Set membership methods for analysing the robustness of complex systems

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1 Interval analysis

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$, a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\mathsf{abs}\left([-7,1]\right) &= [0,7]
\end{aligned}
$$

If f is given

Algorithm f(in: $\mathbf{x} = (x_1, x_2, x_3)$, out: y)

$z := x_1$;
for $k := 0$ to 100
    $z := (\cos x_2)\,(\sin(z) + kx_3)$;
next;
$y := \sin(zx_1)$;

Its interval extension is

Algorithm $[f]$(in: $[\mathbf{x}] = ([x_1], [x_2], [x_3])$, out: $[y]$)
$[z] := [x_1]$;
for $k := 0$ to 100
$[z] := (\cos [x_2]) * (\sin ([z]) + k * [x_3])$;
next;
$[y] := \sin([z] \cdot [x_1])$;

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^{+} \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0$$

Interval analysis can be used to prove properties.

Example

$$
\begin{aligned}
& (f(\mathbf{x}) < 0 \Rightarrow g(\mathbf{x}) \geq 0) \\
\Leftrightarrow \quad & \forall \mathbf{x},\ f(\mathbf{x}) \geq 0 \text{ or } g(\mathbf{x}) \geq 0 \\
\Leftrightarrow \quad & \forall \mathbf{x},\ \max\left(f(\mathbf{x}), g(\mathbf{x})\right) \geq 0
\end{aligned}
$$

2 Vaimos

Vaimos (IFREMER and ENSTA)

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}, \mathbf{z}\right),$$

where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the input vector, $\mathbf{z} \in [\mathbf{z}]$ is the perturbation vector

With the controller $\mathbf{u} = \mathbf{g}(\mathbf{x}, \mathbf{z})$,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x}, \mathbf{z}), \mathbf{z}) \\ &= \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{z}). \end{aligned}$$

Define

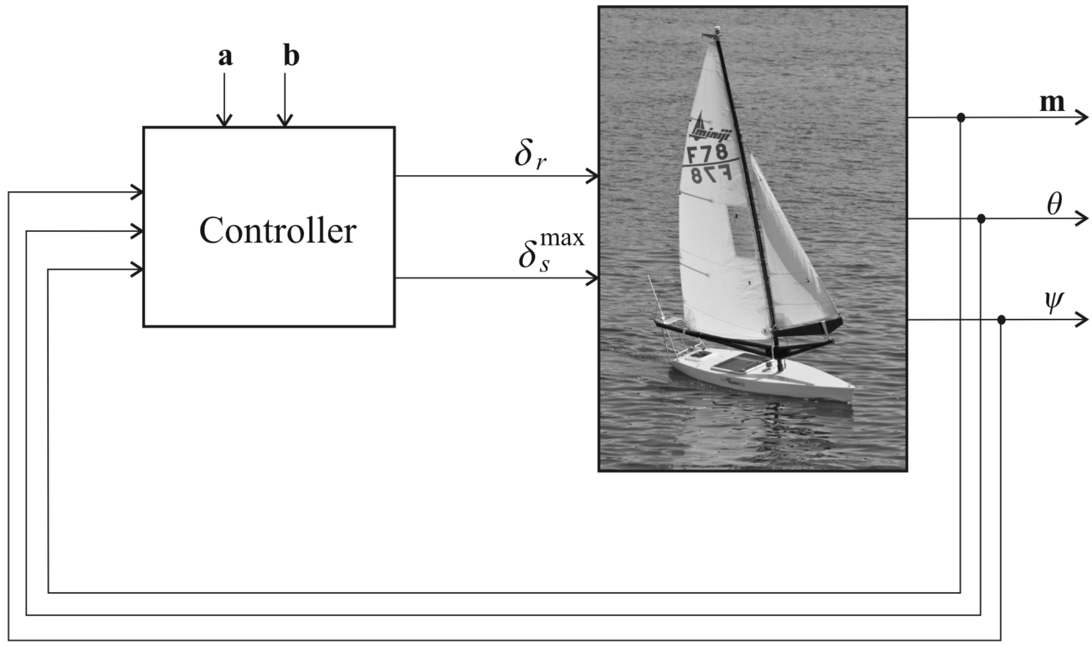
$$\mathbf{F}(\mathbf{x}) = \left\{ \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{z}), \mathbf{z} \in [\mathbf{z}] \right\}.$$

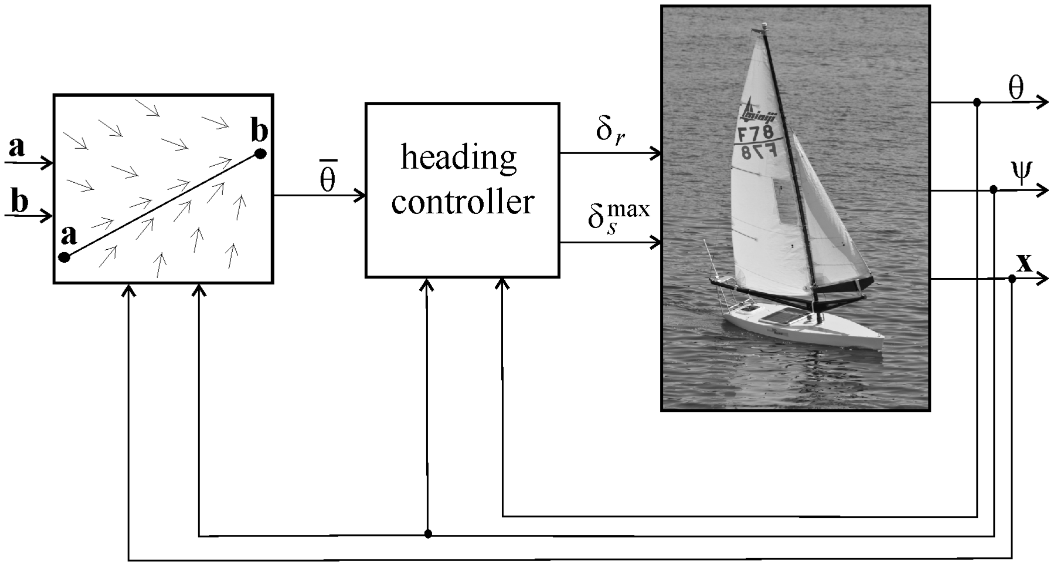
The robot satisfies.

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

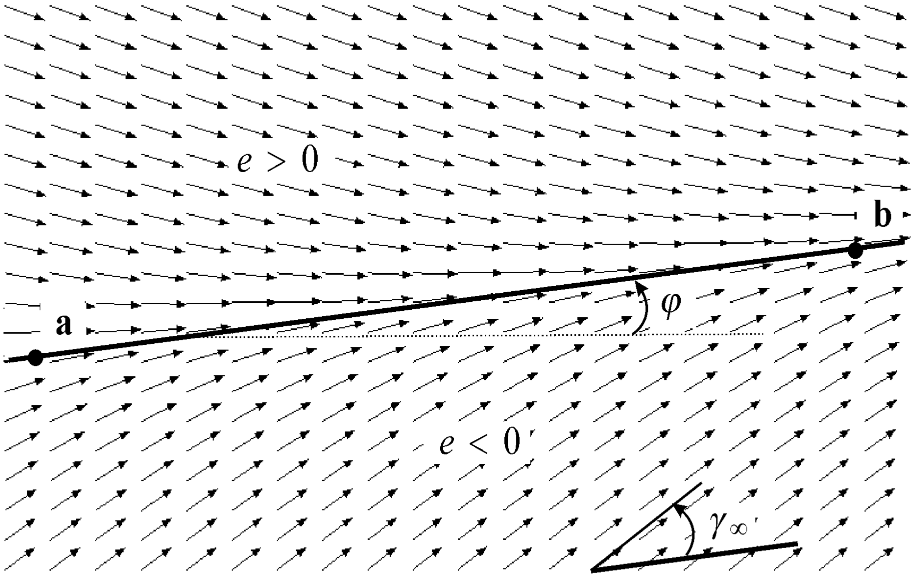
which is *a differential inclusion*.

2.1 Line following


a
b
Controller
δ_r
δ_s^{max}
m
θ
ψ

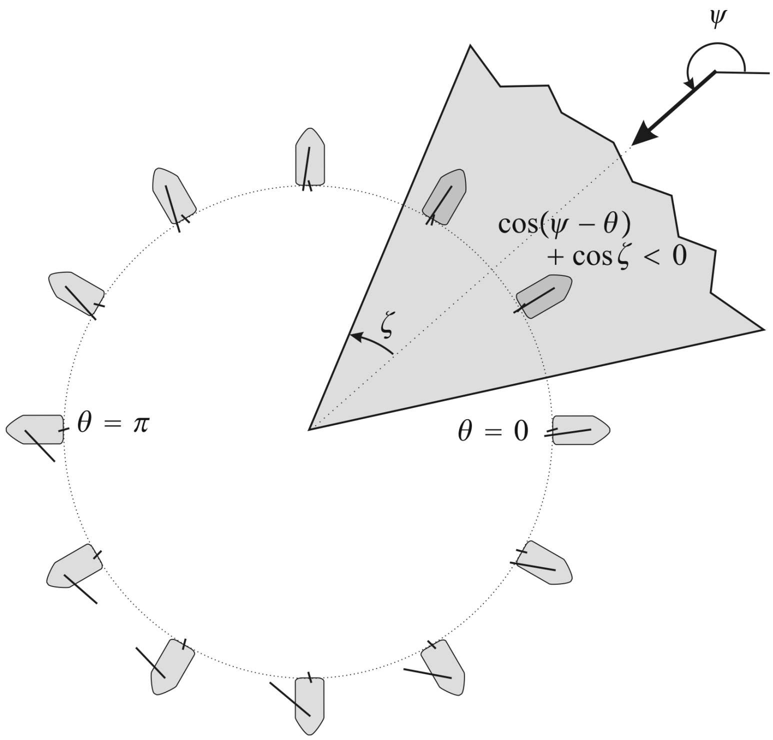


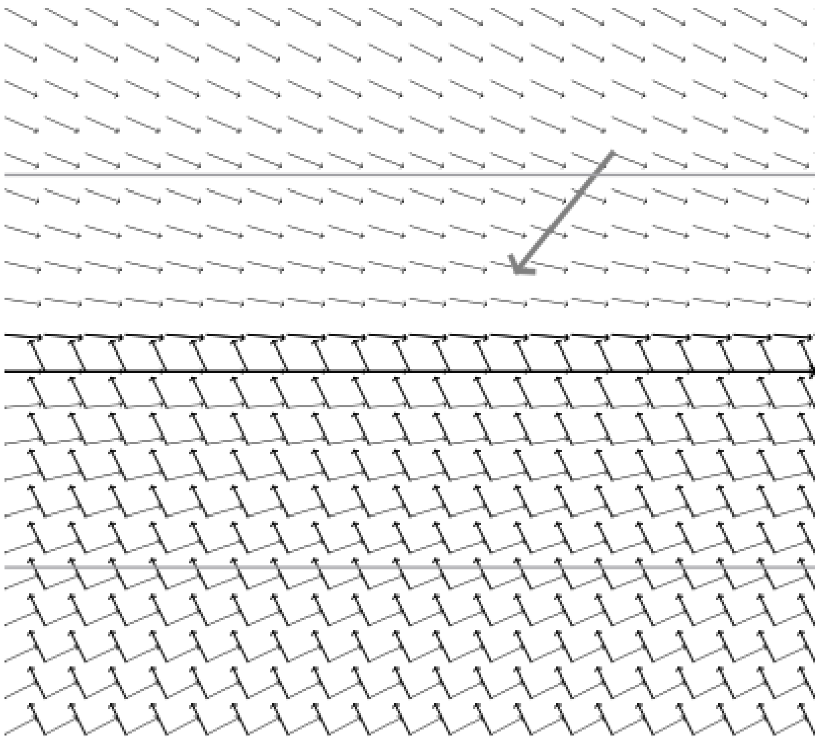
a
b
a
b
$\bar{\theta}$
heading
controller
δ_r
δ_s^{max}
F78
θ
ψ
x



Nominal vector field: $\theta^* = \varphi - \frac{2.\gamma_\infty}{\pi}.\text{atan}\left(\frac{e}{r}\right).$

A course θ^* may be unfeasible





2.2 Controller

Controller in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$; out: $\delta_r, \delta_s^{\text{max}}$; inout: q

$e = \det\left(\frac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}, \mathbf{m}-\mathbf{a}\right)$
if $|e| > \frac{r}{2}$ then $q = \text{sign}(e)$
$\varphi = \text{atan2}(\mathbf{b}-\mathbf{a})$
$\theta^* = \varphi - \frac{2.\gamma_\infty}{\pi}.\text{atan}\left(\frac{e}{r}\right)$
if $\cos(\psi - \theta^*) + \cos\zeta < 0$
  or $(|e| < r$ and $(\cos(\psi - \varphi) + \cos\zeta < 0))$
   then $\bar{\theta} = \pi + \psi - q.\zeta.$
   else $\bar{\theta} = \theta^*$
end
if $\cos\left(\theta - \bar{\theta}\right) \geq 0$ then $\delta_r = \delta_r^{\text{max}}.\sin\left(\theta - \bar{\theta}\right)$
else $\delta_r = \delta_r^{\text{max}}.\text{sign}\left(\sin\left(\theta - \bar{\theta}\right)\right)$
$\delta_s^{\text{max}} = \frac{\pi}{2}.\left(\frac{\cos(\psi - \bar{\theta}) + 1}{2}\right)^q.$

The sailboat robot satisfies

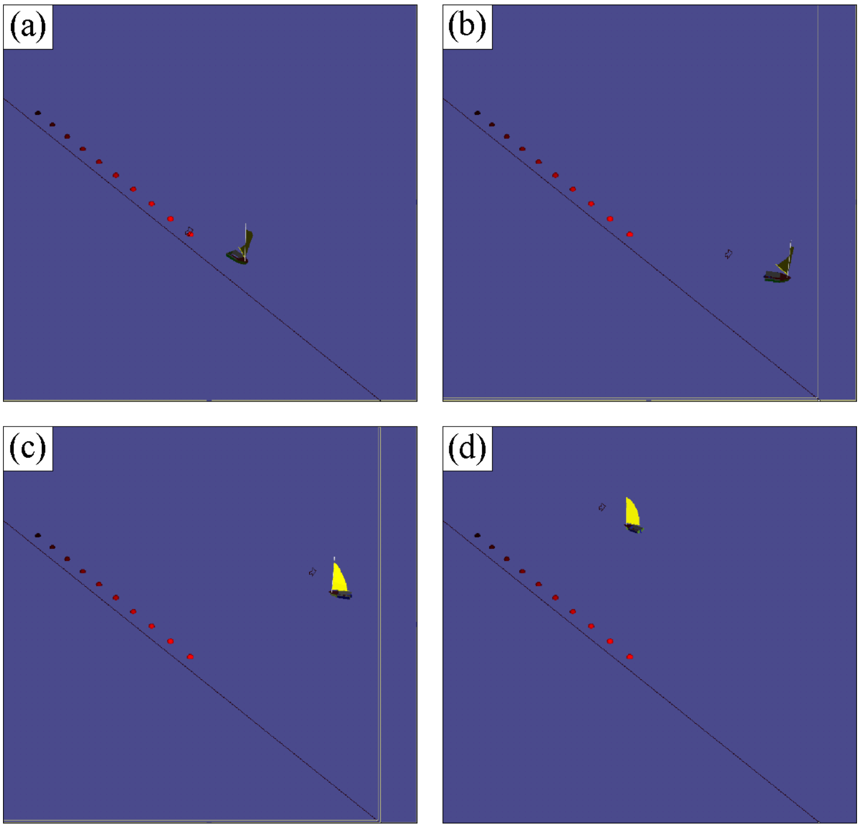
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \psi, q)$$

or equivalently

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is a differential inclusion.

2.3 Validation by simulation


(a)
(b)
(c)
(d)

3 V-stability

3.1 Principle

Stability. The system

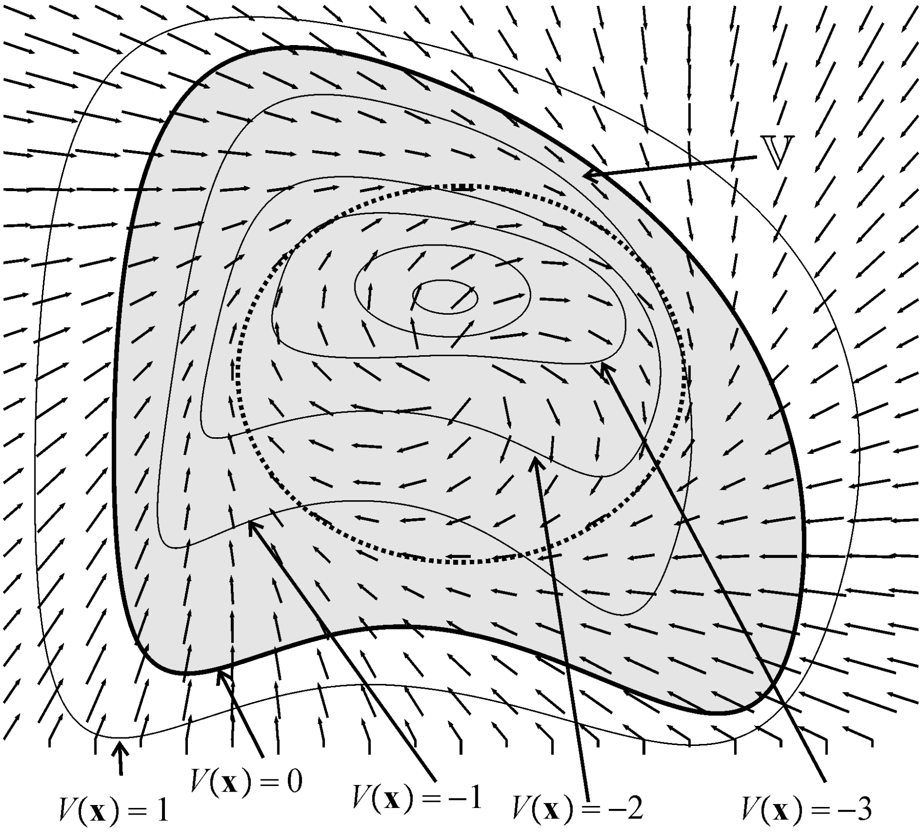
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) is there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned} \dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}. \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0} \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system is V-stable if

$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\right).$$



𝕍
$V(\mathbf{x})=1$
$V(\mathbf{x})=0$
$V(\mathbf{x})=-1$
$V(\mathbf{x})=-2$
$V(\mathbf{x})=-3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$

(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$
\begin{aligned}
& \left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\;\; & \left(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\;\; & \forall \mathbf{x},\; \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow\;\; & \forall \mathbf{x},\; \min\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}), V(\mathbf{x})\right) < 0
\end{aligned}
$$

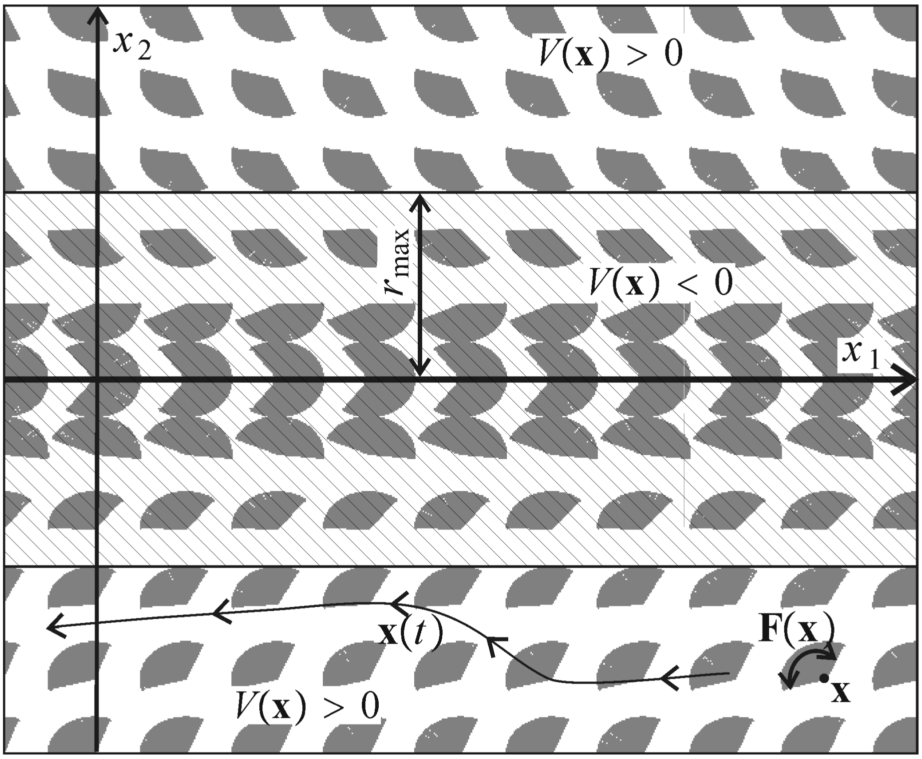
Theorem. We have

$$\underbrace{\left\{\begin{array}{c}\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{f}(\mathbf{x})\geq 0\\ V(\mathbf{x})\geq 0\end{array}\right. \text{ inconsistent }}_{\Leftrightarrow\ \forall \mathbf{x},\ \min\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{f}(\mathbf{x}),V(\mathbf{x})\right)<0} \Leftrightarrow \dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

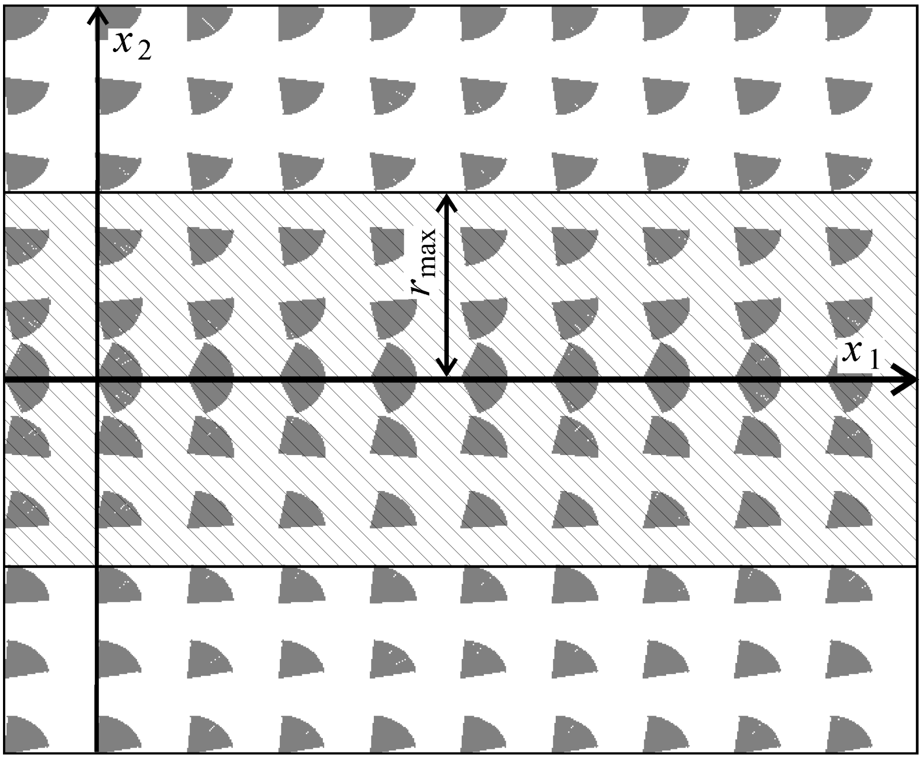
Interval method could easily prove the V-stability.

Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{a} \geq 0 \\ \mathbf{a} \in \mathbf{F}\left(\mathbf{x}\right) \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} \in \mathbf{F}\left(\mathbf{x}\right) \text{ is } V\text{-stable}$$



Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.
$V(x) = x_2^2 - r_{\text{max}}^2$.



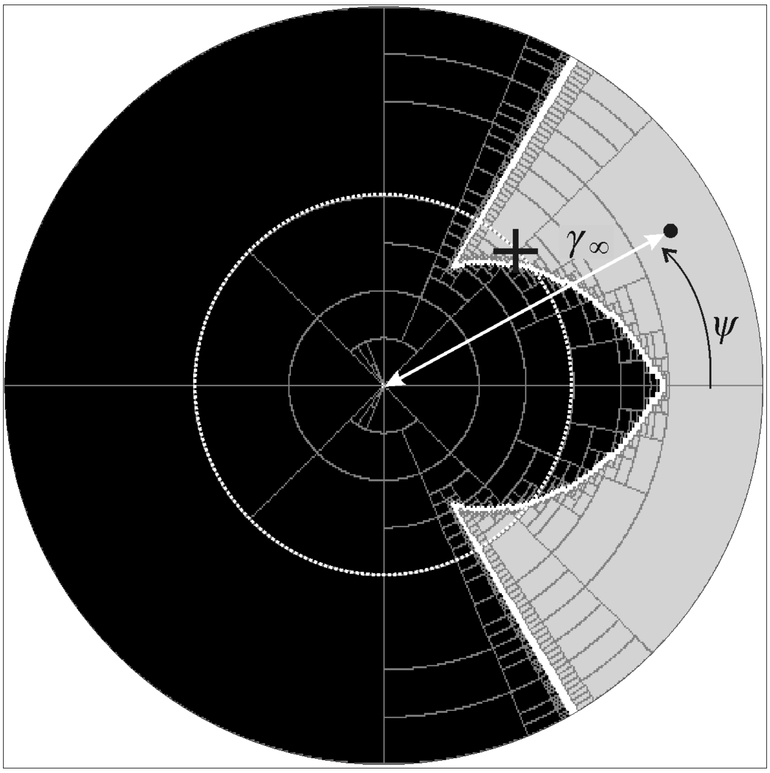
x_2
r_{max}
x_1

3.2 Parametric case

Consider the differential inclusion

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, \mathbf{p}).$$

We characterize the set $\mathbb{P}$ of all $\mathbf{p}$ such that the system is V-stable.



+
γ_∞
ψ

For Vaimos, we have found parameters for the controller such that

Property 1. If $|e(\mathbf{x})| < r_{\mathsf{max}}$ then, it will be the case for ever.

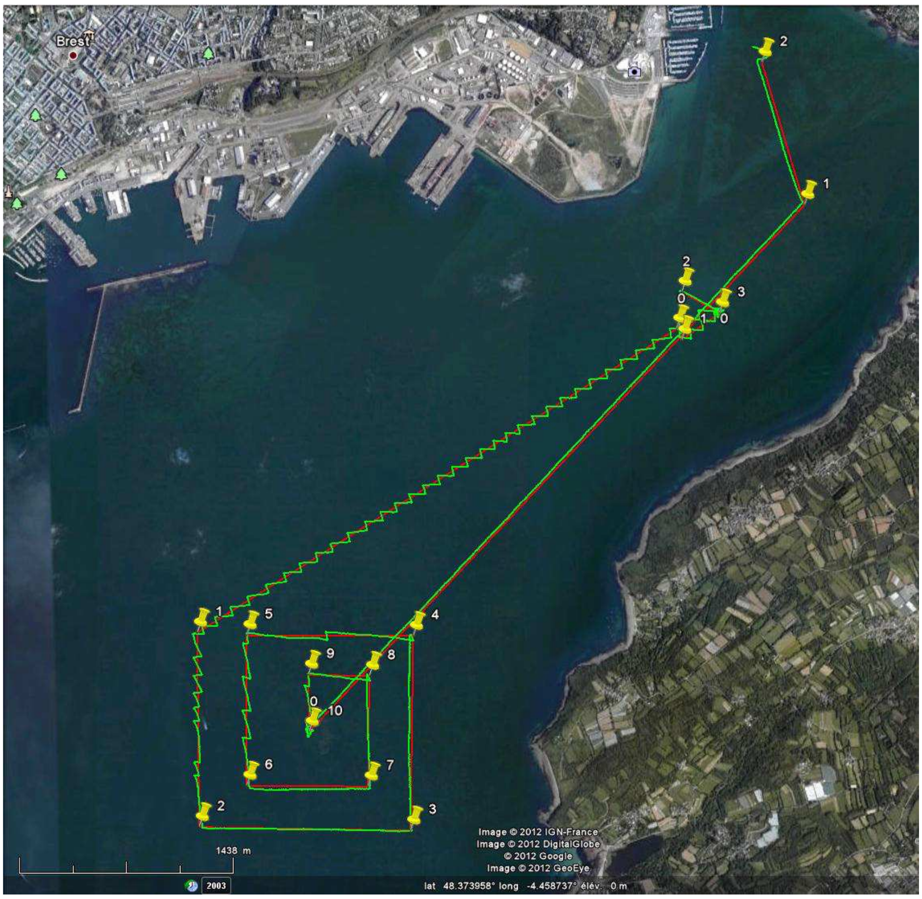
Property 2. If $|e(\mathbf{x})| > r_{\mathsf{max}}$ then $|e(\mathbf{x})|$ will decrease until $|e(\mathbf{x})| < r_{\mathsf{max}}$.

Property 3. The course is be feasible, i.e.,

$$\cos\left(\psi - \bar{\theta}\right) + \cos\zeta \geq 0.$$

3.3 Experimental validation

Brest

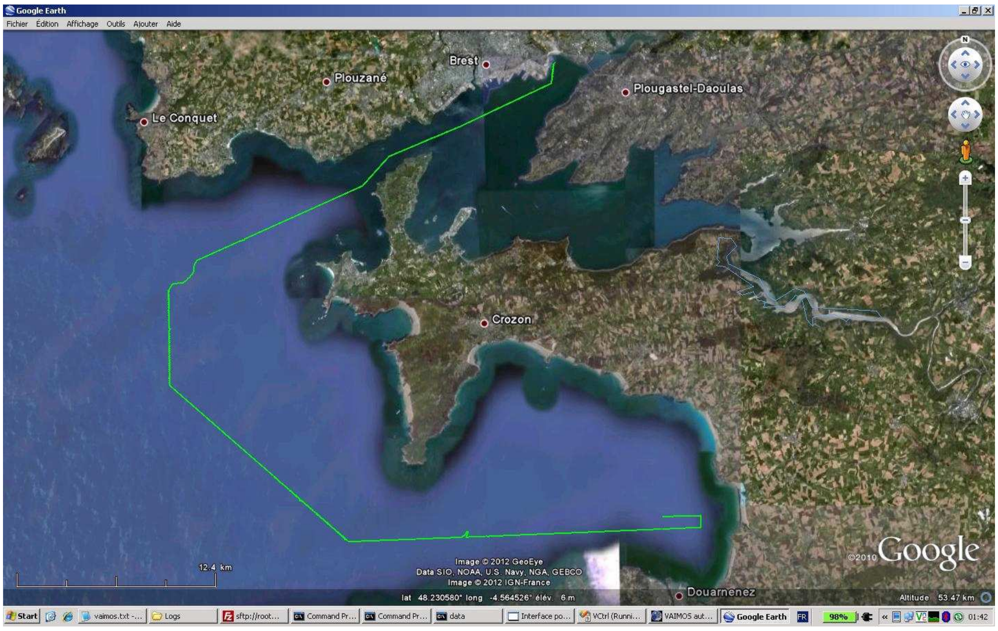


Brest: Vaimos did 27 km

Brest-Douarnenez. January 17, 2012, 8am


Brest
BR D98 305
YAMAHA


miniji
F78


Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Le Conquet
Plougastel-Daoulas
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google

Montrer la mise à l'eau

Middle of Atlantic ocean

350 km made by Vaimos in 53h, September 6-9, 2012.

Show the dashboard

Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.

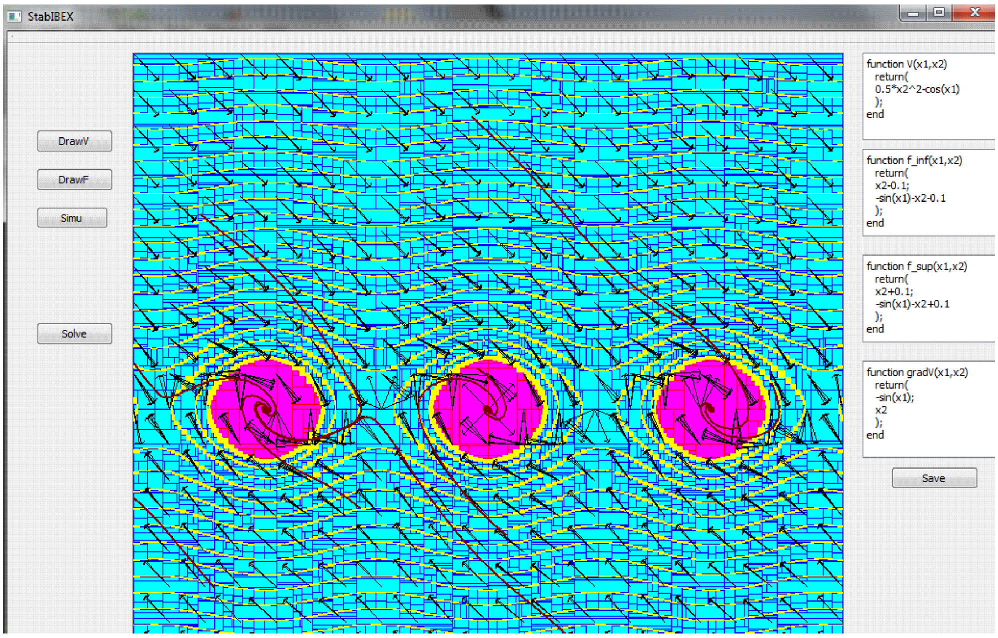
Main reference: L. Jaulin, F. Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE Transaction on Robotics, Volume 27, Issue 5.

4 Stabibex

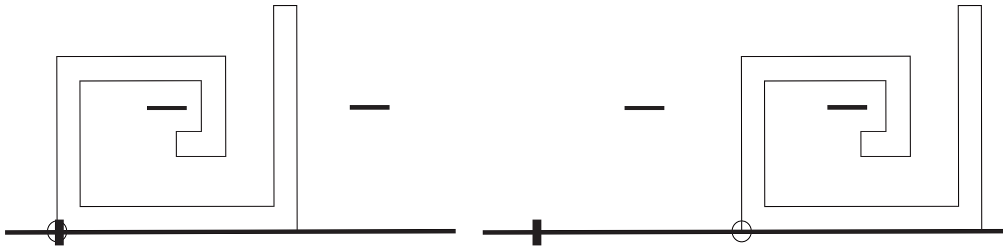
Solver using interval analysis to prove the V-stability.

Made by students from ENSTA-Bretagne.

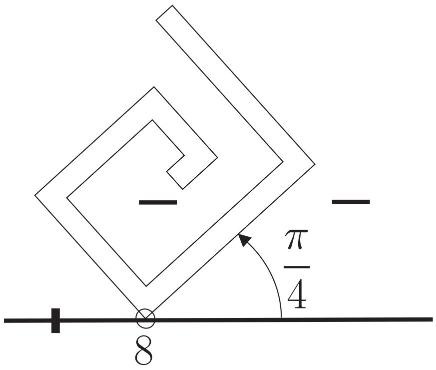
Uses the IBEX library.


StabIBEX
DrawV
DrawF
Simu
Solve
function V(x1,x2)
return(
0.5*x2^2-cos(x1)
);
end
function f_inf(x1,x2)
return(
x2-0.1;
-sin(x1)-x2-0.1
);
end
function f_sup(x1,x2)
return(
x2+0.1;
-sin(x1)-x2+0.1
);
end
function gradV(x1,x2)
return(
-sin(x1);
x2
);
end
Save

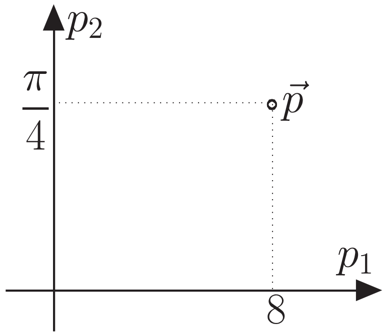
5 Path planning



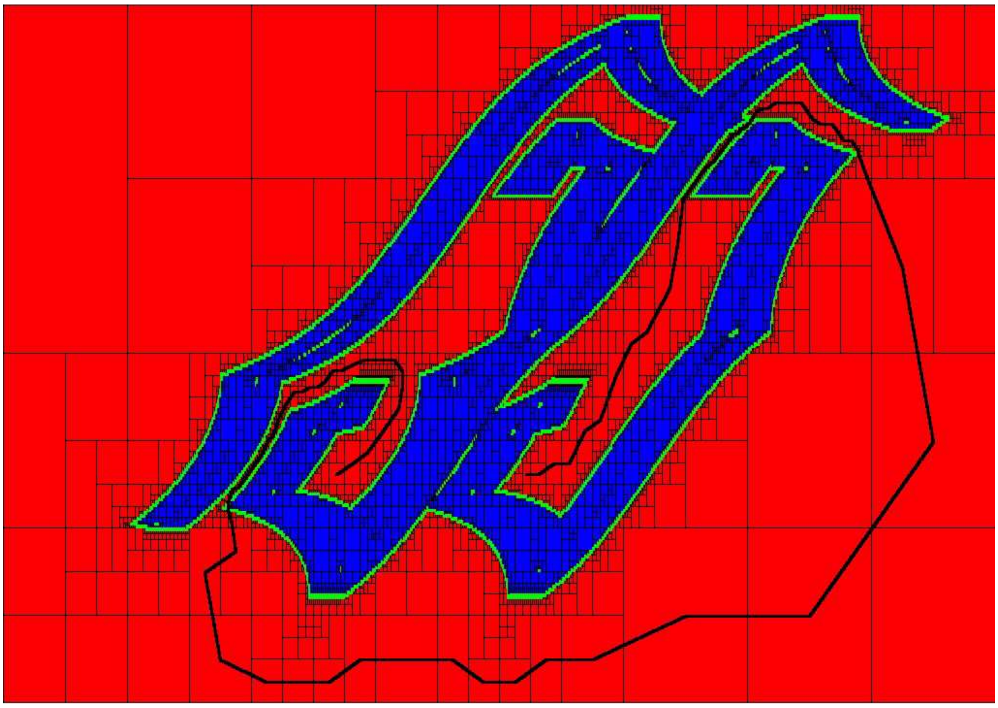
Initial configuration: $\vec{p} = (0\ 0)^{\mathrm{T}}$ Goal configuration: $\vec{p} = (17\ 0)^{\mathrm{T}}$

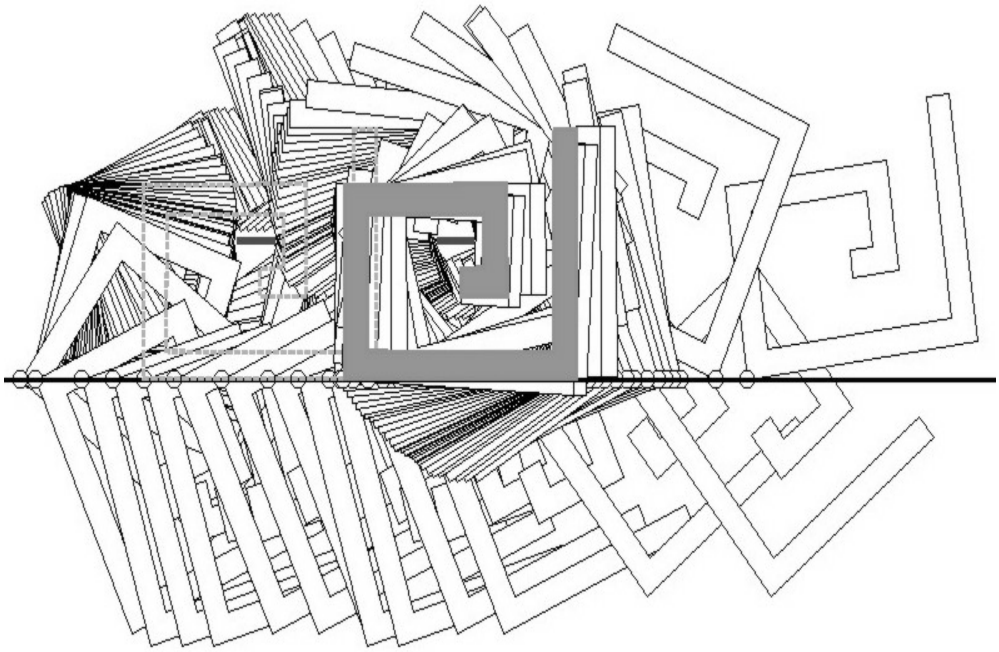


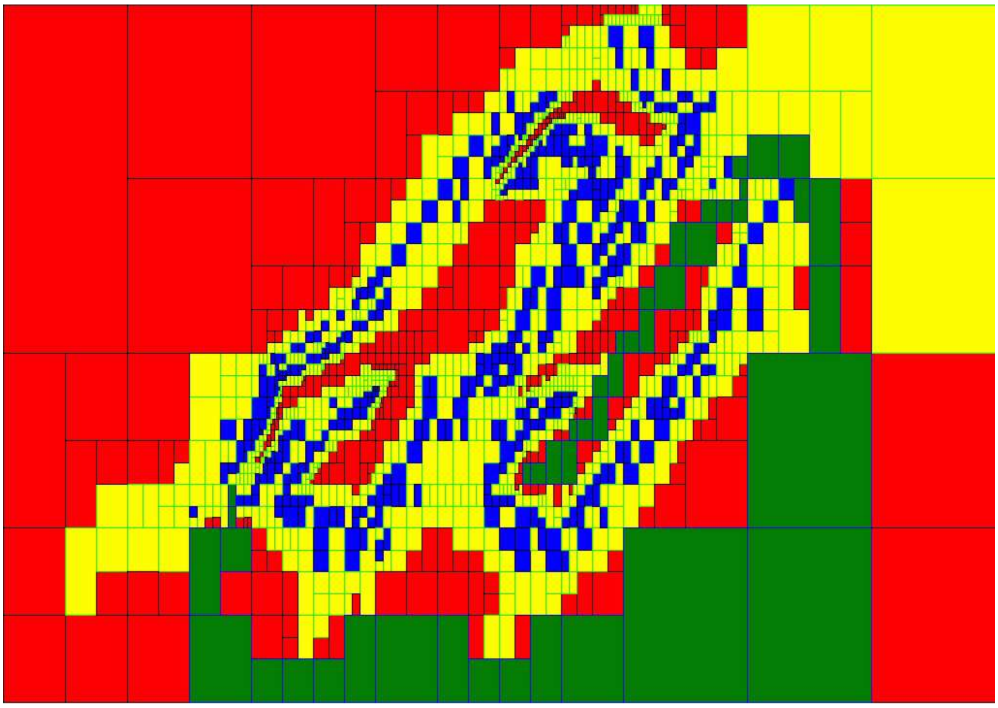
Room

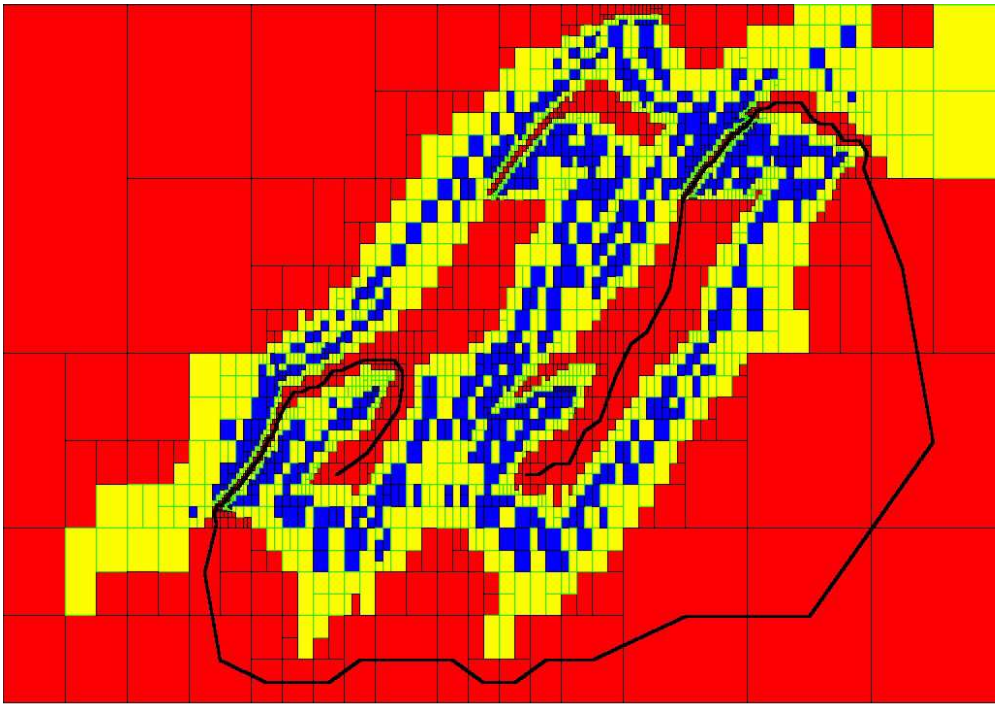


Configuration space









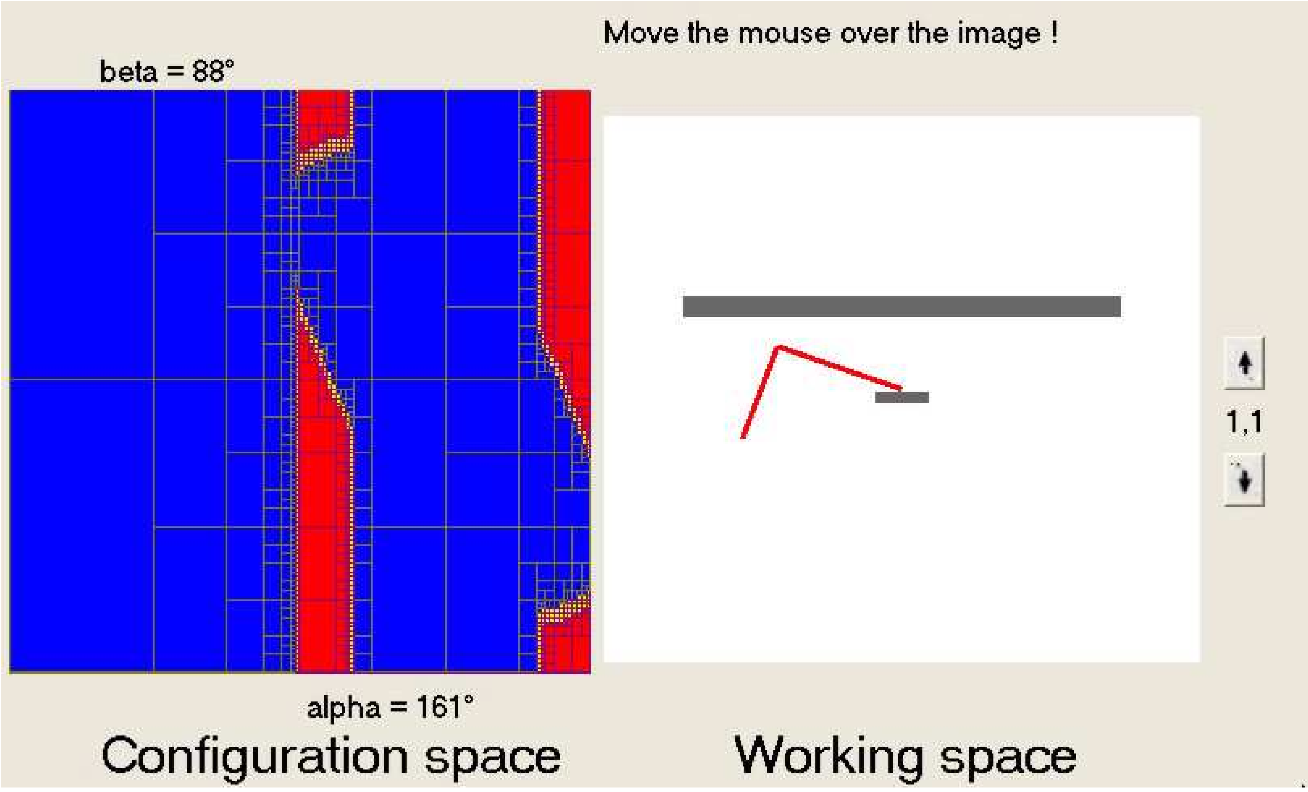
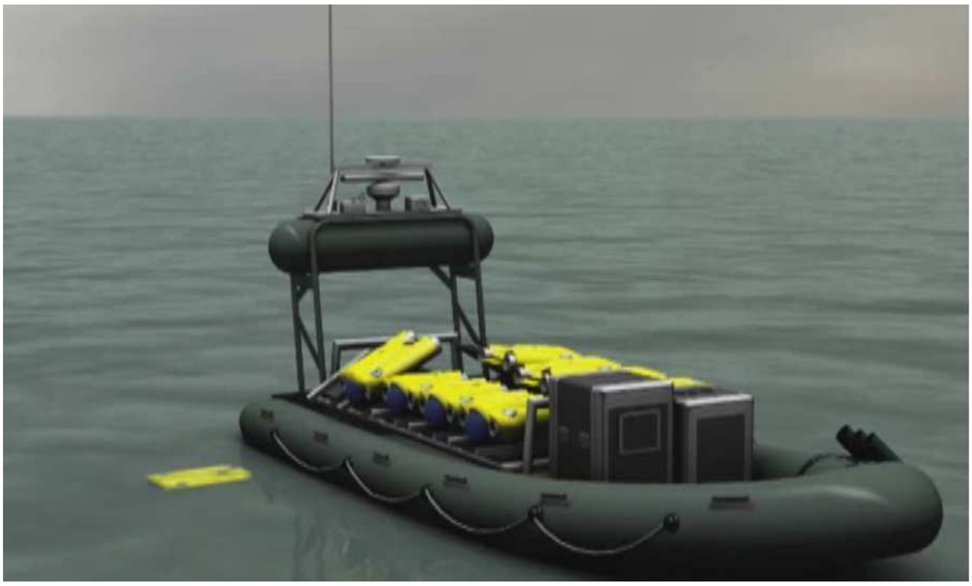
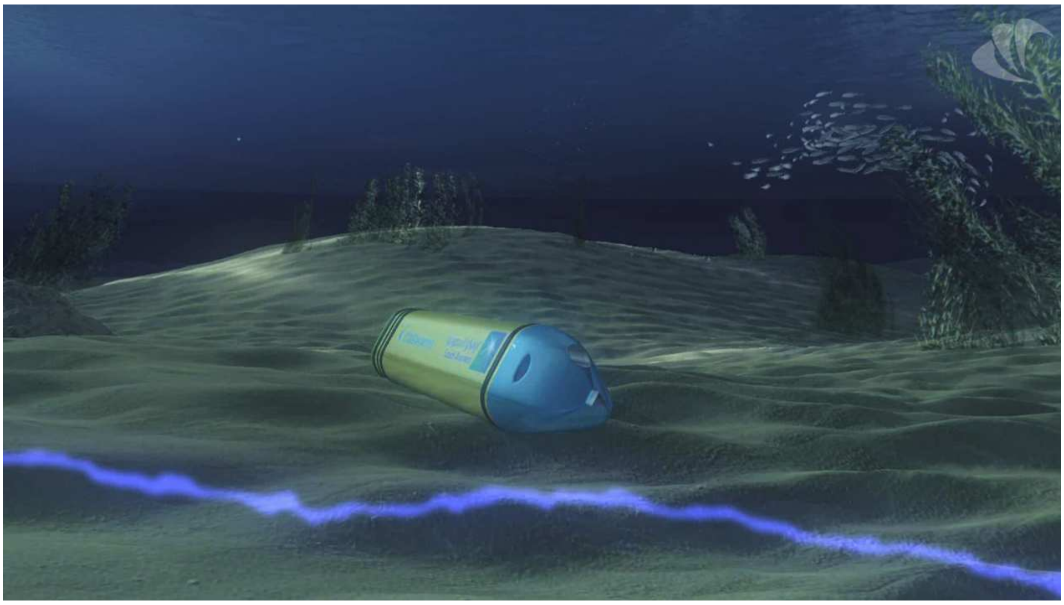


Figure 1:

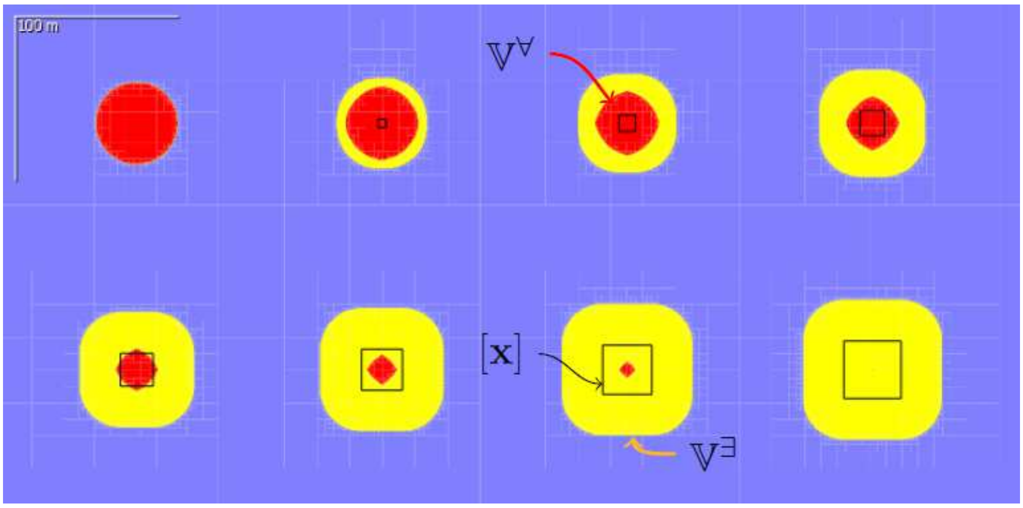
6 Reliable exploration with squads



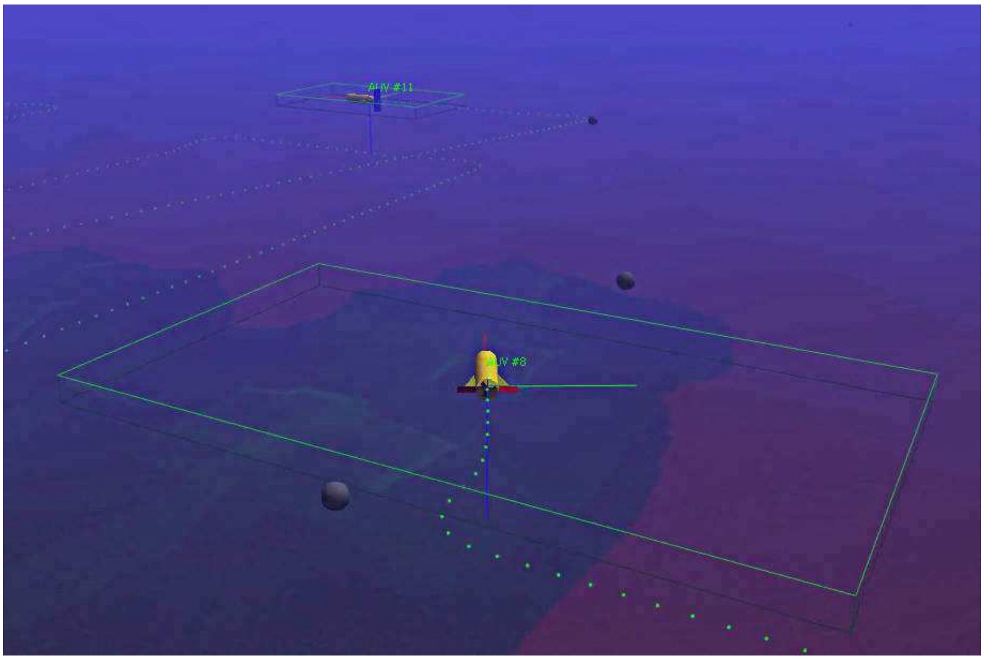
With Thales

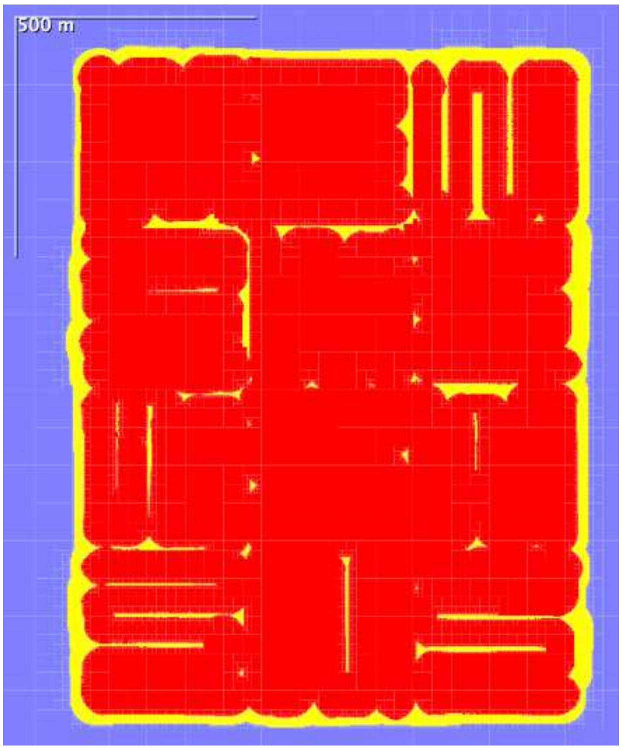


Spicerack, with CGG



100 m
$\mathbb{V}^{\forall}$
$[\mathbf{x}]$
$\mathbb{V}^{\exists}$


AUV #11
AUV #8


500 m