Interval analysis for proving stability properties of robots; Application to sailboat robotics

L. Jaulin
Labsticc, IHSEV, OSM, ENSTA

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Presentation available at *http://youtu.be/GwWiIYsR5AA*

1 Interval analysis

Problem. Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}] \, , f \left(\mathbf{x} \right) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\mathrm{abs}\,([-7,1]) &= [0,7]
\end{aligned}
$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned} [f]([x_1], [x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ &\quad + \sin [x_1] \cdot \sin [x_2] + 2. \end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

2 Sailboat robotics


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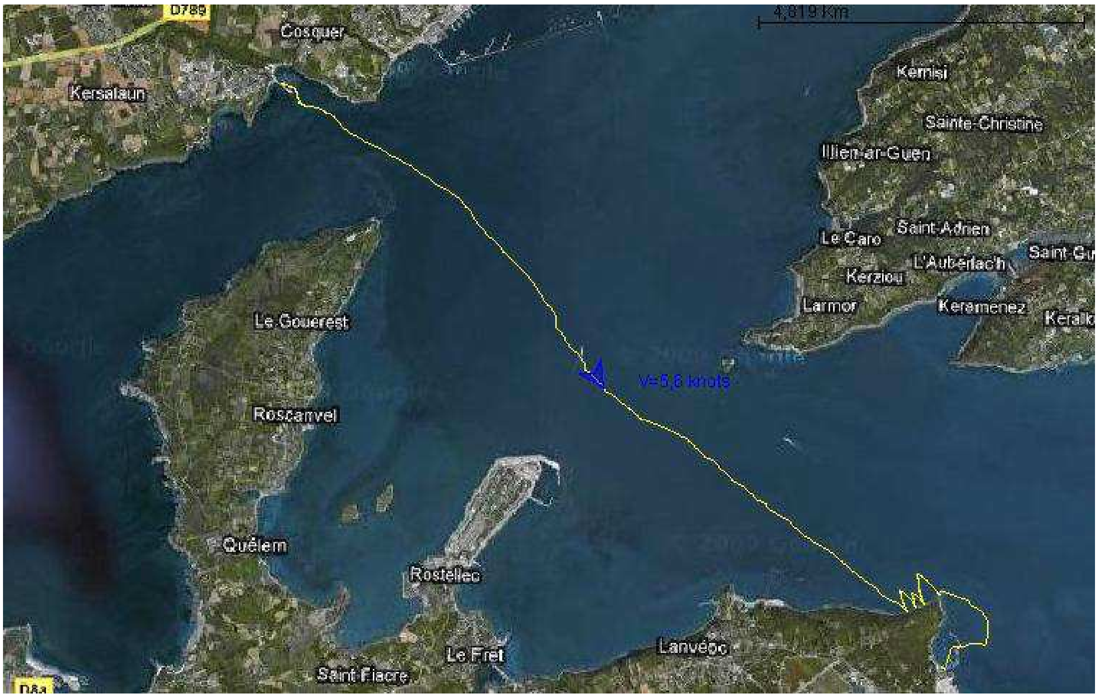
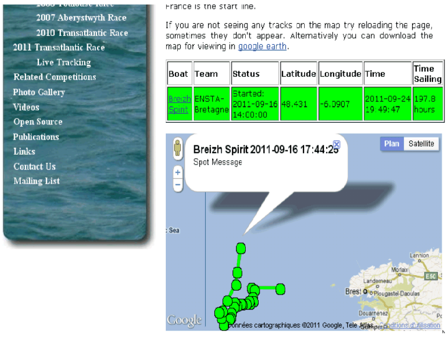


D789
Cosquer
4,819 Km
Kernisi
Kersalaun
Sainte-Christine
Illien-ar-Guen
Le Caro
Saint-Adrien
L'Auberlac'h
Kerziou
Larmor
Keramenez
Le Gouerest
V=5,6 knots
Roscanvel
Quélern
Rostellec
Le Fret
Lanvéoc
Saint Fiacre



2007 Aberystwyth Race
2010 Transatlantic Race
2011 Transatlantic Race
Live Tracking
Related Competitions
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Publications
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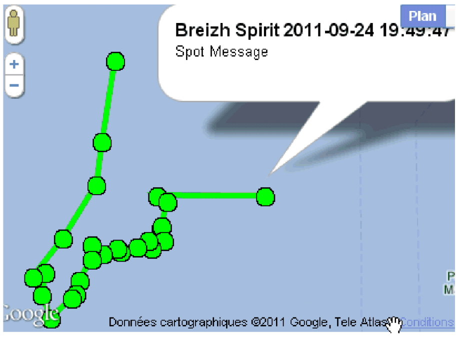
France is the start line.
If you are not seeing any tracks on the map try reloading the page, sometimes they don't appear. Alternatively you can download the map for viewing in google earth.
Boat | Team | Status | Latitude | Longitude | Time | Time Sailing
Breizh Spirit | ENSTA-Bretagne | Started: 2011-09-16 14:00:00 | 48.431 | -6.0907 | 2011-09-24 19:49:47 | 197.8 hours
Breizh Spirit 2011-09-16 17:44:25
Spot Message
Plan
Satellite
Sea
Lannion
Morlaix
Landerneau
Brest
Plougastel-Daoulas
Douarnenez
E50
Google
Données cartographiques ©2011 Google, Tele Atlas
Conditions d'utilisation



Breizh Spirit 2011-09-24 19:49:47
Spot Message
Plan
Google
Données cartographiques ©2011 Google, Tele Atlas
Conditions

3 Vaimos

Collaboration ENSTA/IFREMER

Vaimos at the WRSC (ENSTA-IFREMER-Ecole Navale).

$$
\left\{
\begin{array}{lcc}
\dot{x} & = & v\cos\theta + p_1 a\cos\psi \\
\dot{y} & = & v\sin\theta + p_1 a\sin\psi \\
\dot{\theta} & = & \omega \\
\dot{v} & = & \frac{f_s\sin\delta_s - f_r\sin u_1 - p_2 v^2}{p_9} \\
\dot{\omega} & = & \frac{f_s(p_6 - p_7\cos\delta_s) - p_8 f_r\cos u_1 - p_3\omega}{p_{10}} \\
f_s & = & p_4 a\sin(\theta - \psi + \delta_s) \\
f_r & = & p_5 v\sin u_1 \\
\sigma & = & \cos(\theta - \psi) + \cos(u_2) \\
\delta_s & = & \left\{
\begin{array}{cc}
\pi - \theta + \psi & \text{si } \sigma \leq 0 \\
sign(\sin(\theta - \psi)).u_2 & \text{sinon.}
\end{array}
\right.
\end{array}
\right.
$$

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right).$$

With the controller $\mathbf{u} = \mathbf{g}\left(\mathbf{x}\right)$, the robot satisfies

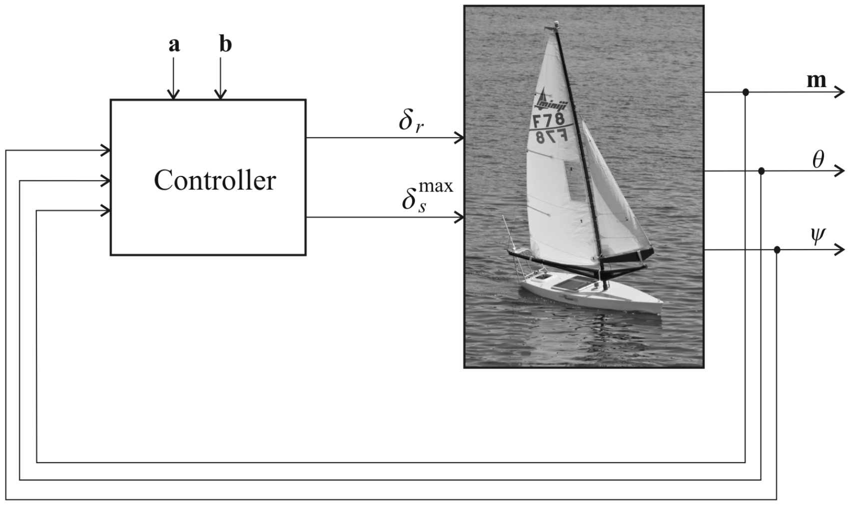
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right).$$

With all uncertainties, the robot satisfies.

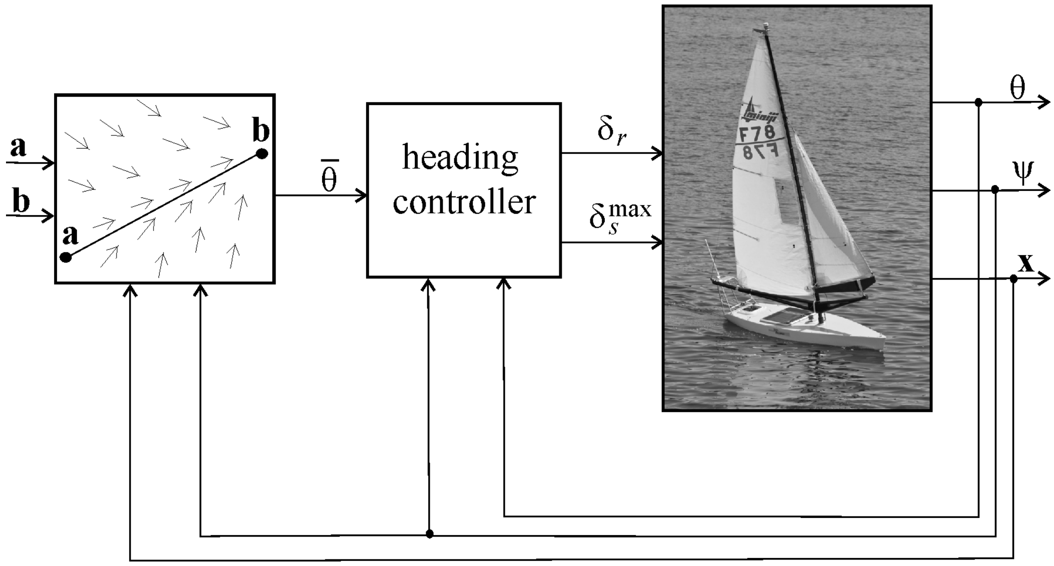
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

4 Line following



Controller of a sailboat robot



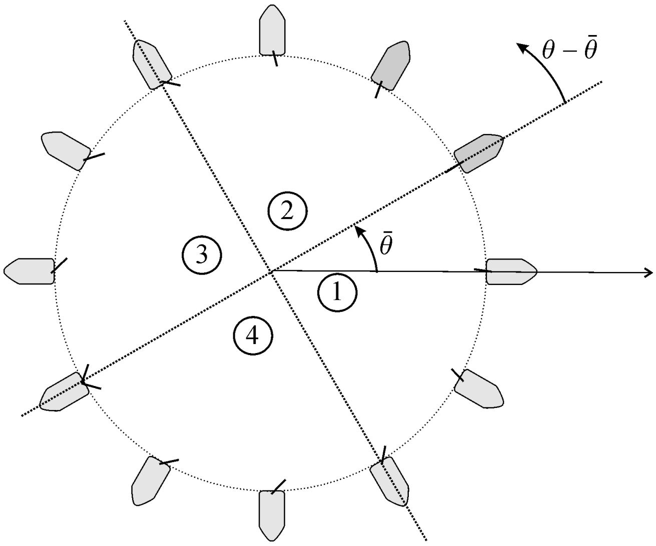
a
b
b
a
$\bar{\theta}$
heading controller
δ_r
$\delta_s^{\max}$
F78
θ
ψ
x

Heading controller

$$\begin{cases} \delta_r & = & \frac{\delta_r^{\mathsf{max}}}{\pi}.\mathrm{atan}(\tan \frac{\theta - \bar{\theta}}{2}) \\ \delta_s^{\mathsf{max}} & = & \frac{\pi}{2}.\left(\frac{\cos(\psi - \bar{\theta}) + 1}{2}\right). \end{cases}$$

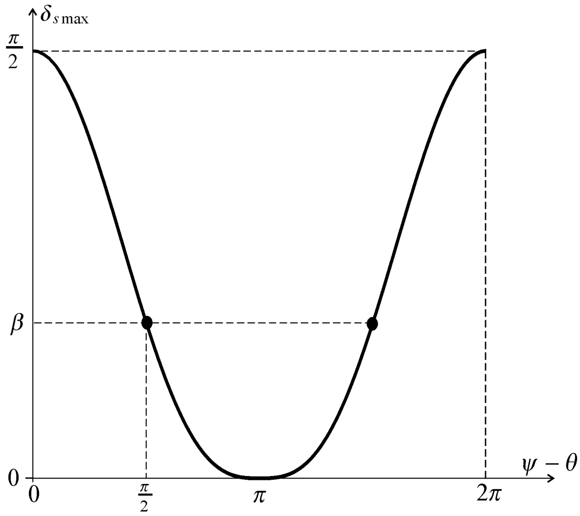
Rudder

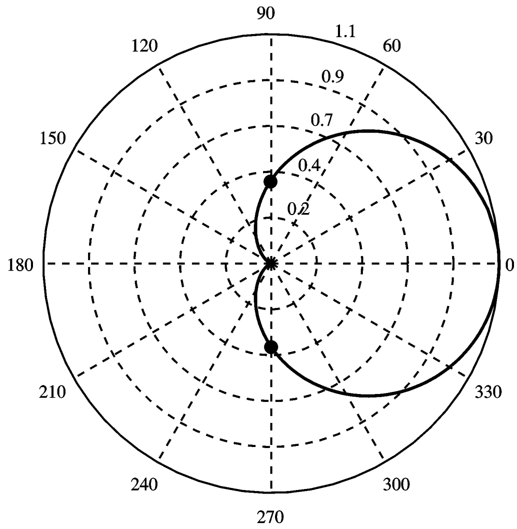
$$\left\{ \begin{array}{lll} \delta_r & = & \frac{\delta_r^{\text{max}}}{\pi}.\text{atan}(\tan\frac{\theta-\bar{\theta}}{2}) \end{array} \right.$$



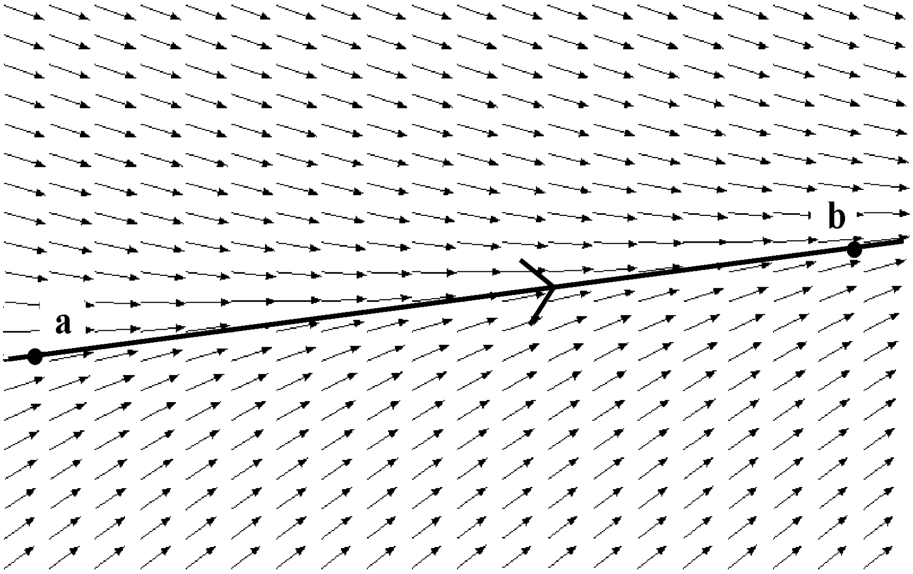
Sail

$$\delta_s^{\mathsf{max}} = \frac{\pi}{2} \cdot \left(\frac{\cos\left(\psi - \bar{\theta}\right) + 1}{2} \right)$$

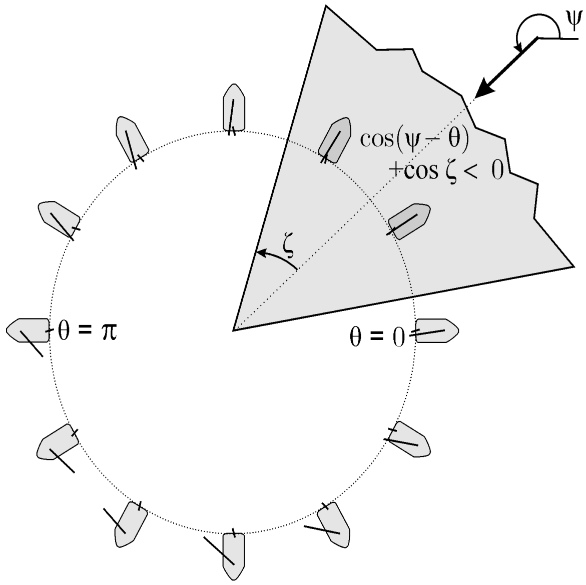




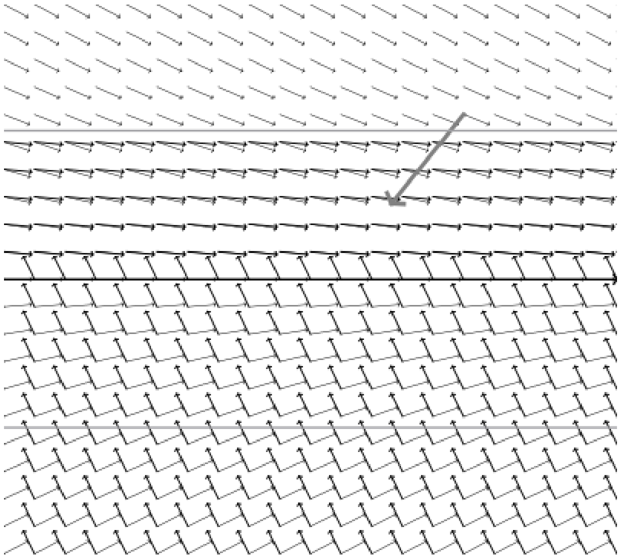
4.1 Vector field



Nominal vector field: $\theta^* = \varphi - \frac{1}{2}.\mathrm{atan}\left(\frac{e}{r}\right)$



A course θ^* may be unfeasible



4.2 Controller

Controlleur : in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$; out: $\delta_r, \delta_s^{\mathsf{max}}$; inout: q

$e = \frac{\det(\mathbf{b}-\mathbf{a},\mathbf{m}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|}$

if $|e| > \frac{r}{2}$ then $q = \mathrm{sign}(e)$

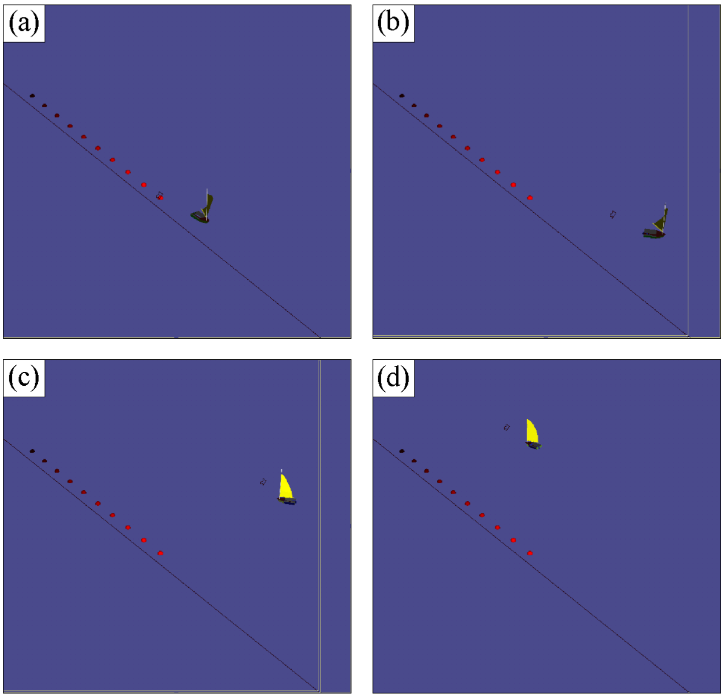
$\bar{\theta} = \mathrm{atan2}(\mathbf{b}-\mathbf{a}) - \frac{1}{2}.\mathrm{atan}\left(\frac{e}{r}\right)$

if $\cos\left(\psi - \bar{\theta}\right) + \cos\zeta < 0$ then $\bar{\theta} = \pi + \psi - q.\zeta$.

$\delta_r = \frac{\delta_r^{\mathsf{max}}}{\pi}.\mathrm{atan}(\tan\frac{\theta-\bar{\theta}}{2})$

$\delta_s^{\mathsf{max}} = \frac{\pi}{2}.\left(\frac{\cos(\psi-\bar{\theta})+1}{2}\right)$.

5 Validation by simulation


(a)
(b)
(c)
(d)

6 Theoretical validation

*Jaulin, Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE TRO.

When the wind is known, the sailboat with the heading controller is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

The system

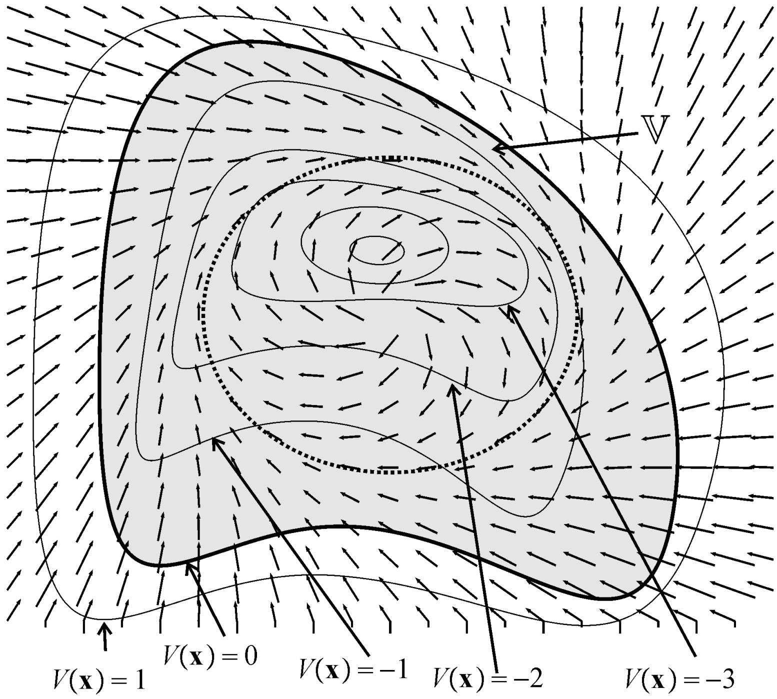
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) is there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned} \dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}, \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0}. \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable if

$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) \leq \varepsilon < 0 \right).$$


$\mathbb{V}$
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$
\begin{aligned}
& \left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\; & \left(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\; & \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow\; & \max\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}), V(\mathbf{x})\right) < 0
\end{aligned}
$$

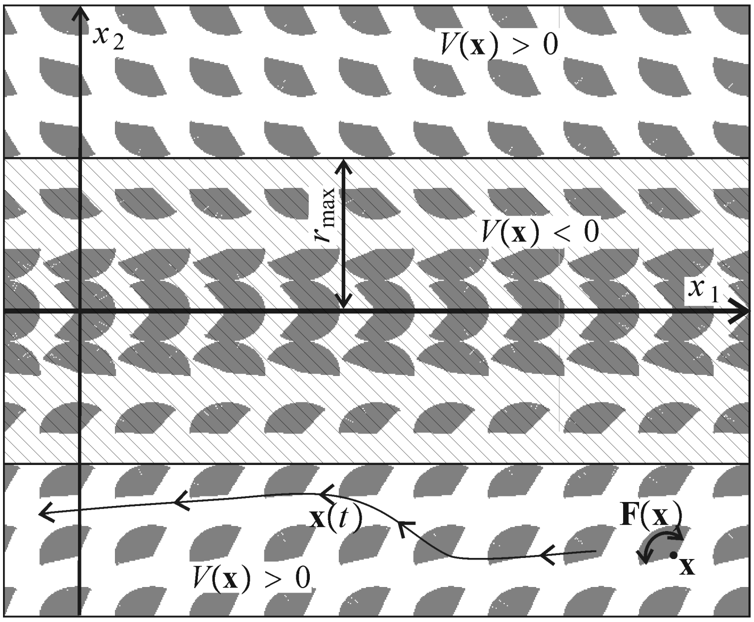
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V-stability.

Theorem. We have

$$\left\{\begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a}\geq 0 \\ \mathbf{F}^{-}(\mathbf{x})\leq \mathbf{a}\leq \mathbf{F}^{+}(\mathbf{x}) \\ V(\mathbf{x})\geq 0 \end{array}\right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}}\in \mathbf{F}(\mathbf{x}) \text{ is } V\text{-stable}$$

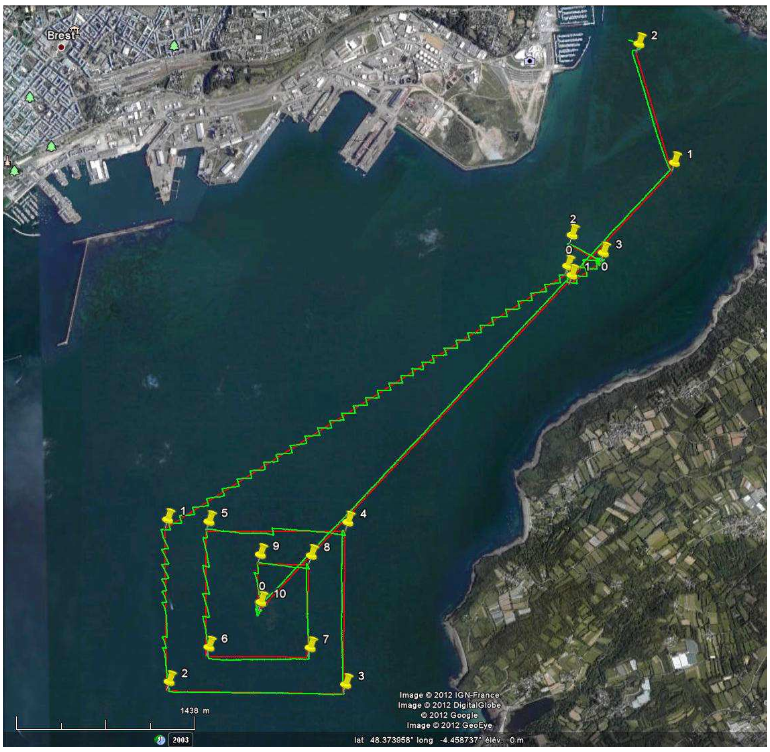


Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(\mathbf{x}) = x_2^2 - r_{\mathsf{max}}^2.$$

7 Experimental validation

Collaboration ENSTA-Ifremer. Fabrice Le Bars, Olivier Ménage, Patrick Rousseau, . . .



Rade de Brest

Brest-Douarnenez. January 17, 2012, 8am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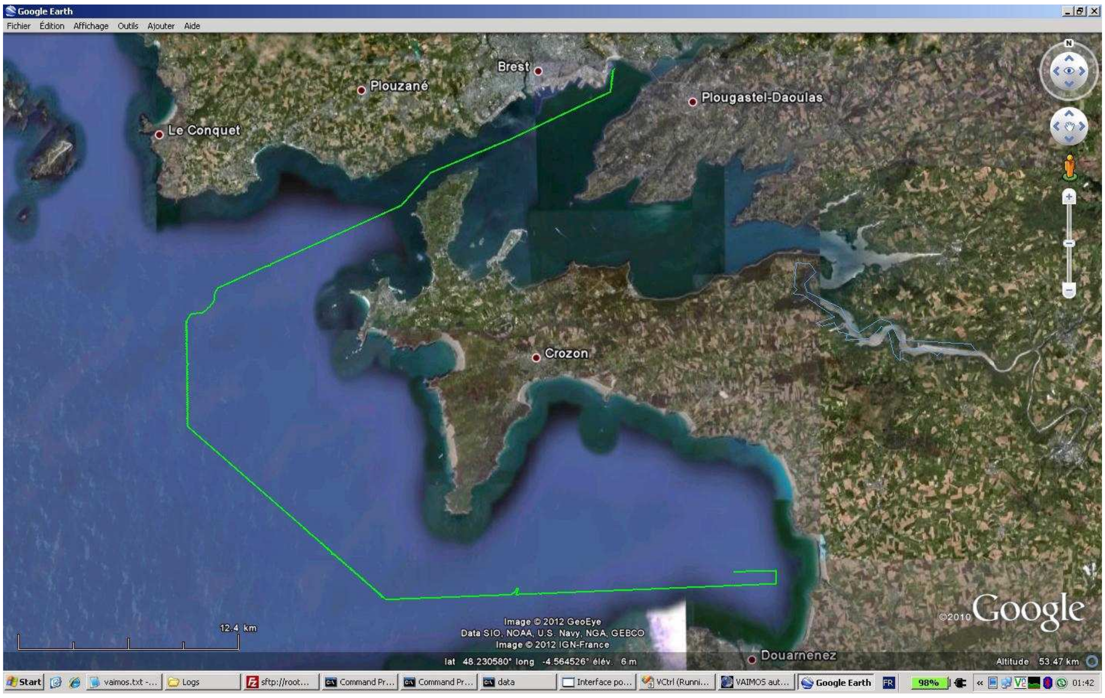
YAMAHA


minij
F78


Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Plougastel-Daoulas
Le Conquet
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google
Start
Logs
Command Pr...
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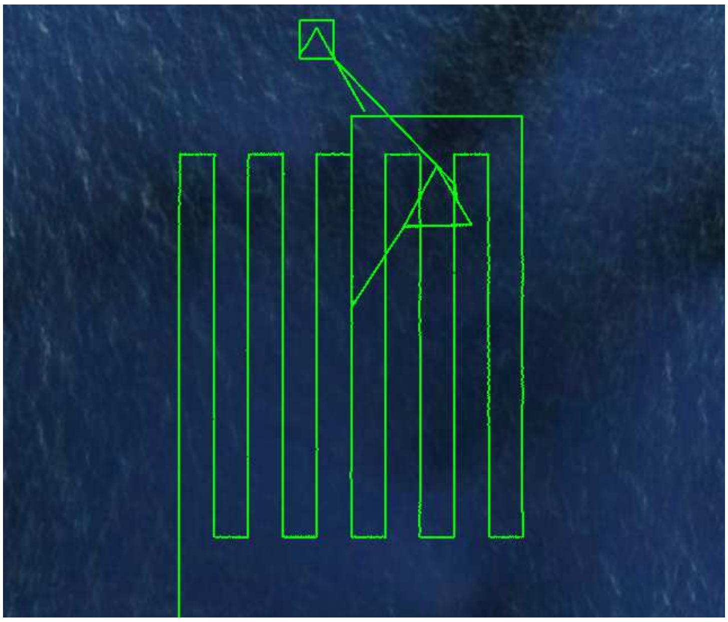

Middle of Atlantic ocean, 350 km made by Vaimos in 53h, September 6-9, 2012.

Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.