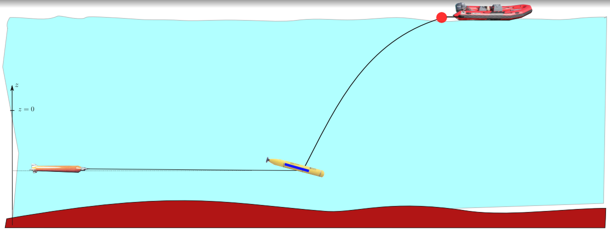
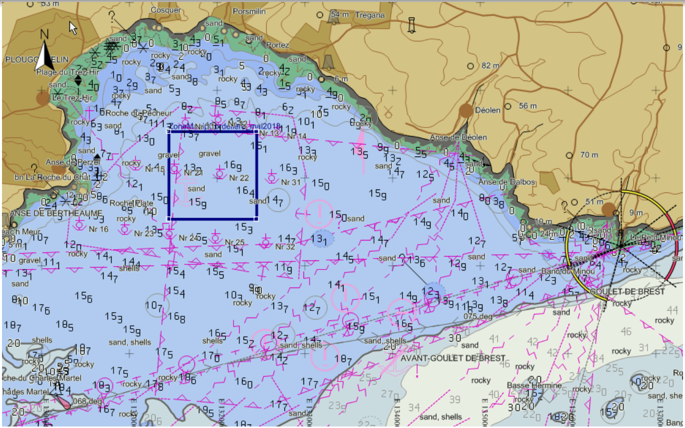
Intervals of a Kleene algebra used to compute inner and outer approximations of invariant sets of a dynamical system

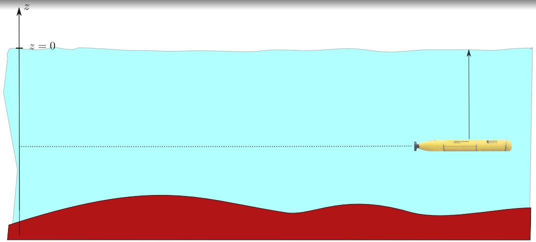
T. Le Mézo, **L. Jaulin**, B. Zerr, D. Massé, S. Rohou, V. Drevelle
Lab-STICC, ENSTA-Bretagne
GT VS-CPS - GT MEA, CNAM Paris, July 3, 2018

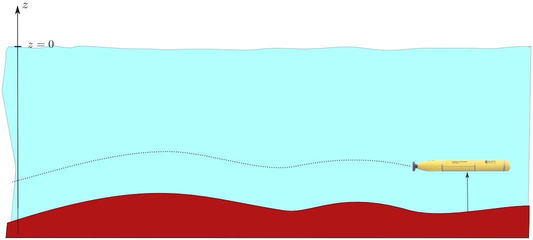
Navigating underwater

La Cordelière


z
z = 0


Cosquer
Porsmilin
Tregana
Portez
PLOUGONVELIN
Plage du Trez-Hir
Le Trez-Hir
Roche du Pêcheur
Anse de Perzel
La Roche du Chat
ANSE DE BERTHEAUME
Roche Plate
Déolen
Anse de Déolen
Anse de Dalbos
Banc du Minou
GOULET DE BREST
AVANT-GOULET DE BREST
Basse Hermine
gravel
sand
rocky
shells
sand, shells
075 deg
068 deg


z
z = 0

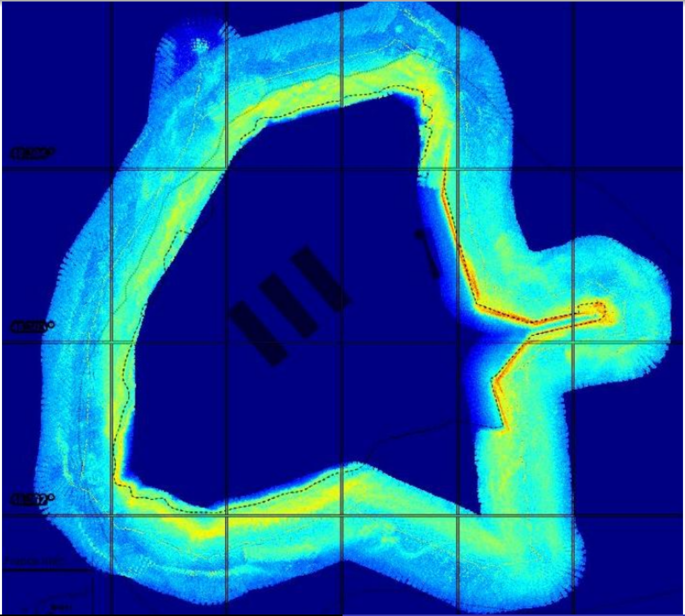

z
z = 0

Ile des morts experiment

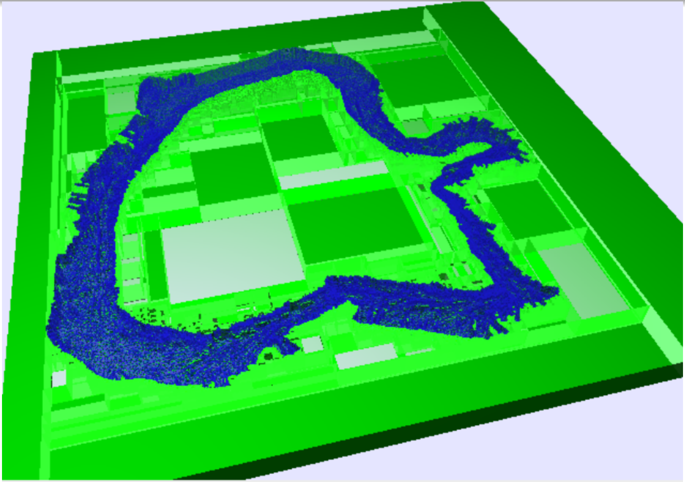

Saint-Renan
Bohars
Guilers
Lambézellec
Le Relecq-Kerhuon
La Forest-Landerneau
L'Élorn
Ploumoguer
Saint-Marc
Brest
N165
Dirinon
Saint-Urbain
Plouzané
Saint-Pierre
E60
Plougastel-Daoulas
Loperhet
Irvillac
Locmaria-Plouzané
Daoulas
Plougonvelin
Hôpital-Camfrout
Logonna-Daoulas
Roscanvel
Landévennec
Lanvéoc
Camaret-sur-Mer
Rosnoën
Crozon
Argol
Rue de Dinan
Telgruc-sur-Mer
Saint-Nic
Ar Valare
Plomodiern
E50


Île des Morts
Celtic Sea
ROSCANVEL
Camping Municipal
Route de la Fraternité
Route de Quélern
D355
Route de Keruzou
Le Disloup
Route du Milieu
Route du Ménez
Persuel
Taladec'h
Rue Auguste


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
50m



24 juillet 2013



Test-case

Consider an underwater robot:

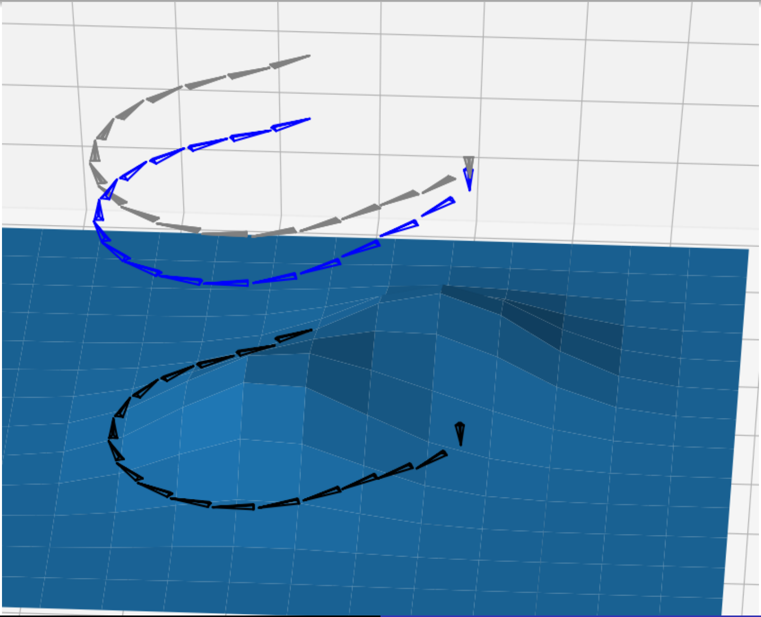
$$\begin{cases} \dot{x} & = & \cos\psi \\ \dot{y} & = & \sin\psi \\ \dot{z} & = & u_1 \\ \dot{\psi} & = & u_2 \end{cases}$$

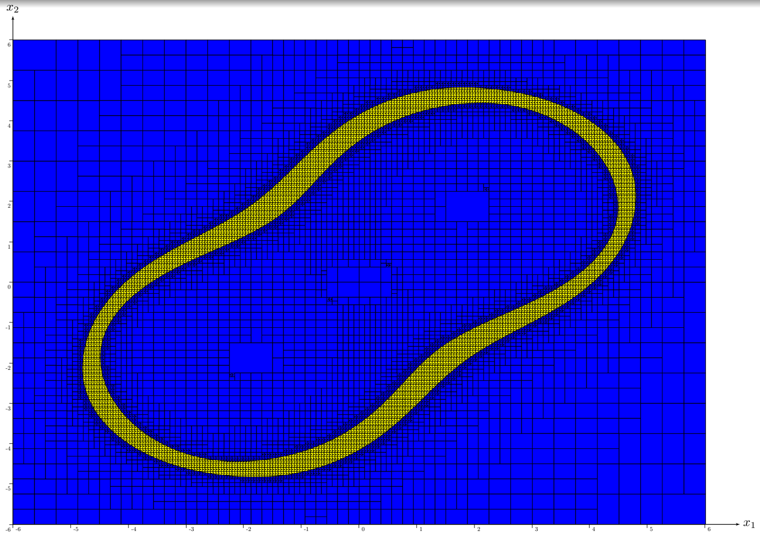
The robot is able to measure its altitude, the angle of the gradient of h and its depth

$$\begin{cases} y_1 & = & z - h(x,y) \\ y_2 & = & angle(\nabla h(x,y)) - \psi \\ y_3 & = & -z \end{cases}$$

We take the controller

$$\mathbf{u} = \begin{pmatrix} y_3 - \overline{y}_3 \\ \text{-tanh}(h_0 + y_3 + y_1) + \text{sawtooth}(y_2 + \frac{\pi}{2}) \end{pmatrix}.$$




x_2
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
x_1

Computing invariant sets

Guaranteed integration[5][4]

Consider the system

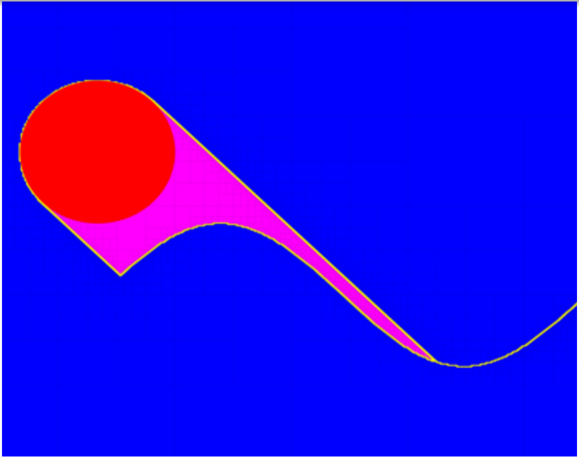
$$\mathscr{S} : \dot{\mathsf{x}}(t) = \gamma(\mathsf{x}(t))$$

Denote by $\varphi_\gamma(t, \mathsf{x})$ the flow map.

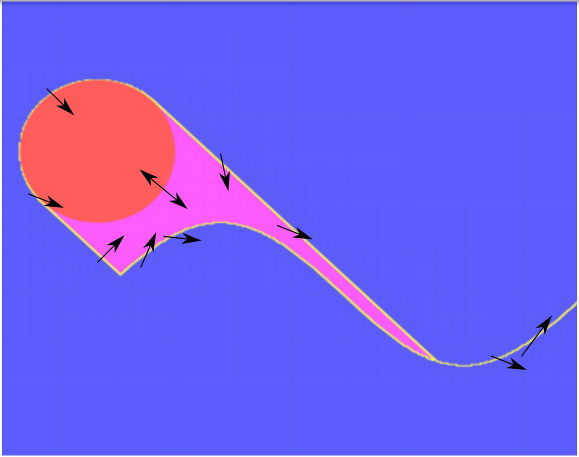
The *forward reach set* of $\mathbb{X} \subset \mathbb{R}^n$ is:

$$\text{Forw}(\mathbb{X}) = \left\{ \mathbf{x} \mid \exists \mathbf{x}_0 \in \mathbb{X}, \exists t \geq 0, \mathbf{x} = \varphi_\gamma(t, \mathbf{x}_0) \right\}.$$

$$\begin{aligned}\dot{x}_1 &= 1\\ \dot{x}_2 &= \operatorname{sign}(\sin(x_1) - x_2)\end{aligned}$$



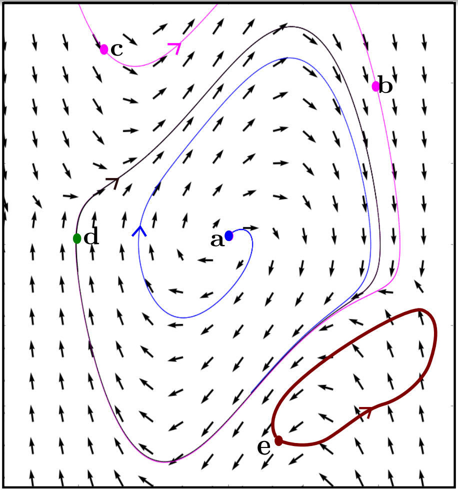
$\mathbb{D}$: the red disk
$\text{Forw}(\mathbb{D}) \in [\mathbb{F}^-, \mathbb{F}^+]$.
$\mathbb{F}^-$: red+magenta
$\mathbb{F}^+$: red+magenta+yellow: positive invariant.



Largest positive invariant sets

Example: Consider

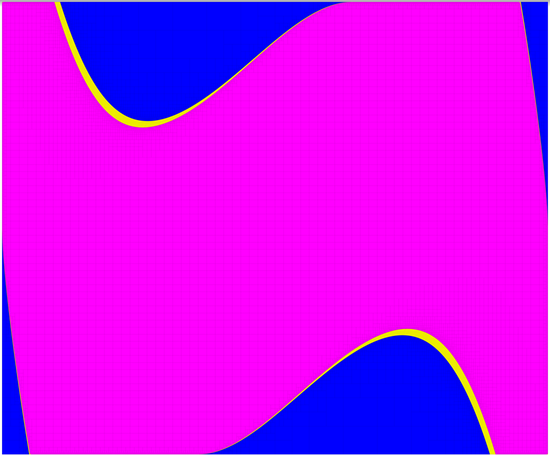
$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \left(1 - x_1^2\right) \cdot x_2 - x_1 \end{cases}$$



Positive invariant sets: $\text{Inv}^{+}(\mathbb{X})$ with $\mathbb{X} = [-4,4] \times [-4,4]$.

The *largest positive invariant set* in $\mathbb{X} \subset \mathbb{R}^n$ is:

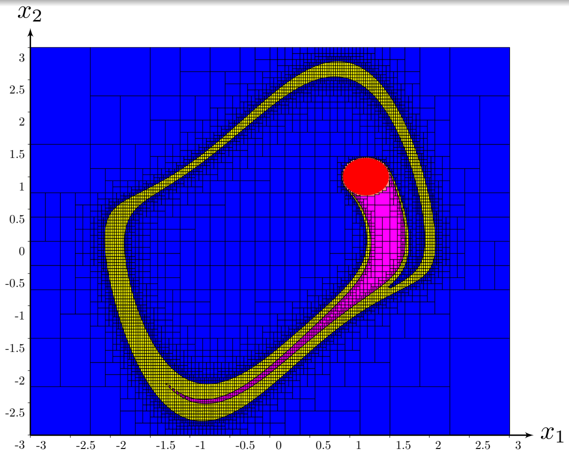
$$\mathrm{Inv}^{+}(\mathbb{X}) = \{\mathbf{x}_0 \mid \forall t \geq 0, \varphi(t, \mathbf{x}_0) \in \mathbb{X}\}.$$



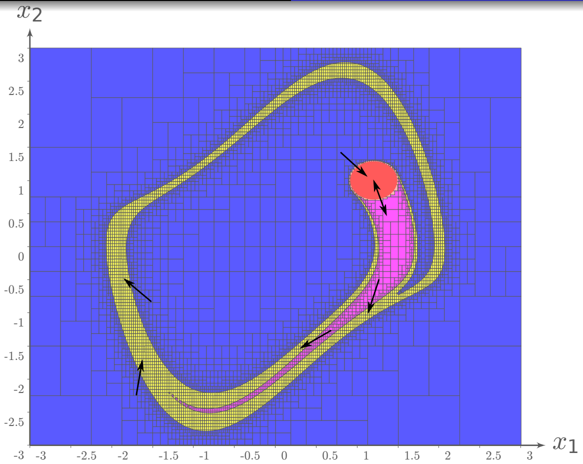
$\mathrm{Inv}^{+}(\mathbb{X})$ with $\mathbb{X} = [-4,4] \times [-4,4]$.

We have

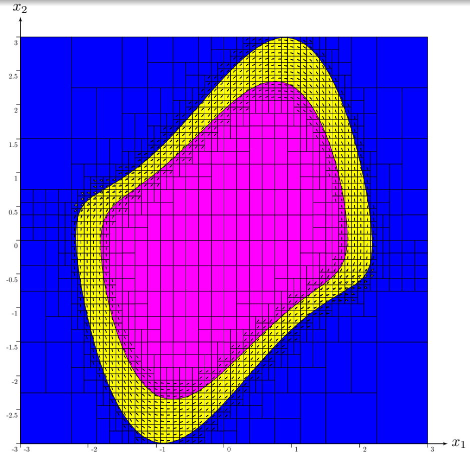
$$\begin{aligned}\text{Forw}(\mathbb{A}) &= \overline{\left\{\mathbf{x} \mid \exists t \geq 0, \varphi_\gamma(-t,\mathbf{x}) \in \mathbb{A}\right\}} \\ &= \overline{\left\{\mathbf{x} \mid \forall t \geq 0, \varphi_\gamma(-t,\mathbf{x}) \in \overline{\mathbb{A}}\right\}} \\ &= \overline{\text{Inv}^-\left(\overline{\mathbb{A}}\right)}\end{aligned}$$



Forw($\mathbb{A}$)



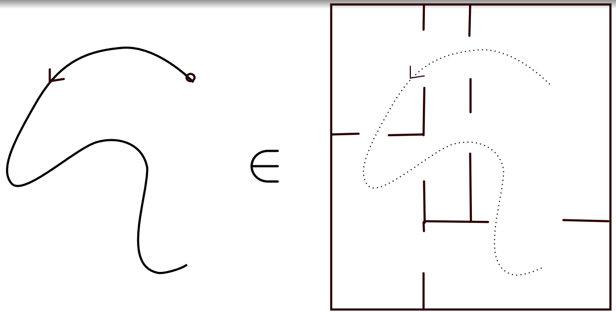
Forw($\mathbb{A}$)



$$\text{Forw}(\mathbb{X}), \mathbb{X} = [-1,1] \times [-1,1]$$

Mazes

A maze [5][2] is a set of trajectories.



The trajectory $\mathbf{x}(\cdot)$ belongs to the maze $[\mathbf{x}](\cdot)$

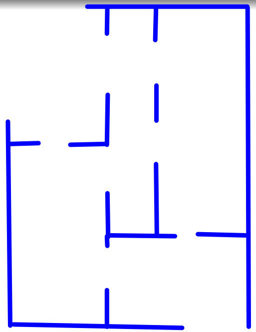
Here, a **maze** $\mathscr{L}$ is composed of

- A paving $\mathscr{P}$
- Doors between adjacent boxes

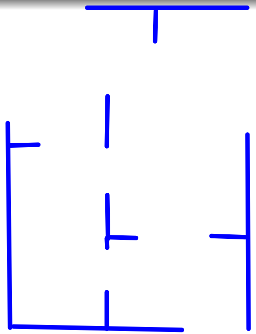
The set of mazes forms a lattice with respect to $\subset$.

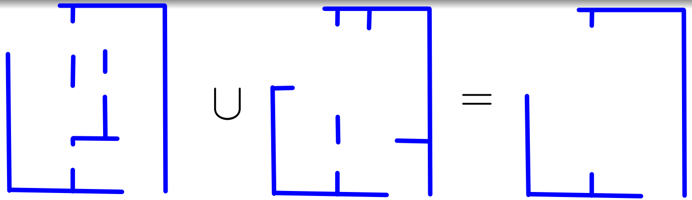
$\mathscr{L}_a \subset \mathscr{L}_b$ means :

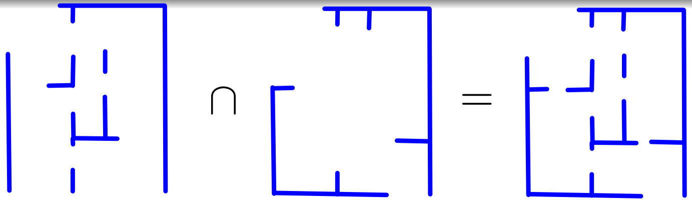
- the boxes of $\mathscr{L}_a$ are subboxes of the boxes of $\mathscr{L}_b$.
- The doors of $\mathscr{L}_a$ are thinner than those of $\mathscr{L}_b$.



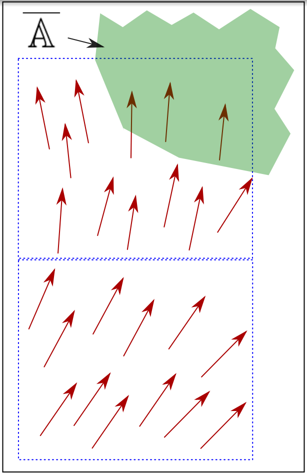
$\supset$

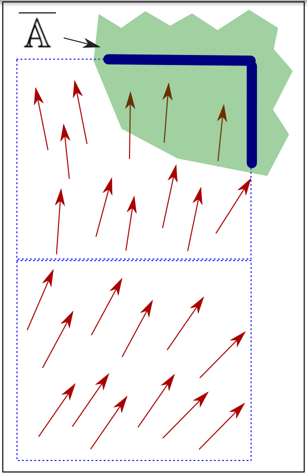


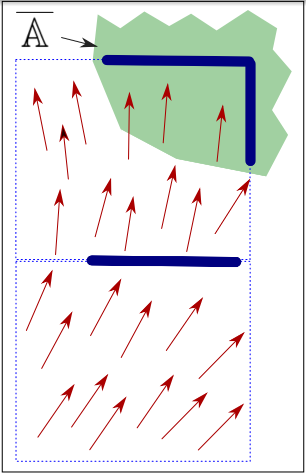


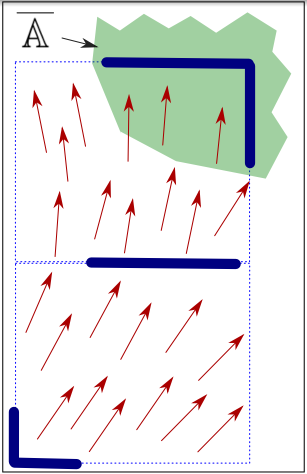


Contract trajectories that never go to $\overline{\mathbb{A}}$

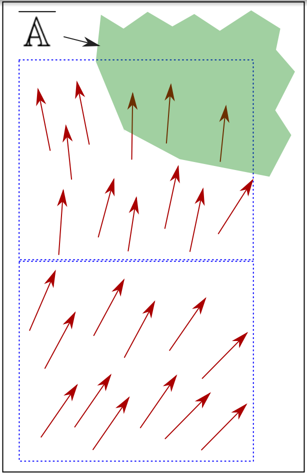

$\overline{\mathbb{A}}$

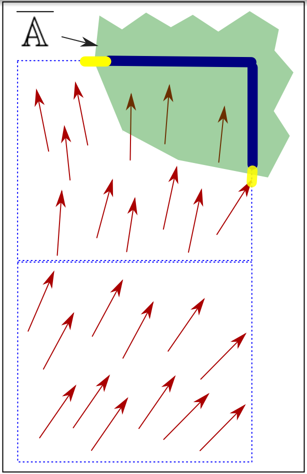

A

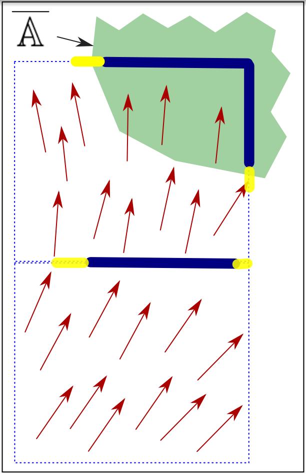

A

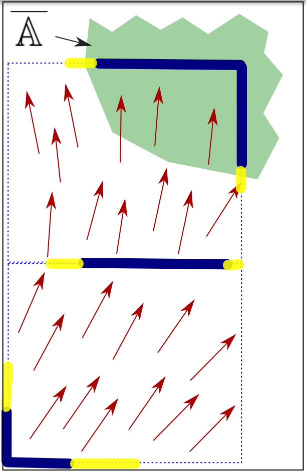

A

Contract trajectories that possibly go to $\overline{\mathbb{A}}$

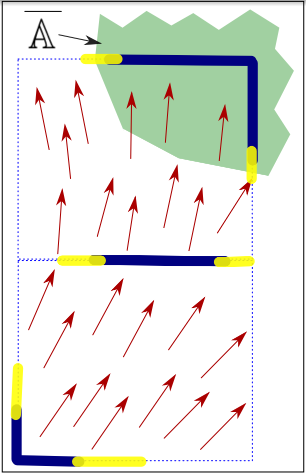

$\overline{\mathbb{A}}$

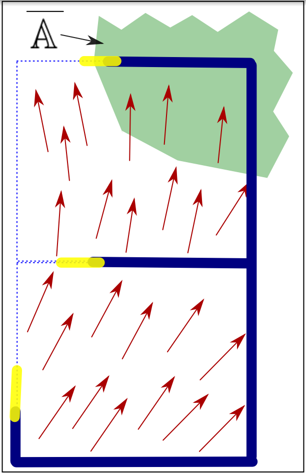

$\overline{\mathbb{A}}$

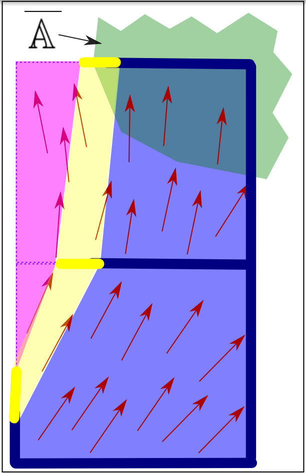

A


A

Getting the largest positive invariant set

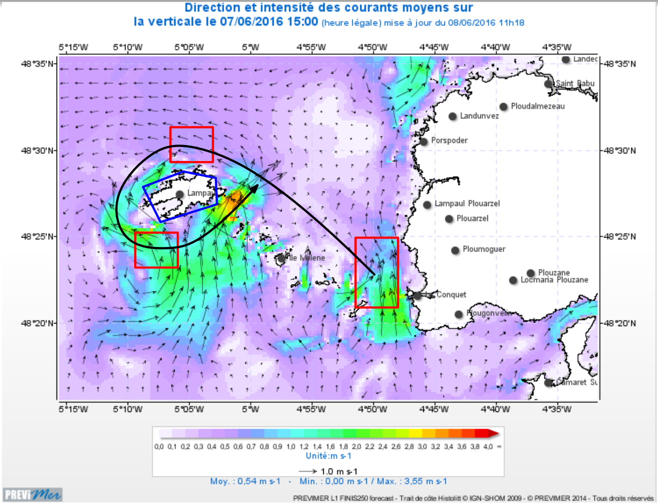

A


A

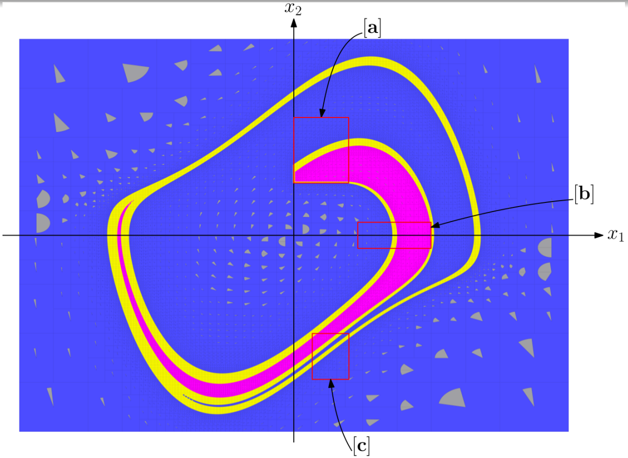

A

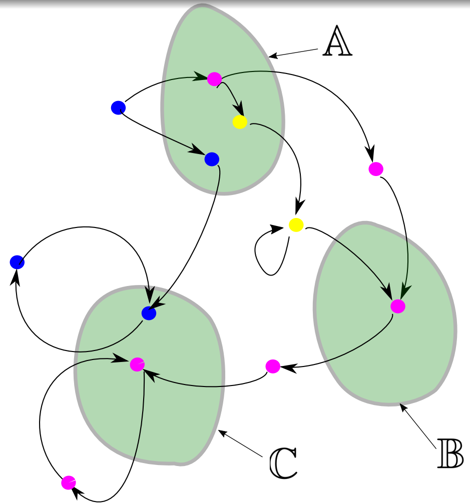
Kleene approach

Motivation



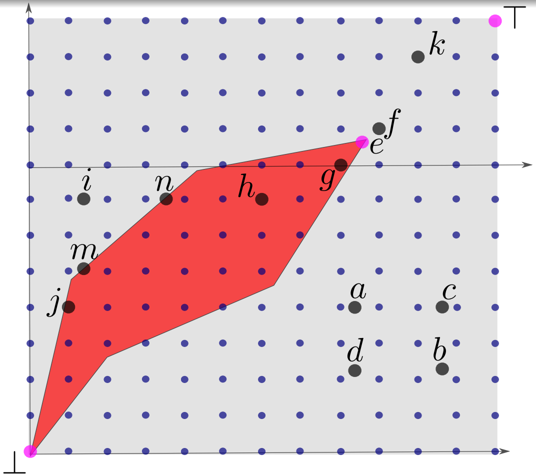
Visiting the three red boxes using a buoy that follows the currents is an Eulerian state estimation problem


x_2
[a]
[b]
x_1
[c]


A
B
C

Lattice

A *lattice* $(\mathscr{L}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds [1].
A *machine lattice* $(\mathscr{L}_M, \leq)$ of $\mathscr{L}$ is complete sublattice of $(\mathscr{L}, \leq)$ which is finite.


⊤
k
f
e
g
i
n
h
m
j
a
c
d
b
⊥

Kleene algebra

Kleene algebra	$(\mathscr{K},+,\cdot,*)$
Addition	$a+b$
Product	$a\cdot b$
Associativity	$a+(b+c)=(a+b)+c$
	$a\cdot(b\cdot c)=(a\cdot b)\cdot c$
Commutativity	$a+b=b+a$
Distributivity	$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$
	$(b+c)\cdot a=(b\cdot a)+(c\cdot a)$
zero	$a+\perp=a$
One	$a\cdot\top=\top\cdot a=a$
Annihilation	$a\cdot\perp=\perp\cdot a=\perp$
Idempotence	$a+a=a$
Partial order	$a\leq b\Leftrightarrow a+b=b$
Kleene star	$a^*=\top+a+a\cdot a+a\cdot a\cdot a+\ldots$

A Kleene algebra $\mathscr{K}\ (\leq, +, \cdot, *, \perp, \top)$ is a lattice.
We can also define the machine Kleene algebra $(\mathscr{K}_M, \leq)$ of $\mathscr{K}$.

Automorphism

Given a lattice $(\mathscr{L}, \wedge, \vee, \bot, \top)$, an *automorphism* of $\mathscr{L}$ is a function $f : \mathscr{L} \to \mathscr{L}$ such that

$$\begin{array}{llcl} \text{(i)} & f(\top) & = & \top \\ \text{(ii)} & f(a \wedge b) & = & f(a) \wedge f(b) \end{array}$$

We denote by $\mathscr{A}(\mathscr{L})$ the set of automorphisms of $\mathscr{L}$.

Example. $f(\mathbb{A}) = \varphi(1, \mathbb{A})$ is an automorphism of $(\mathscr{P}(\mathbb{R}^n), \cap, \cup, \perp, \top)$

- $f(\mathbb{A} \cap \mathbb{B}) = f(\mathbb{A}) \cap f(\mathbb{B})$
- $f(\mathbb{R}^n) = \mathbb{R}^n$

- $\mathbb{A} \cap \varphi(1, \mathbb{A})$ is an automorphism
- $\mathbb{A} \cap \varphi(1, \mathbb{A}) \cap \varphi^2(1, \mathbb{A})$ is an automorphism
- $(\varphi(1, \mathbb{A}))^*$ is an automorphism

The set of automorphisms forms a Kleene algebra

Kleene algebra	$(\mathscr{A}(\mathscr{L}),\wedge,\circ,*)$
Addition	$f\wedge g$
Product	$f\circ g$
Associativity	$f\wedge(g\wedge h)=(f\wedge g)\wedge h$
	$f\circ(g\circ h)=(f\circ g)\circ h$
Commutativity	$f\wedge g=f\wedge g$
Distributivity	$f\circ(g\wedge h)=(f\circ g)\wedge(f\circ h)$
	$(g\wedge h)\circ f=(g\circ f)\wedge(h\circ f)$
zero	$f\wedge\top=f$
One	$f\circ\mathrm{Id}=\mathrm{Id}\circ f=f$
Annihilation	$f\circ\top=\top$
Idempotency	$f\wedge f=f$
Partial order	$f\geq g\Leftrightarrow f\wedge g=g$
Kleene star	$f^*=\mathrm{Id}\wedge f\wedge f^2\wedge f^3\wedge\ldots$

Factorization

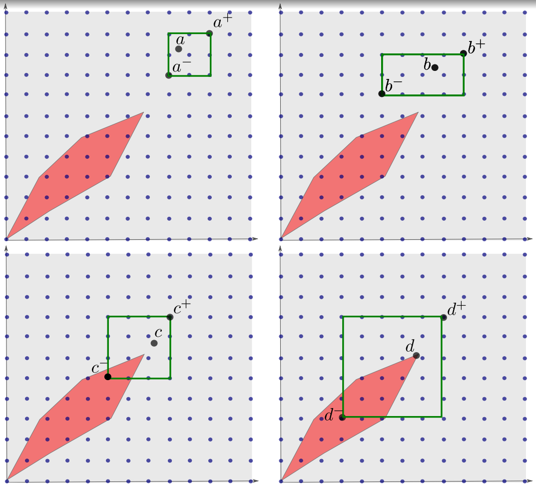
We want to compute expressions, such as

$$f^{*}(a) \wedge (g^{*}(b) \vee h^{*}(a))^{*}.$$

We have

$$f^* \wedge f^* = f^*$$
$$(f^*)^* = f^*$$
$$(f^* \wedge g^*)^* = (f \wedge g)^*$$
$$f^* \circ (f \circ g^*)^* = (f \wedge g)^*$$

Algorithm

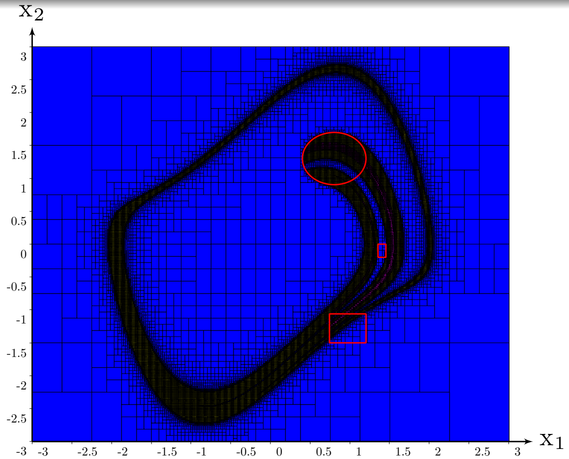

a^+
a
a^-
b^+
b
b^-
c^+
c
c^-
d^+
d
d^-

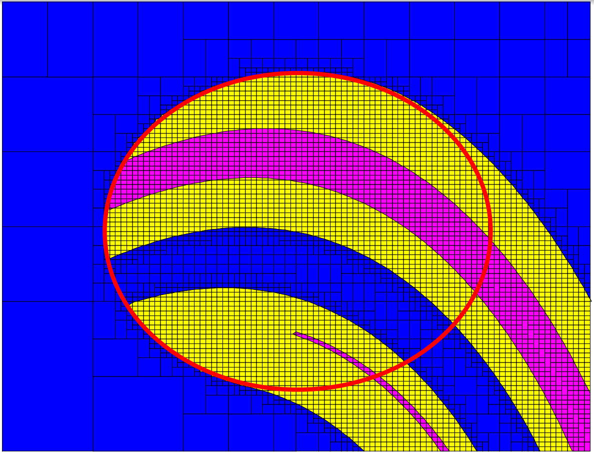
Dynamical systems

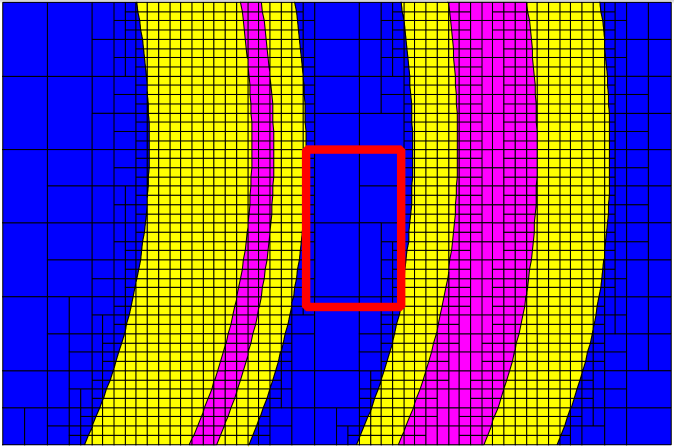
Path planning reach set

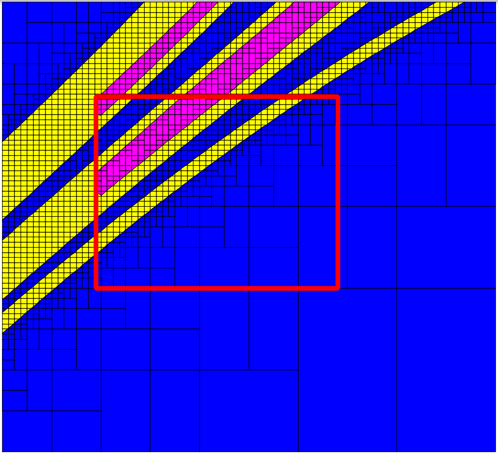
We want the set $\mathbb{X}$ of all paths that start in $\mathbb{A}$, avoid $\mathbb{B}$ and reach $\mathbb{C}$. We have

$$\mathbb{X} = \left(\overline{\overleftarrow{f_{\gamma|\overline{\mathbb{B}}}}^{*}\left(\overline{\mathbb{A}}\right)} \cap \overline{\overrightarrow{f_{\gamma|\overline{\mathbb{B}}}}^{*}\left(\overline{\mathbb{C}}\right)}\right)$$


x_2
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
x_1







Control reach set

Consider the system:

$$\mathscr{S} : \dot{\mathbf{x}}(t) = \gamma(\mathbf{x}(t), u),\ u \in \{0, 1\}$$

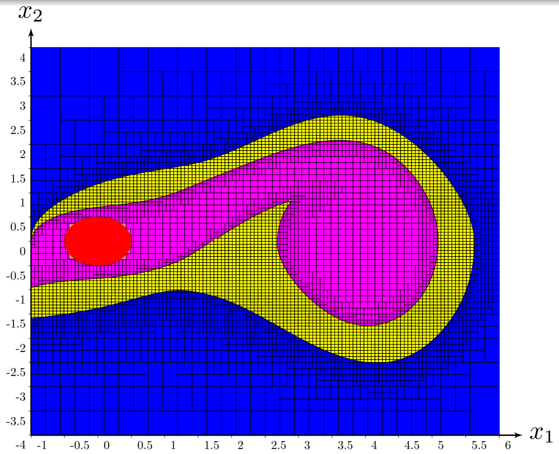
We want to compute the largest set $\mathbb{X}$ that can be reached from the set $\mathbb{A}$.

We have

$$\mathbb{X} = \overline{\left(\overleftarrow{f_1} \circ \overleftarrow{f_0}\right)^* (\overline{\mathbb{A}})}$$

Car on the hill system [3] :

$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -9.81\sin(0.55\sin(1.2x_1) - 0.6\sin(1.1x_1)) - 0.7x_2 + u \end{cases}$$


x_2
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
x_1

B. A. Davey and H. A. Priestley.
Introduction to Lattices and Order.
Cambridge University Press, (ISBN 0521784514), 2002.

T. Le Mézo L. Jaulin and B. Zerr.
Bracketing the solutions of an ordinary differential equation with uncertain initial conditions.
Applied Mathematics and Computation, 318:70–79, 2018.

M. Lhommeau, L. Jaulin, and L. Hardouin.
Capture Basin Approximation using Interval Analysis.
International Journal of Adaptative Control and Signal Processing, 25(3):264–272, 2011.

T. Le Mézo, L. Jaulin, and B. Zerr.
Inner approximation of a capture basin of a dynamical system.
In *Abstracts of the 9th Summer Workshop on Interval Methods*. Lyon, France, June 19-22, 2016.

T. Le Mézo, L. Jaulin, and B. Zerr.
An interval approach to compute invariant sets.
IEEE Transaction on Automatic Control, 62:4236–4243, 2017.