Interval Contractors to Solve Dynamical Geometrical Equations with Application to Underwater Robotics

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1. Secure a zone

INFO OBS. Un sous-marin nucléaire russe repéré dans le Golfe de Gascogne

Le navire a été repéré en janvier. Ce serait la première fois depuis la fin de la Guerre Froide qu'un tel sous-marin, doté de missiles nucléaires, se serait aventuré dans cette zone au large des côtes françaises.

Bay of Biscay 220 000 km^2

An intruder

Several robots $\mathscr{R}_1, \dots, \mathscr{R}_n$ at positions $\mathbf{a}_1, \dots, \mathbf{a}_n$ are moving in the ocean.
If the intruder is in the visibility zone of one robot, it is detected.[5]

Complementary approach

We assume that a virtual intruder exists inside $\mathbb{G}$.
We localize it with a set-membership observer inside $\mathbb{X}(t)$.
The secure zone corresponds to the complementary of $\mathbb{X}(t)$.

Assumptions

The intruder satisfies

$$\dot{\mathbf{x}} \in \mathbb{F}(\mathbf{x}(t)).$$

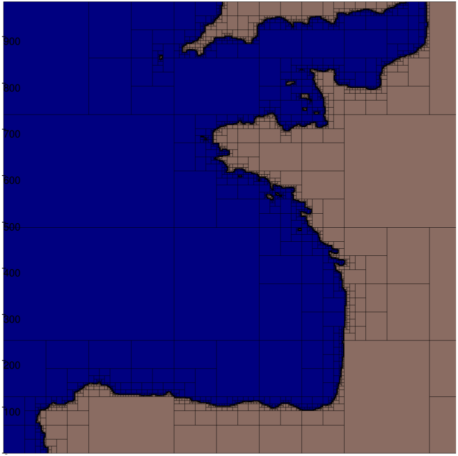
Each robot $\mathscr{R}_i$ has the visibility zone $g_{\mathbf{a}_i}^{-1}([0, d_i])$ where d_i is the scope.

Theorem. An (undetected) intruder has a state vector $\mathbf{x}(t)$ inside the set

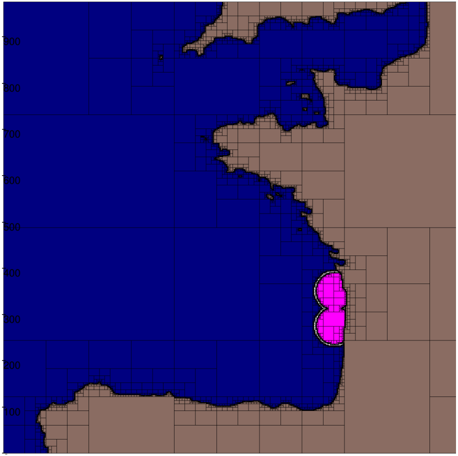
$$\mathbb{X}(t) = \mathbb{G} \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt)) \cap \bigcap_i g^{-1}_{\mathbf{a}_i(t)}([d_i(t), \infty]),$$

where $\mathbb{X}(0) = \mathbb{G}$. The secure zone is

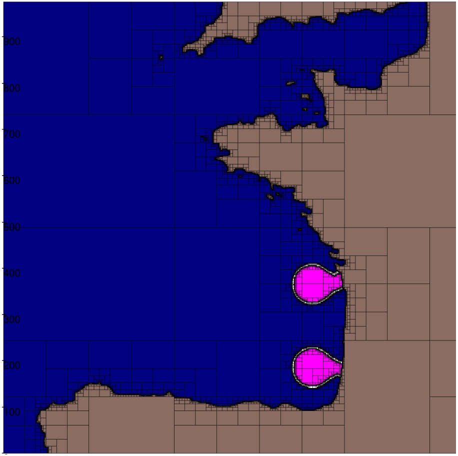
$$\mathbb{S}(t) = \overline{\mathbb{X}(t)}.$$



Set $\mathbb{G}$ in blue

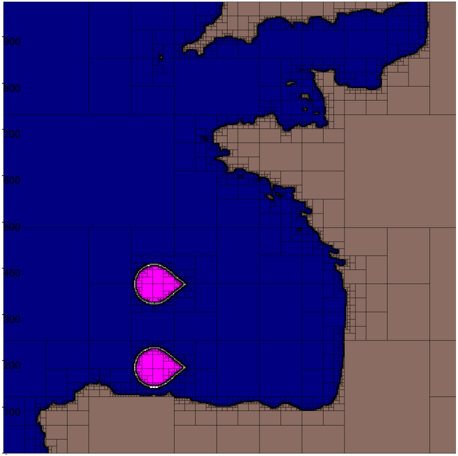


Magenta: $\mathbb{G} \cap \bigcup_i g_{\mathbf{a}_i(t)}^{-1}([0, d_i(t)])$ Blue:$\mathbb{G} \cap \bigcap_i g_{\mathbf{a}_i(t)}^{-1}([d_i(t), \infty])$



Blue:

$$\mathbb{X}(t) = \mathbb{G} \cap (\mathbb{X}(t-dt) + dt \cdot \mathbb{F}(\mathbb{X}(t-dt))) \cap \bigcap_i g^{-1}_{\mathbf{a}_i(t)}([d_i(t), \infty]).$$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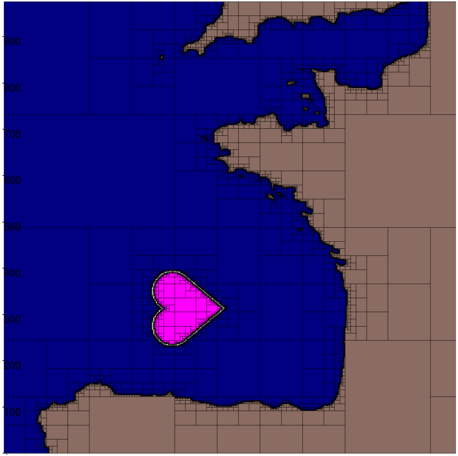
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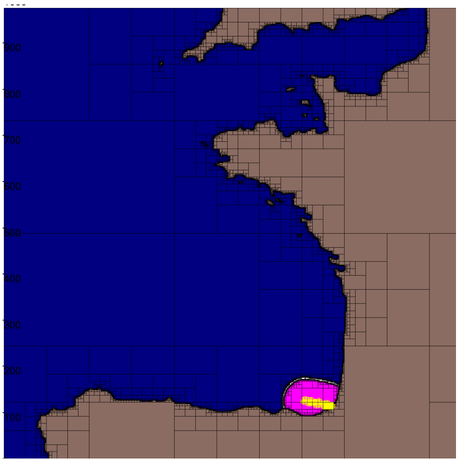
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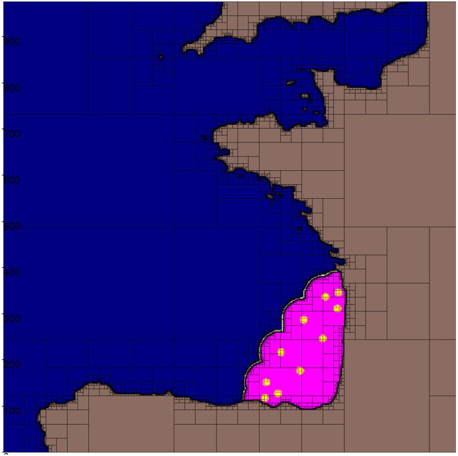
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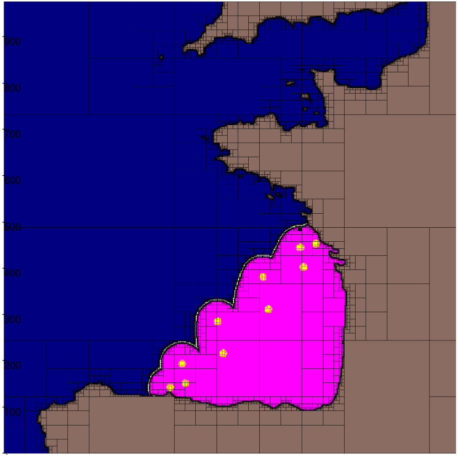
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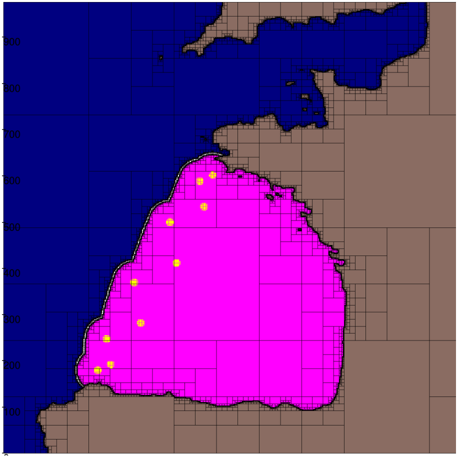


Video : youtube

Idea: Take into account the future.
The feasible set can be obtained by the following contractions

$$\begin{array}{rcl} \overrightarrow{\mathbb{X}}(t) & = & \overrightarrow{\mathbb{X}}(t)\cap(\mathbb{X}(t-dt)+dt\cdot\mathbb{F}(\mathbb{X}(t-dt))) \\ \overleftarrow{\mathbb{X}}(t) & = & \overleftarrow{\mathbb{X}}(t)\cap(\mathbb{X}(t+dt)-dt\cdot\mathbb{F}(\mathbb{X}(t+dt))) \\ \mathbb{X}(t) & = & \overrightarrow{\mathbb{X}}(t)\cap\overleftarrow{\mathbb{X}}(t) \end{array}$$

with the initialization

$$\mathbb{X}(t)=\overrightarrow{\mathbb{X}}(t)=\overleftarrow{\mathbb{X}}(t)=\mathbb{G}.$$

2. Intervals

Problem. Given $f:\mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\text{abs}([-7,1]) &= [0,7]
\end{aligned}
$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned} [f]([x_1], [x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ &\quad + \sin [x_1] \cdot \sin [x_2] + 2. \end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^+ \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Contractors

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* [2] for the equation $f(\mathbf{x}) = 0$, if

$$\begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathscr{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

Tubes

A trajectory is a function $\mathbf{f}:\mathbb{R}\rightarrow\mathbb{R}^{n}$. [4, 3]. For instance

$$\mathbf{f}(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$$

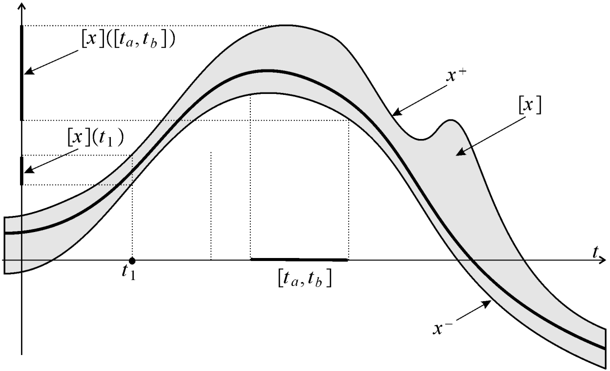
is a trajectory.

Order relation

$$\mathbf{f} \leq \mathbf{g} \Leftrightarrow \forall t, \forall i, f_i(t) \leq g_i(t).$$

We have

$$\begin{aligned} \mathbf{h} &= \mathbf{f} \wedge \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i(t) = \min(f_i(t), g_i(t)), \\ \mathbf{h} &= \mathbf{f} \vee \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i(t) = \max(f_i(t), g_i(t)). \end{aligned}$$



The set of trajectories is a lattice. Interval of trajectories (tubes) can be defined.

Propagation

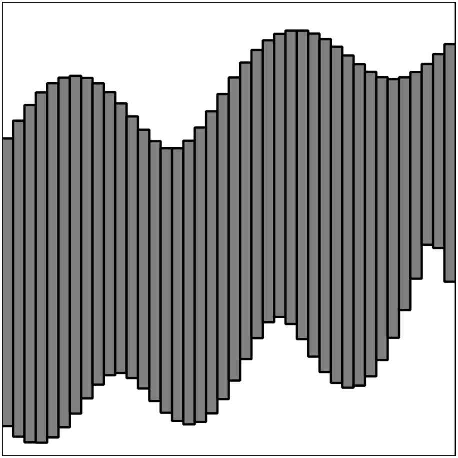
Example. Consider $x(t) \in [x](t)$ with the constraint

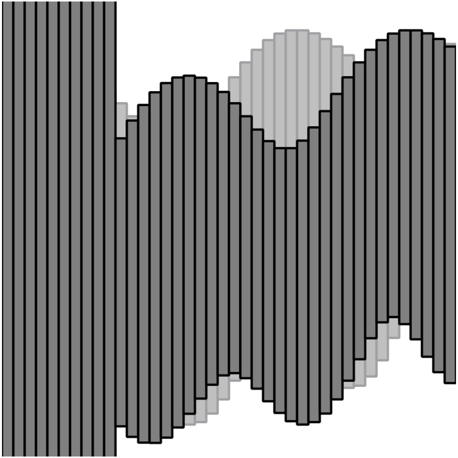
$$\forall t,\ x(t) = x(t+1)$$

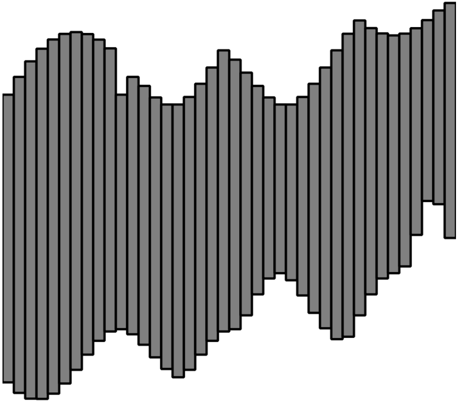
Contract the tube $[x](t)$.

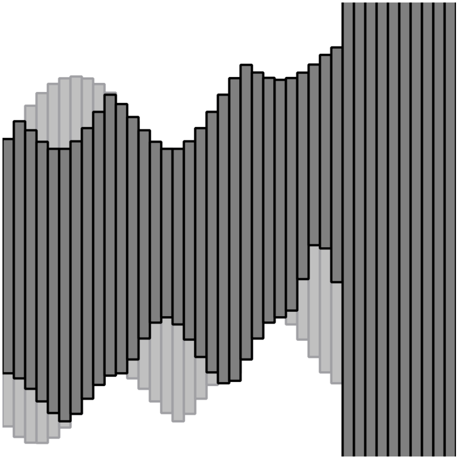
We first decompose into primitive trajectory constraints

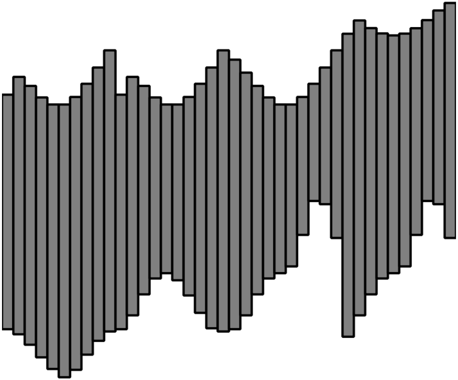
$$\begin{aligned} x(t) &= a(t+1) \\ x(t) &= a(t). \end{aligned}$$

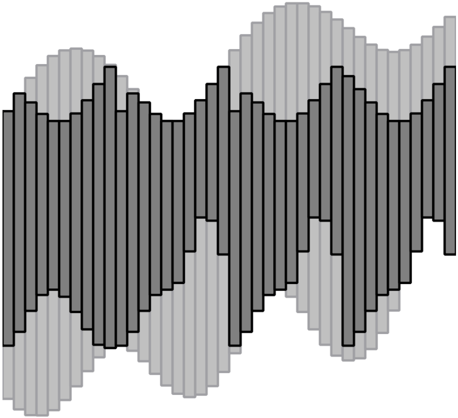


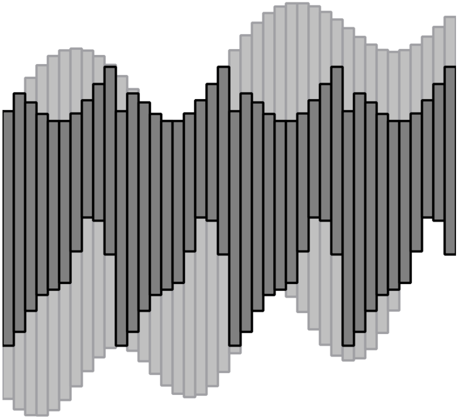


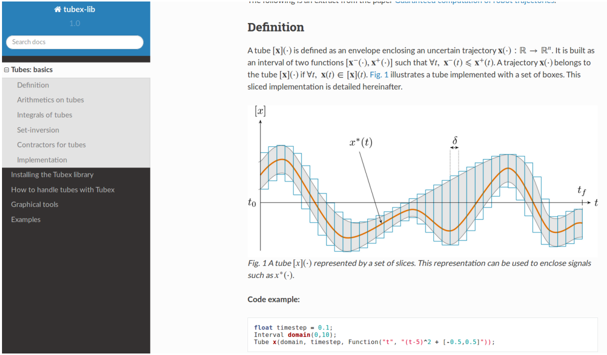












tubex-lib
1.0
Search docs

Tubes: basics
- Definition
- Arithmetics on tubes
- Integrals of tubes
- Set-inversion
- Contractors for tubes
- Implementation

Installing the Tubex library
How to handle tubes with Tubex
Graphical tools
Examples

Definition

A tube $[\mathbf{x}](\cdot)$ is defined as an envelope enclosing an uncertain trajectory $\mathbf{x}(\cdot) : \mathbb{R} \to \mathbb{R}^n$. It is built as an interval of two functions $[\mathbf{x}^-(\cdot), \mathbf{x}^+(\cdot)]$ such that $\forall t,\ \mathbf{x}^-(t) \leqslant \mathbf{x}^+(t)$. A trajectory $\mathbf{x}(\cdot)$ belongs to the tube $[\mathbf{x}](\cdot)$ if $\forall t,\ \mathbf{x}(t) \in [\mathbf{x}](t)$. Fig. 1 illustrates a tube implemented with a set of boxes. This sliced implementation is detailed hereinafter.

Fig. 1 A tube $[x](\cdot)$ represented by a set of slices. This representation can be used to enclose signals such as $x^*(\cdot)$.

Code example:

```
float timestep = 0.1;
Interval domain(0,10);
Tube x(domain, timestep, Function("t", "(t-5)^2 + [-0.5,0.5]"));
```

http://codac.io/ [4]

3. Cooperative localization

Classical state estimation

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t),t) & t \in \mathbb{T} \subset \mathbb{R}. \end{cases}$$

Space constraint $\mathbf{g}(\mathbf{x}(t),t) = 0$.

Example.

$$\begin{cases} \dot{x}_1 = x_3 \cos x_4 \\ \dot{x}_2 = x_3 \cos x_4 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = u_2 \\ (x_1(5) - 1)^2 + (x_2(5) - 2)^2 - 4 = 0 \\ (x_1(7) - 1)^2 + (x_2(7) - 2)^2 - 9 = 0 \end{cases}$$

With time-space constraints

$$\left\{ \begin{array}{lll} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) \qquad\qquad t \in \mathbb{R} \\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t),\mathbf{x}(t'),t,t') \qquad (t,t') \in \mathbb{T} \subset \mathbb{R} \times \mathbb{R}. \end{array} \right.$$

Example. An ultrasonic underwater robot with state

$$\mathbf{x} = (x_1, x_2, \dots) = (x, y, \theta, v, \dots)$$

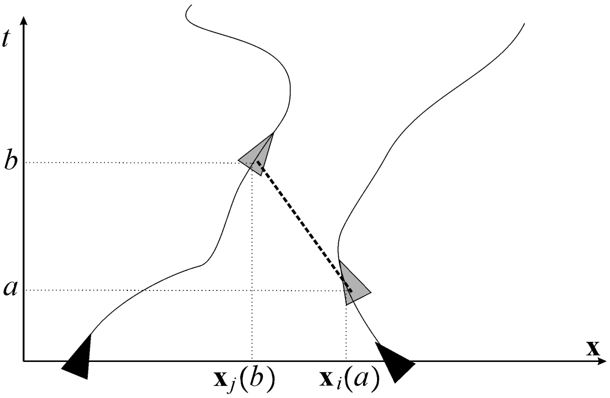
At time t the robot emits an omni-directional sound. At time t' it receives it

$$\left(x_1 - x_1^{'}\right)^2 + \left(x_2 - x_2^{'}\right)^2 - c\left(t - t'\right)^2 = 0.$$

Consider n robots $\mathscr{R}_1,\dots,\mathscr{R}_n$ described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i].$$

Omnidirectional sounds are emitted and received.
A *ping* is a 4-uple (a, b, i, j) where a is the emission time, b is the reception time, i is the emitting robot and j the receiver.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b
a
x
$\mathbf{x}_j(b)$
$\mathbf{x}_i(a)$

With the time space constraint

$$\begin{array}{l} \dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\ g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \end{array}$$

where

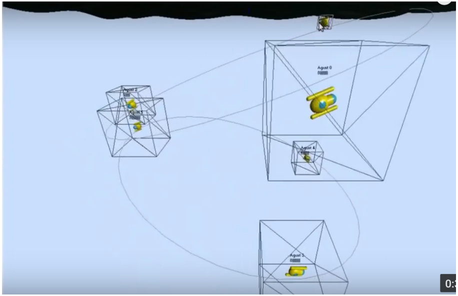
$$g(\mathbf{x}_i, \mathbf{x}_j, a, b) = \|x_1 - x_2\| - c(b - a).$$

Clocks are uncertain. We only have measurements $\tilde{a}(k), \tilde{b}(k)$ of $a(k), b(k)$ thanks to clocks h_i. Thus

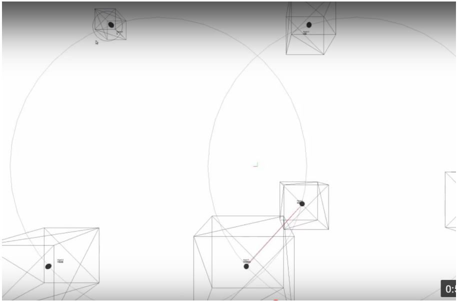
$$\begin{array}{l} \dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\ g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\ \tilde{a}(k) = h_{i(k)}(a(k)) \\ \tilde{b}(k) = h_{j(k)}(b(k)) \end{array}$$

The drift of the clocks is bounded

$$\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}(a(k)) \\
\tilde{b}(k) = h_{j(k)}(b(k)) \\
\dot{h}_i = 1 + n_h,\ n_h \in [n_h]
\end{array}$$



Youtube [1]



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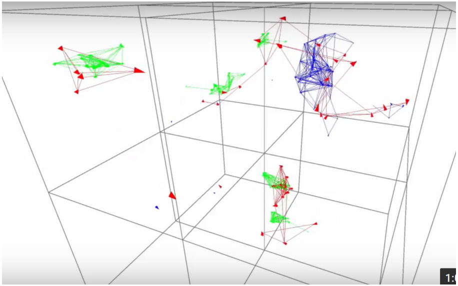

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