Computing capture tubes

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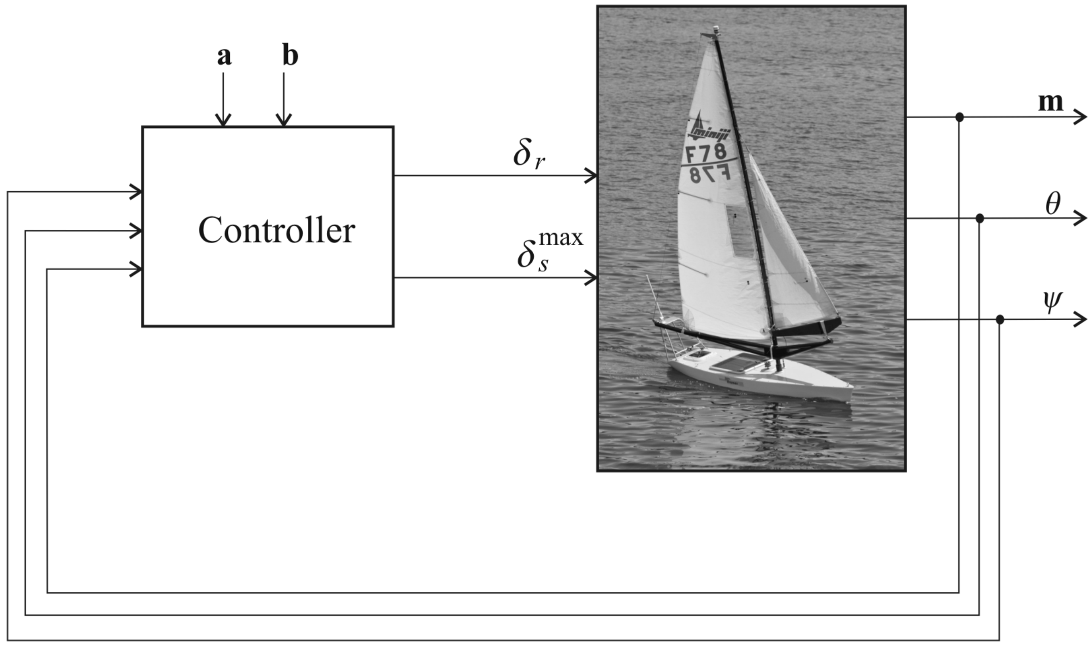
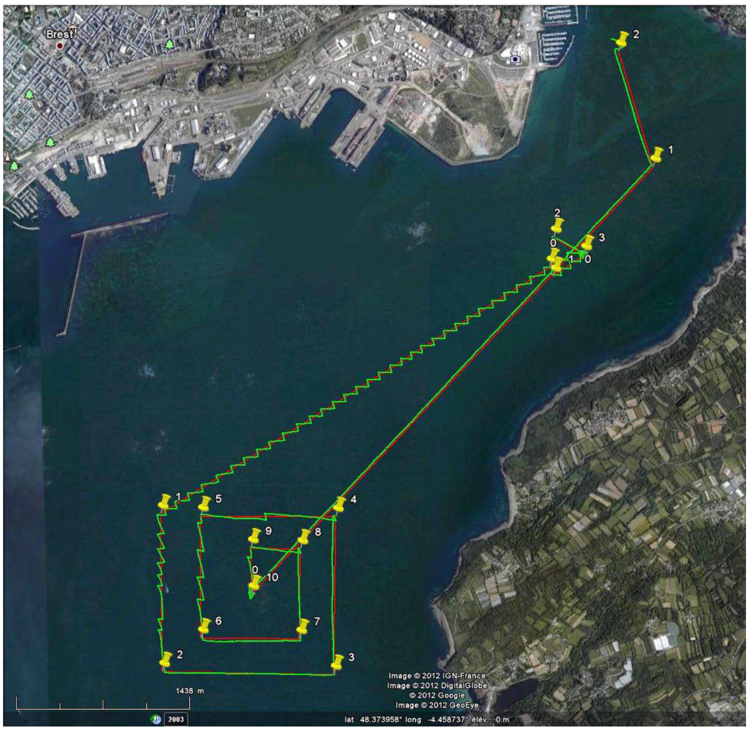
Presentation available at
http://youtu.be/wDF-RBbHepg

1 V-stability

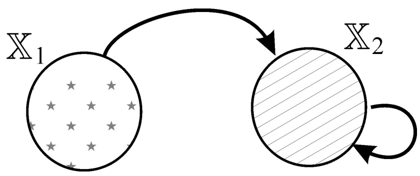
Vaimos (IFREMER and ENSTA)



a
b
Controller
δ_r
δ_s^{max}
F78
m
θ
ψ


Brest
2
1
2
0
3
1
0
1
5
4
9
8
0
10
6
7
2
3
1438 m
Image © 2012 IGN-France
Image © 2012 DigitalGlobe
© 2012 Google
Image © 2012 GeoEye
2003
lat 48.373958° long -4.458737° élév. 0 m

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$



$\mathbb{X}_1$: outside the corridor.

$\mathbb{X}_2$: inside the corridor.

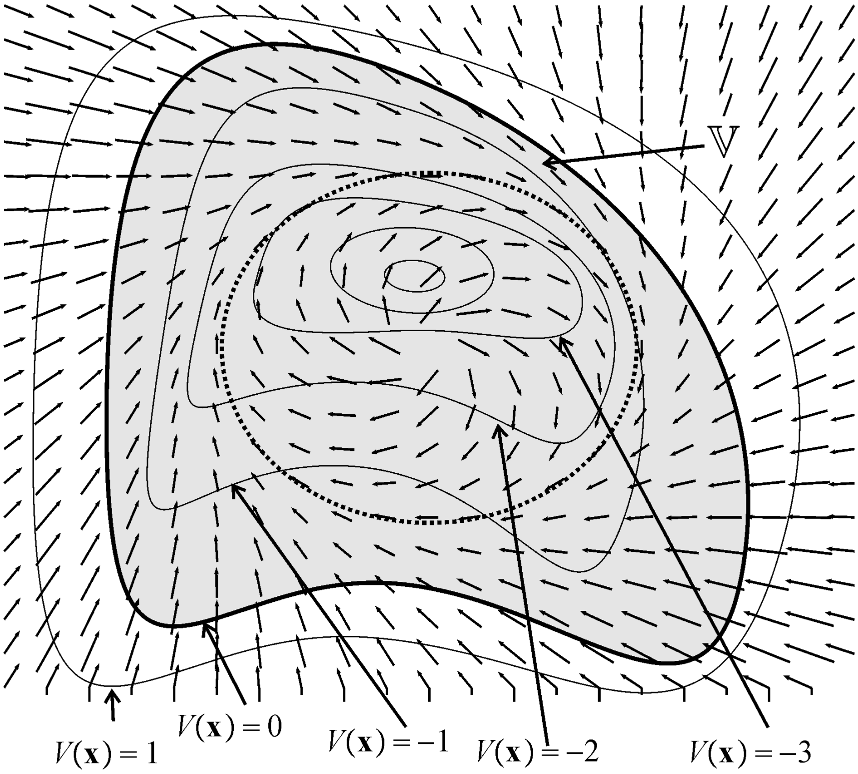
Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$. The system is V-stable if

$$\left(V(\mathbf{x}) \geq 0 \ \Rightarrow \ \dot{V}(\mathbf{x}) < 0 \right).$$

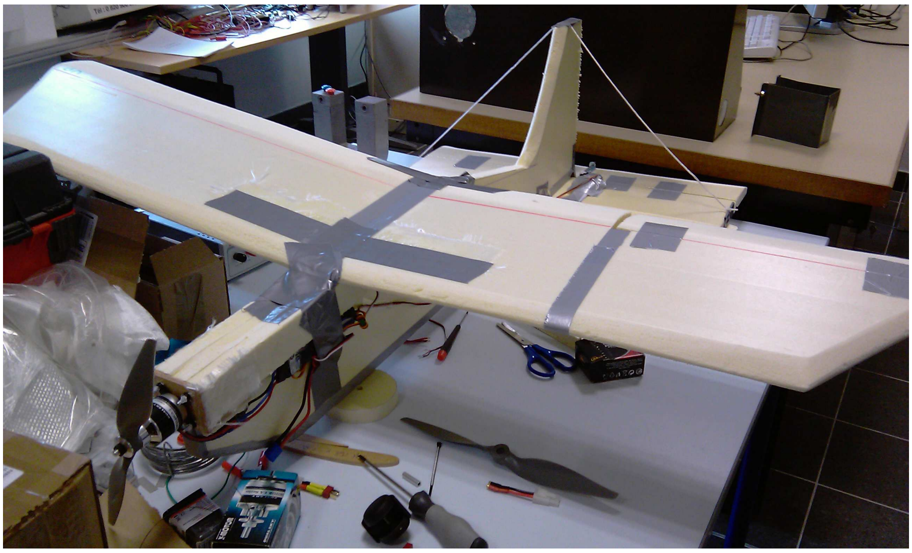
Since

$$\dot{V}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$$

Checking the V-stability can be done using using interval analysis.



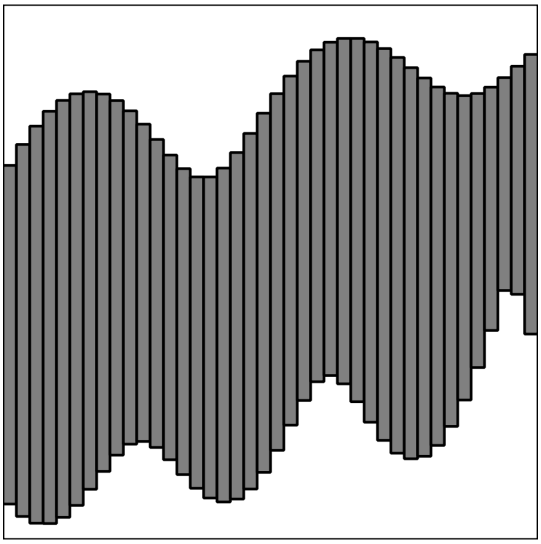
$\mathbb{V}$
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$



Non-holonomic system

2 Tubes

A tube is a function which associates to any $t \in \mathbb{R}$ a subset of $\mathbb{R}^n$.



In the machine a tube can be represented by two stair functions

Example of tubes

$$
\begin{aligned}
[f](t) &= [1,2]\cdot t + \sin([1,3]\cdot t) \\
[g](t) &= [a_0] + [a_1]\,t + [a_2]\,t^2 + [a_3]\,t^3 \\
\int_0^t [g](\tau)\,d\tau &= [a_0]\,t + [a_1]\,\frac{t^2}{2} + [a_2]\,\frac{t^3}{3} + [a_3]\,\frac{t^4}{4}.
\end{aligned}
$$

3 Capture tubes

Consider the time dependant system

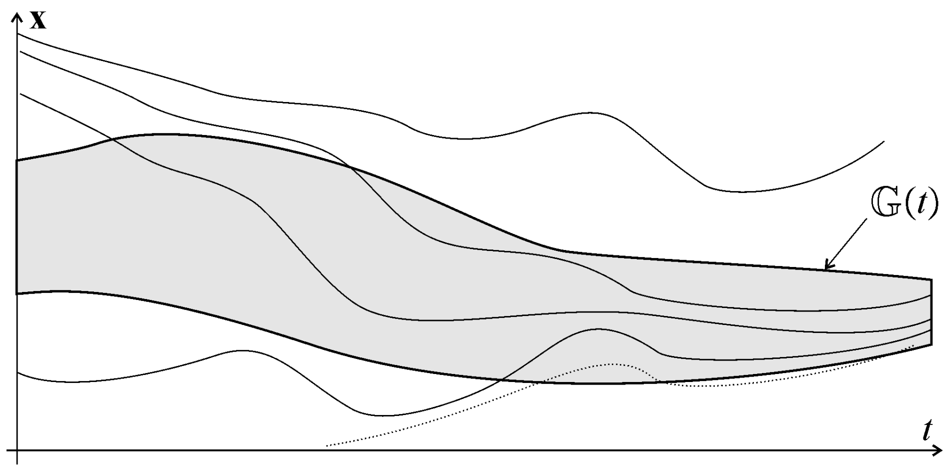
$$\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, t\right)$$

and a *tube*

$$\mathbb{G}(t) \subset \mathbb{R}^n, t \in \mathbb{R}.$$

The tube $\mathbb{G}(t)$ is said to be a *capture tube* if

$$\mathbf{x}(t) \in \mathbb{G}(t), \tau > 0 \Rightarrow \mathbf{x}(t+\tau) \in \mathbb{G}(t+\tau).$$



Theorem. Consider the tube

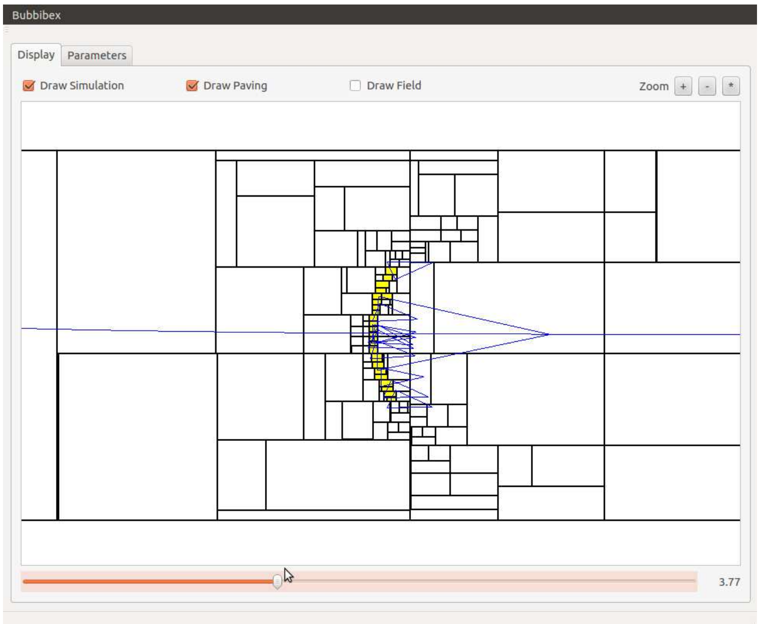
$$\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\}$$

where $\mathbf{g} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$. If the *cross out* condition

$$\begin{cases} \underbrace{\frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t) \, .\mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t)}_{\dot{g}_i(\mathbf{x},t)} \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq 0 \end{cases}$$

is inconsistent for all $(\mathbf{x}, t, i)$, then $\mathbb{G}(t)$ is a capture tube for $\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

A software Bubbibex (using Ibex) made by students from ENSTA Bretagne uses interval analysis to prove the inconsistency.



4 Test-case

Robot

$$\begin{cases} \dot{x} &= u_1 \\ \dot{y} &= u_2 \\ \dot{\theta} &= -\theta. \end{cases}$$

Target $(x_d, y_d) = (t, 0)$. We choose the control

$$u_1 = -x + t,\ u_2 = -y.$$

The closed loop system satisfies

$$\begin{cases} \dot{x} & = & -x + t \\ \dot{y} & = & -y \\ \dot{\theta} & = & -\theta. \end{cases}$$

Target tube. The tube we want is

$$\mathbb{G}(t) = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x},t) \leq \mathbf{0}\},$$

with

$$\begin{cases} g_1(\mathbf{x},t) & = & (x_1 - t)^2 + x_2^2 - r^2 \\ g_2(\mathbf{x},t) & = & (\cos x_3 - 1)^2 + \sin^2 x_3 - 0.2. \end{cases}$$

For $r = 4$, Bubbibex proves that $\mathbb{G}(t)$ is a capture tube.

For $r < 1$, some trajectories leave $\mathbb{G}(t)$ forever.

5 Lattice and capture tubes

Consider $\mathcal{S} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

If $\mathbb{T}$ is the set of tubes and $\mathbb{T}_c$ is the set of all capture tubes of $\mathcal{S}$ then $(\mathbb{T}_c, \subset)$ is a sublattice of $(\mathbb{T}, \subset)$.

We have indeed

$$\left\{ \begin{array}{lcl} \mathbb{G}_1(t) & \in & \mathbb{T}_c \\ \mathbb{G}_2(t) & \in & \mathbb{T}_c \end{array} \right. \Rightarrow \left\{ \begin{array}{lcl} \mathbb{G}_1(t) \cap \mathbb{G}_2(t) & \in & \mathbb{T}_c \\ \mathbb{G}_1(t) \cup \mathbb{G}_2(t) & \in & \mathbb{T}_c \end{array} \right.$$

6 Computing capture tubes

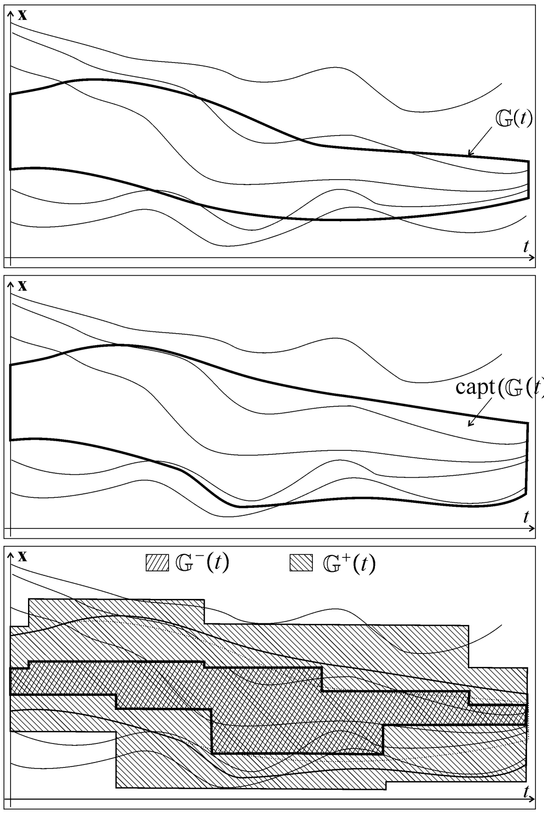
If $\mathbb{G}(t) \in \mathbb{T}$, define

$$\text{capt}\left(\mathbb{G}(t)\right) = \bigcap \left\{ \overline{\mathbb{G}}(t) \in \mathbb{T}_c \mid \mathbb{G}(t) \subset \overline{\mathbb{G}}(t) \right\}.$$

This set is the smallest capture tube enclosing $\mathbb{G}(t)$.

Problem. Given $\mathbb{G}(t) \in \mathbb{T}$, compute an interval $\left[\mathbb{G}^{-}(t), \mathbb{G}^{+}(t)\right] \in \mathbb{IT}$ such that

$$\mathsf{capt}\left(\mathbb{G}(t)\right) \in \left[\mathbb{G}^{-}(t), \mathbb{G}^{+}(t)\right].$$


x
t
𝔾(t)
capt(𝔾(t))
𝔾⁻(t)
𝔾⁺(t)

Flow. The flow associated with the system $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is a function $\phi_{t_0,t_1} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \Rightarrow \phi_{t_0,t_1}(\mathbf{x}(t_0)) = \mathbf{x}(t_1).$$

Proposition. For the system $\mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ and the tube $\mathbb{G}(t)$, we have

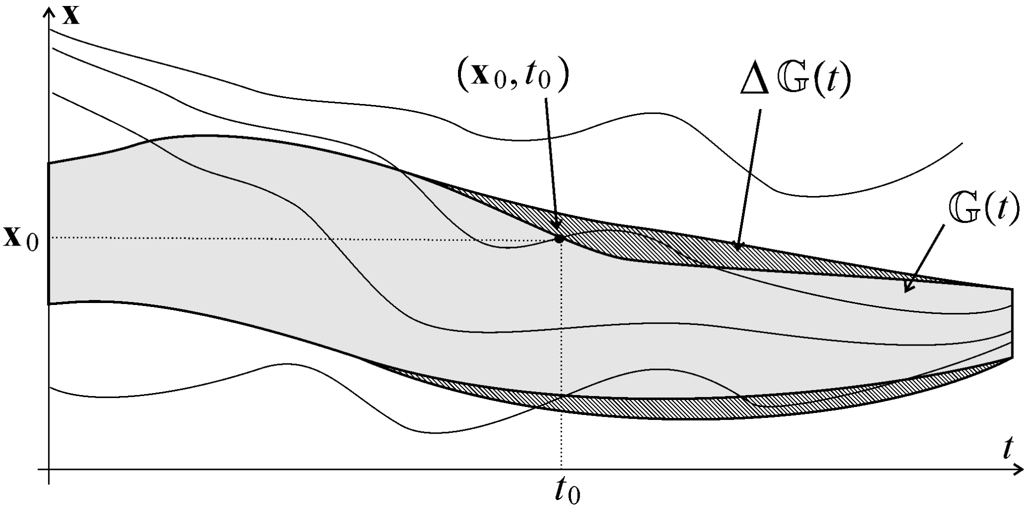
$$\text{capt}\,(\mathbb{G}(t)) = \mathbb{G}(t) \cup \mathbf{\Delta}\mathbb{G}(t),$$

with

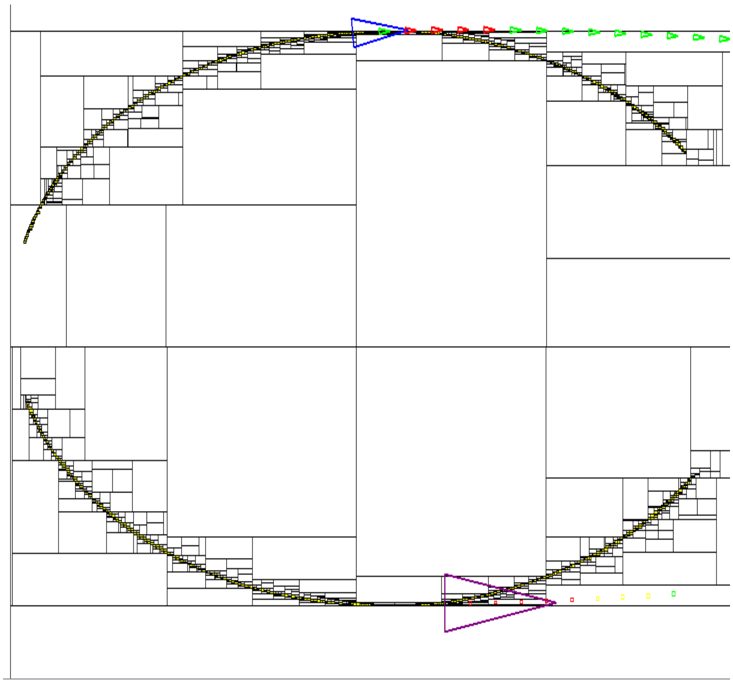
$$\mathbf{\Delta}\mathbb{G}(t) = \left\{ (\mathbf{x}, t) \;\middle|\; \begin{array}{l} \exists\,(\mathbf{x}_0, t_0) \text{ satisfying the cross out condition} \\ t \geq t_0,\ \phi_{t_0,t}(\mathbf{x}_0) \notin \mathbb{G}(t) \end{array} \right\}$$

Recall the cross out condition:

$$\begin{cases} \frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t)\,.\mathbf{f}(\mathbf{x}, t) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t) \geq 0 \\ g_i(\mathbf{x}, t) = 0 \\ \mathbf{g}(\mathbf{x}, t) \leq 0 \end{cases}$$



x
$(\mathbf{x}_0, t_0)$
$\Delta\mathbb{G}(t)$
$\mathbb{G}(t)$
$\mathbf{x}_0$
t
t_0



7 Test case 1

Consider the system

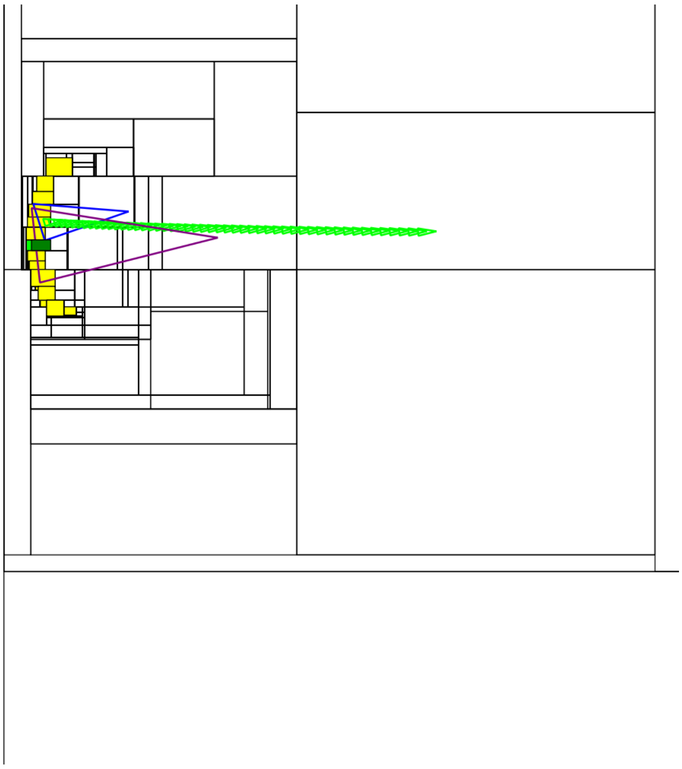
$$\mathcal{S}:\begin{cases}\dot{x} &= -x+t\\ \dot{y} &= -y\\ \dot{\theta} &= -\theta\end{cases}$$

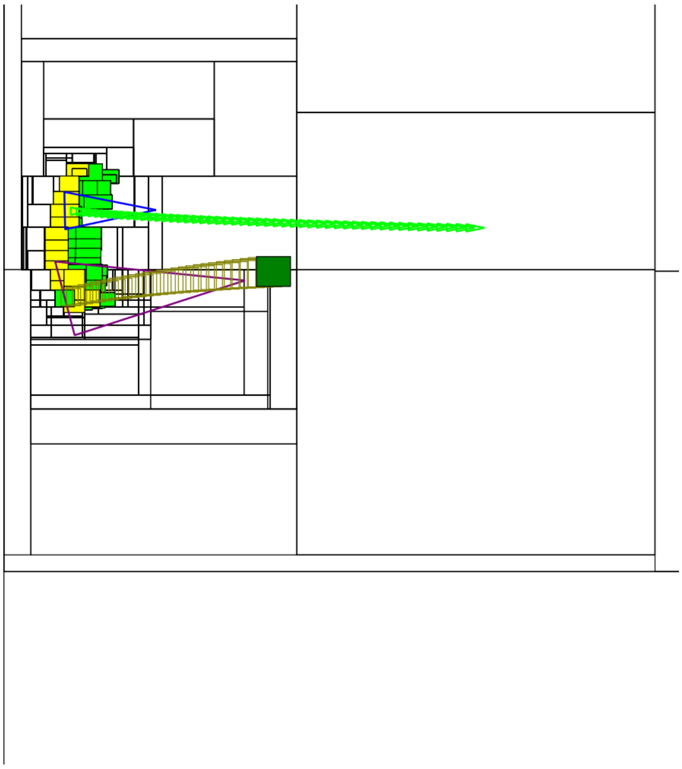
and the tube

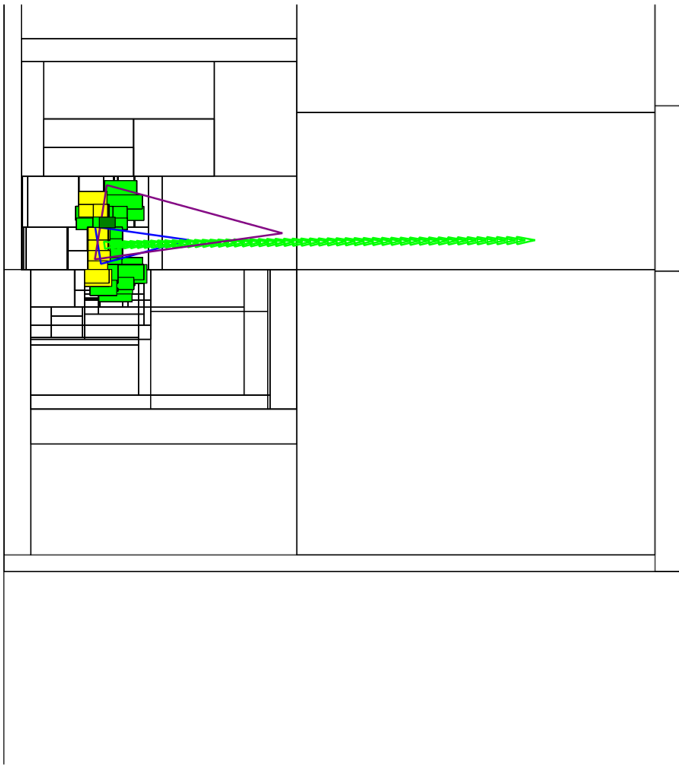
$$\mathbb{G}(t):\begin{cases}g_1(\mathbf{x},t) &= (x_1-t)^2+x_2^2-r(t)\\ g_2(\mathbf{x},t) &= (\cos x_3-1)^2+\sin^2 x_3-0.2.\end{cases}$$

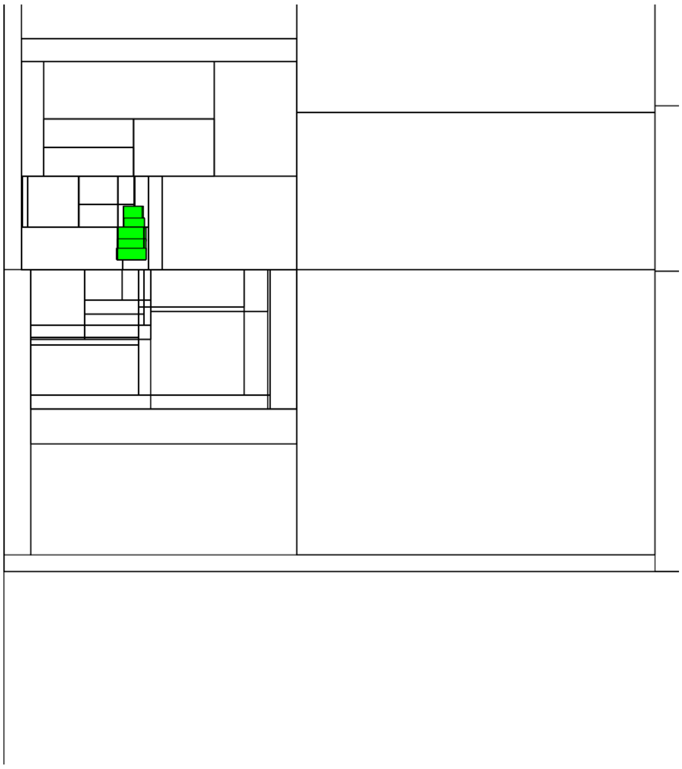
$$r(t)=0.2\cdot(t+1)^2\,.$$

Some trajectories leave and come back to $\mathbb{G}(t)$.









8 Test case 2

Consider the robot (Dubin's car)

$$\begin{cases} \dot{x} &= \cos\theta \\ \dot{y} &= \sin\theta \\ \dot{\theta} &= u \end{cases}$$

where $u \in [-2, 2]$.

To move toward the target (x_d, y_d), we take the controller:

$$\left\{\begin{array}{lll} \mathbf{n} & = & \frac{1}{\sqrt{(x_d-x)^2+(y_d-y)^2}}\left(\begin{array}{c} x_d - x \\ y_d - y \end{array}\right) + \frac{2}{\sqrt{\dot{x}_d^2+\dot{y}_d^2}}\left(\begin{array}{c} \dot{x}_d \\ \dot{y}_d \end{array}\right) \\ \theta_d & = & \text{atan2}\left(\mathbf{n}\right) \\ u & = & -2 \cdot \sin\left(\theta - \theta_d\right). \end{array}\right.$$

Target

$$\begin{cases} x_d(t) &= \rho_x \cos t \\ y_d(t) &= \rho_y \sin t. \end{cases}$$

For the derivative, we get

$$\begin{cases} \dot{x}_d(t) &= -\rho_x \sin t \\ \dot{y}_d(t) &= \rho_y \cos t. \end{cases}$$

Target tube. We want the robot to stay inside the set

$$\mathbb{G}(t) = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, t) \leq \mathbf{0}\},$$

with

$$\begin{cases} g_1(\mathbf{x}, t) &= (x - x_d)^2 + (y - y_d)^2 - \rho^2 \\ g_2(\mathbf{x}, t) &= \left(\cos\theta - \frac{n_x}{\|\mathbf{n}\|}\right)^2 + \left(\sin\theta - \frac{n_y}{\|\mathbf{n}\|}\right)^2 - \alpha^2. \end{cases}$$

Resolution. We used the solver Bubbibex.

The tube is proved to be unsafe.

For given parameters, Bubbibex is able to compute the margin (i.e., width$\Big(\Big[\mathbb{G}^-(t), \mathbb{G}^+(t)\Big]\Big)$).

References

L. Jaulin and F. Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE Transaction on Robotics, Volume 27, Issue 5.

L. Jaulin, J. Ninin, G. Chabert, S. Le Menec, M. Saad, V. Le Doze, A. Stancu (2014) . Computing capture tubes, SCAN 2014, Würzburg.