested by the thick set

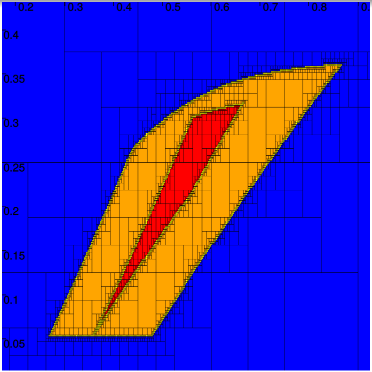
$$[\![\mathbb{Q}]\!] = \left\{\mathbf{q} \in \mathbb{R}^2 | \forall i, y(\mathbf{q}, t_i) \in [y_i]\right\}.$$

If we set

$$\begin{array}{rcl} \mathbf{f}_{[\mathbf{t}]}(\mathbf{q}) & = & \left(\begin{array}{c} y(\mathbf{q},[t_1]) \\ \vdots \\ y(\mathbf{q},[t_{10}]) \end{array}\right) \\ [\mathbf{y}] & = & [y_{10}] \times \cdots \times [y_{10}] \\ [\mathbf{t}] & = & [t_1] \times \cdots \times [t_{10}] \end{array}$$

then, we have

$$[\![\mathbb{Q}]\!] = \mathbf{f}_{[\mathbf{t}]}^{-1}([\mathbf{y}]).$$



Thick set $[\![\mathbb{Q}]\!]$

Chain

Chain. The function $\varphi_{\mathbf{p}}(\mathbf{x})$ is a *chain* if it can be written as

$$\varphi_{\mathbf{p}}(\mathbf{x}) = \mathbf{f}^{\ell}_{\mathbf{p}_\ell} \circ \cdots \circ \mathbf{f}^{2}_{\mathbf{p}_2} \circ \mathbf{f}^{1}_{\mathbf{p}_1}(\mathbf{x})$$

where $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_\ell\}$ are vector components of $\mathbf{p}$ and the $\mathbf{f}^{k}_{\mathbf{p}_k}$ are links.

Consequence: No dependency effect and no wrapping effect.

Example. The function

$$\varphi_{\mathbf{p}}(\mathbf{x}) = \begin{pmatrix} sin(p_3 x_1^2) + p_4 \\ \frac{p_5(p_1 e^{x_2} + p_1 p_2 x_1 x_2)}{p_3 x_1^2} \end{pmatrix}$$

is not a link. Now, if we define

$$\mathbf{f}^1_{\mathbf{p}_1}(\mathbf{x}) = \begin{pmatrix} p_1 e^{x_2} + p_1 p_2 x_1 x_2 \\ p_3 x_1^2 \end{pmatrix}$$

$$\mathbf{f}^2_{\mathbf{p}_2}(\mathbf{z}) = \begin{pmatrix} sin(z_2) + p_4 \\ \frac{p_5 z_1}{z_2} \end{pmatrix}$$

with $\mathbf{p}_1 = (p_1, p_2, p_3)$ and $\mathbf{p}_2 = (p_4, p_5)$ then, we have

$$\varphi_{\mathbf{p}}(\mathbf{x}) = \mathbf{f}^2_{\mathbf{p}_2} \circ \mathbf{f}^1_{\mathbf{p}_1}(\mathbf{x}).$$

Since both $\mathbf{f}^2_{\mathbf{p}_2}, \mathbf{f}^1_{\mathbf{p}_1}$ are links, $\varphi_{\mathbf{p}}(\mathbf{x})$ is a chain.

Test-cases for the backward reach set

Test-case 1: linear system

Consider the linear system

$$\mathbf{x}(k+1) = \mathbf{A} \cdot \mathbf{x}(k)$$

where

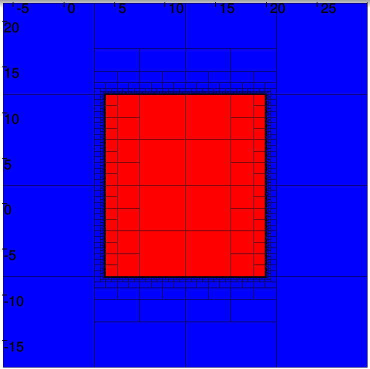
$$\mathbf{A}(k) \in \begin{pmatrix} [2.5, 3] & [2, 3] \\ [4, 4.5] & [-3, -2] \end{pmatrix}$$

plays the role of $\mathbf{p}$.

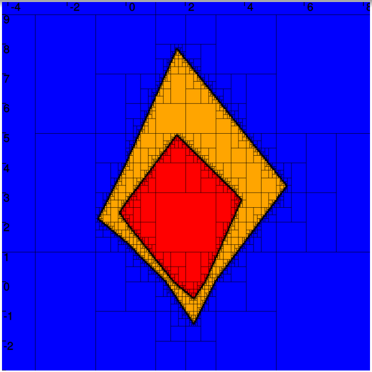
We want to reach the target

$$\mathbb{Y} = [4, 20] \times [-8, 12]$$

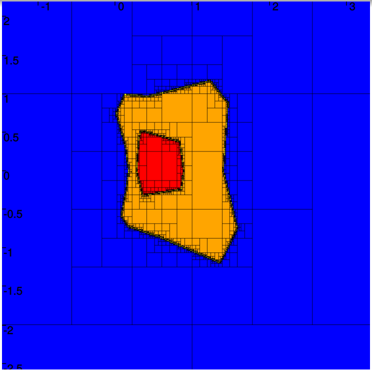
at time $\bar{k} = 3$.



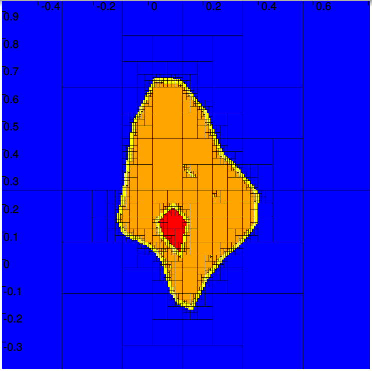
$[\![\mathbb{X}]\!](3) = \mathbb{Y}$ is thin.



$$[\![\mathbb{X}]\!](2) = \mathbf{f}_{[\mathbf{A}]}^{-1}([\![\mathbb{X}]\!](3))$$



$$[\![\mathbb{X}]\!](1) = \mathbf{f}_{[\mathbf{A}]}^{-1}([\![\mathbb{X}]\!](2))$$



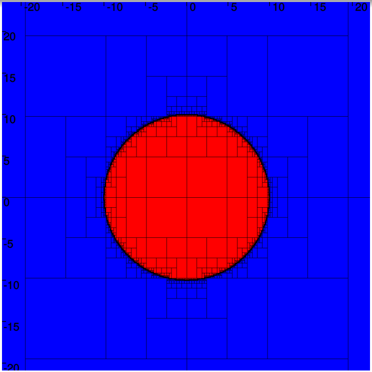
$$[\![\mathbb{X}]\!](0) = \mathbf{f}^{-1}_{[\mathbf{A}]}([\![\mathbb{X}]\!](1))$$

Test-case 2: nonlinear system

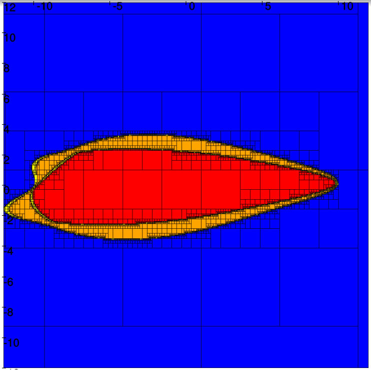
Nonlinear system

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} x_1(k) + x_2^2(k) \cdot u_1(k) \\ \frac{1}{2} \cdot x_1(k) \cdot x_2(k) + u_2(k) \end{pmatrix}$$

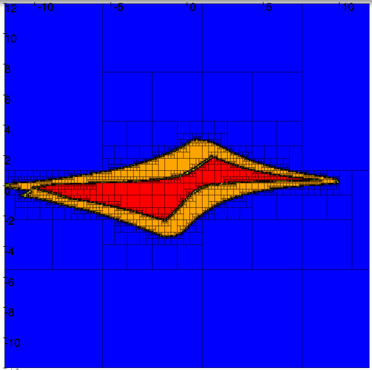
with $u_1 \in [1,2]$ and $u_2 \in [-2,-1]$. The set $\mathbb{Y}$ is assumed to be a centred disk which has to be reached at time $\bar{k} = 3$.



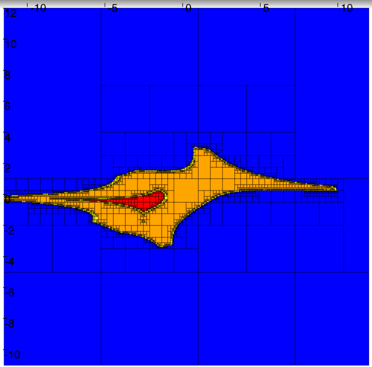
$[\![\mathbb{X}]\!](3) = \mathbb{Y}$ is thin.



$$[\![\mathbb{X}]\!](2) = \mathbf{f}_{[\mathbf{A}]}^{-1}([\![\mathbb{X}]\!](3))$$



$$[\![\mathbb{X}]\!](1) = \mathbf{f}^{-1}_{[\mathbf{A}]}([\![\mathbb{X}]\!](2))$$



$$[\![\mathbb{X}]\!](0) = \mathbf{f}^{-1}_{[\mathbf{A}]}([\![\mathbb{X}]\!](1))$$