Interval analysis for robust control; application to sailboat robotics

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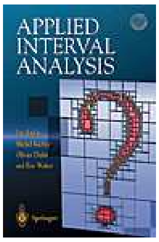
Journée scientifique en hommage à Éric Walter
Jeudi 20 mars 2014.



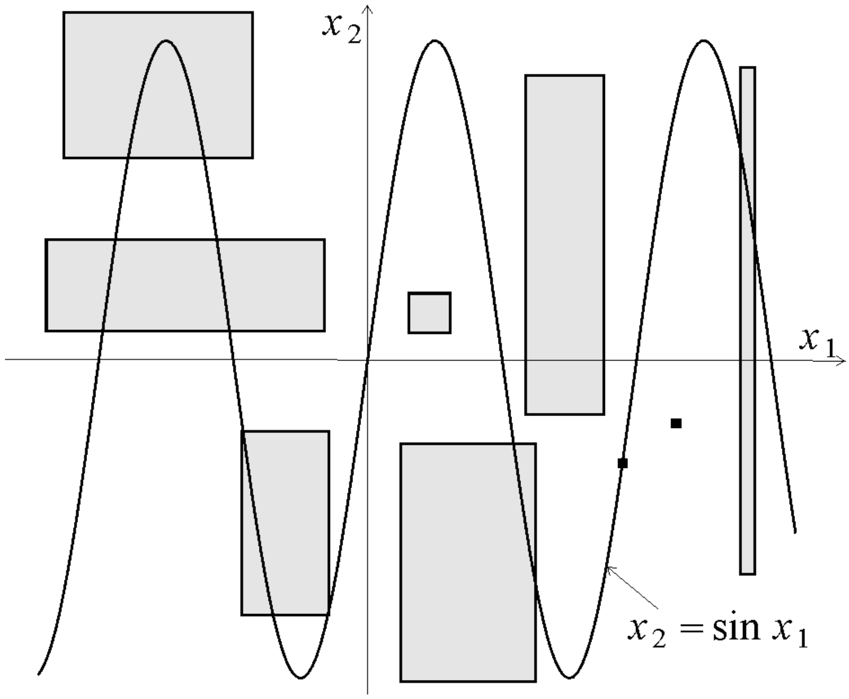
1 Contractors

The operator $\mathcal{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

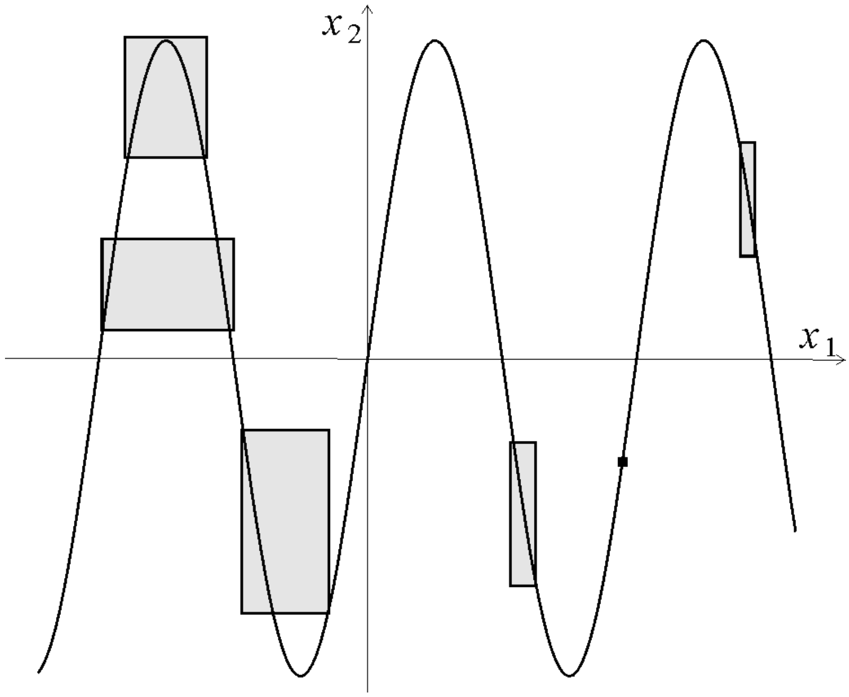
$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$



x_2
x_1
$x_2 = \sin x_1$


x_2
x_1

Contractor algebra

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1([\mathbf{x}]) \cup \mathcal{C}_2([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1(\mathcal{C}_2([\mathbf{x}]))$
reiteration	$\mathcal{C}^\infty \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

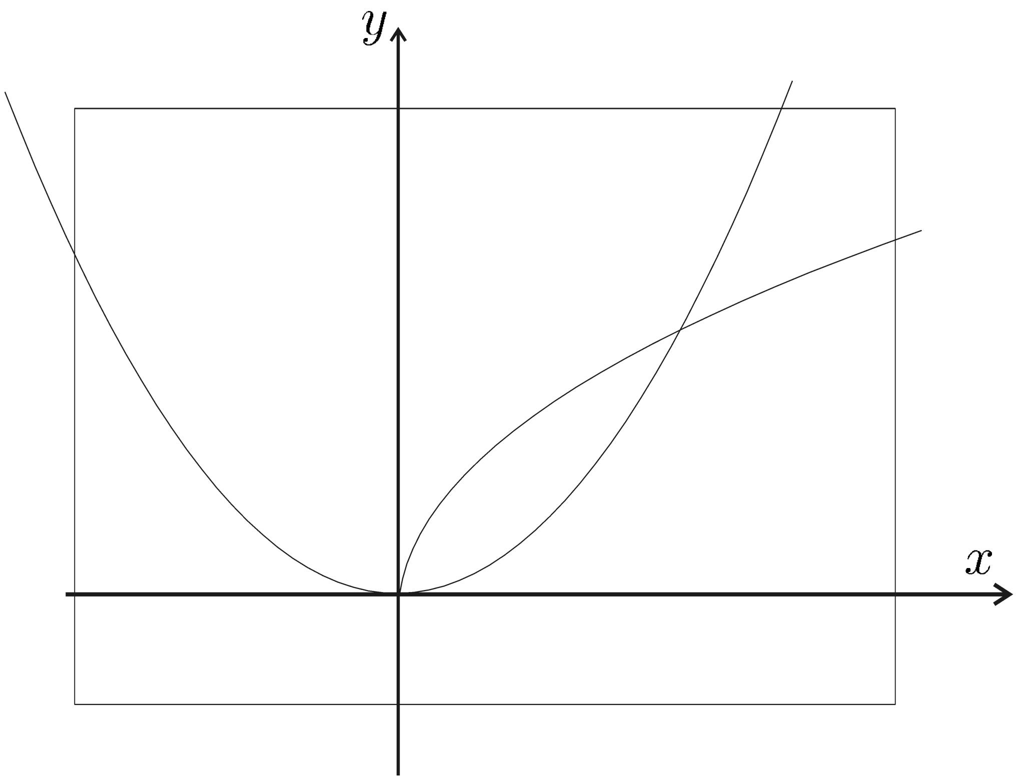
Example. Solve the system

$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

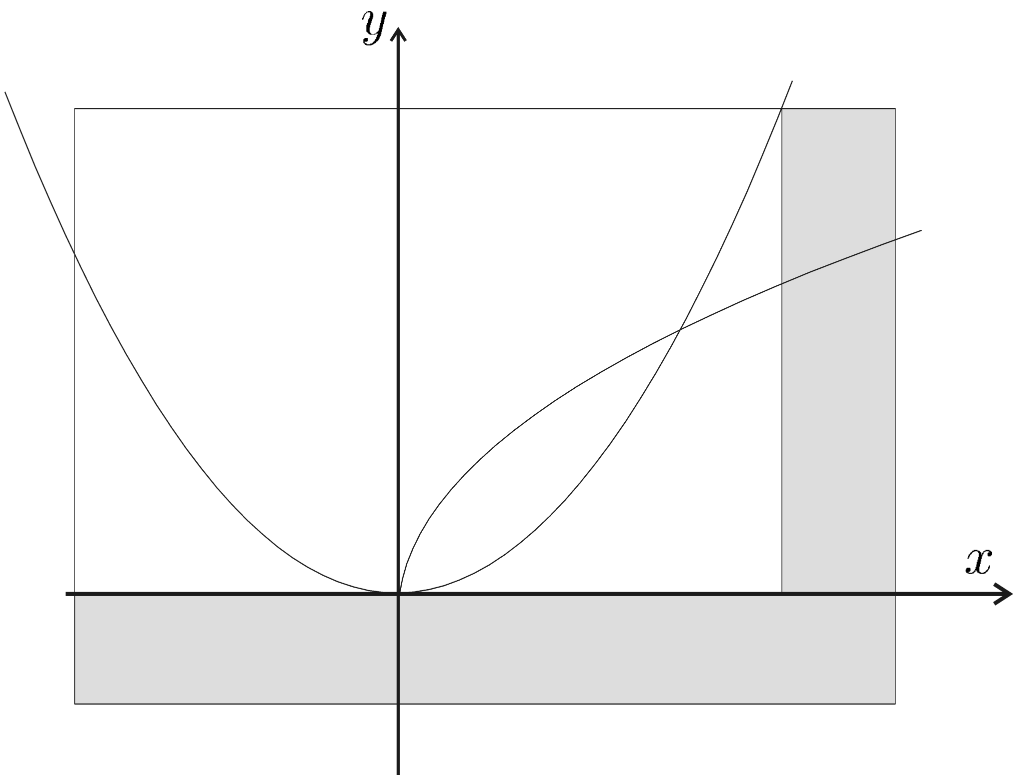
We build two contractors

$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated with } y = x^2$$

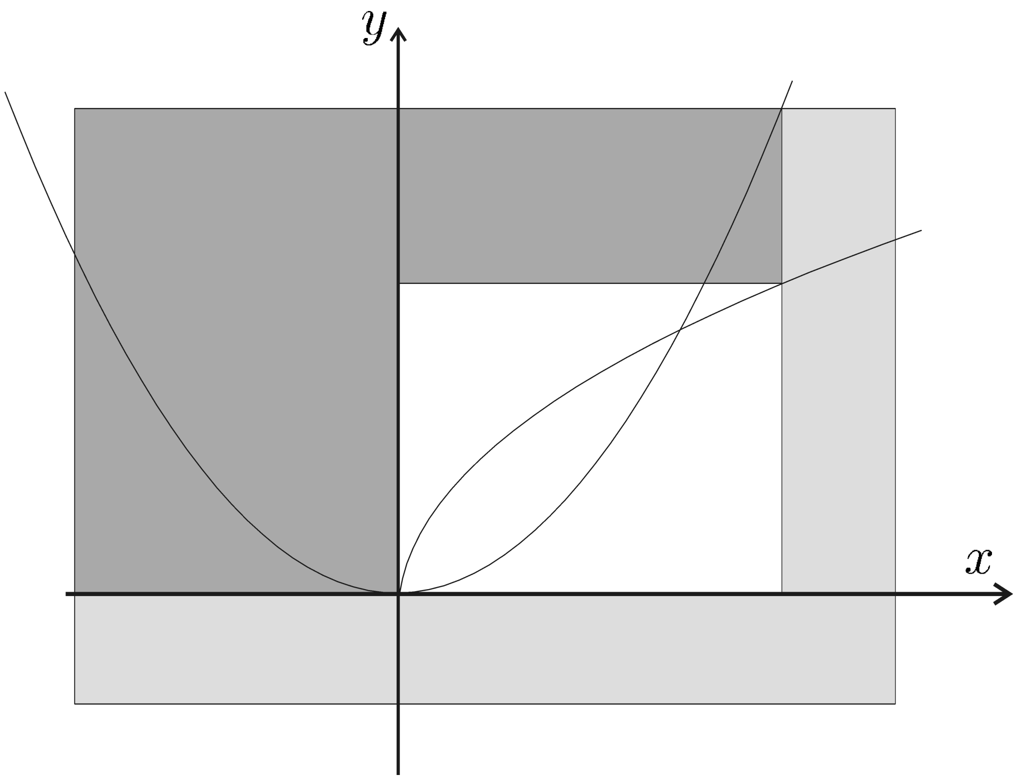
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated with } y = \sqrt{x}$$

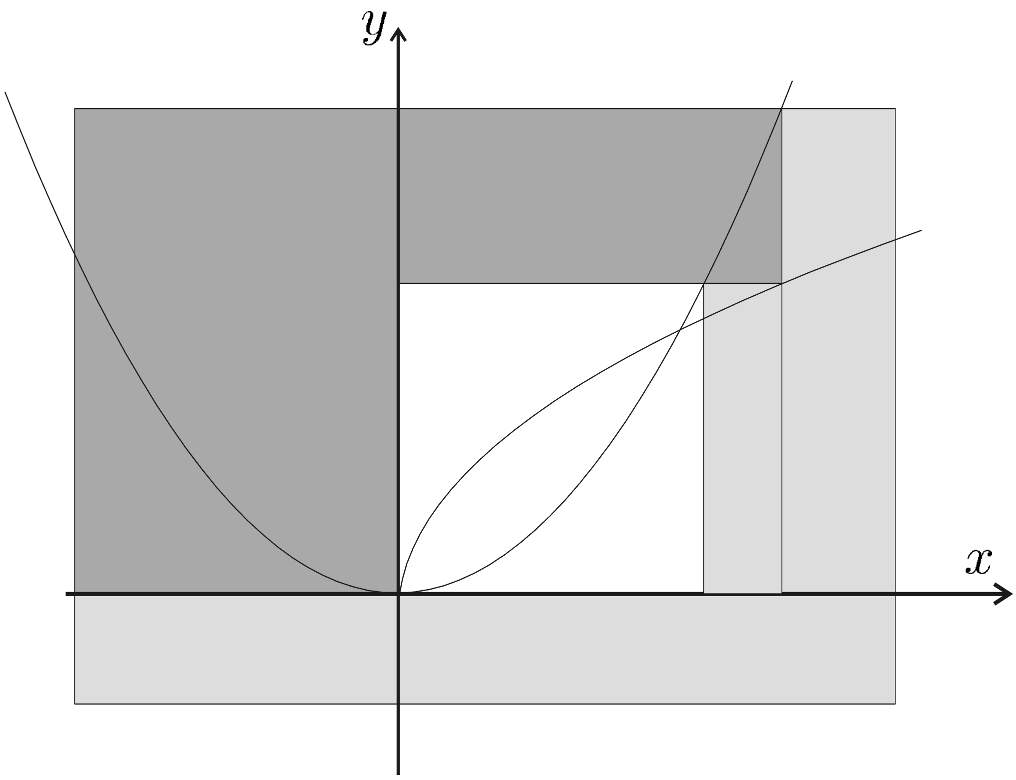


y
x

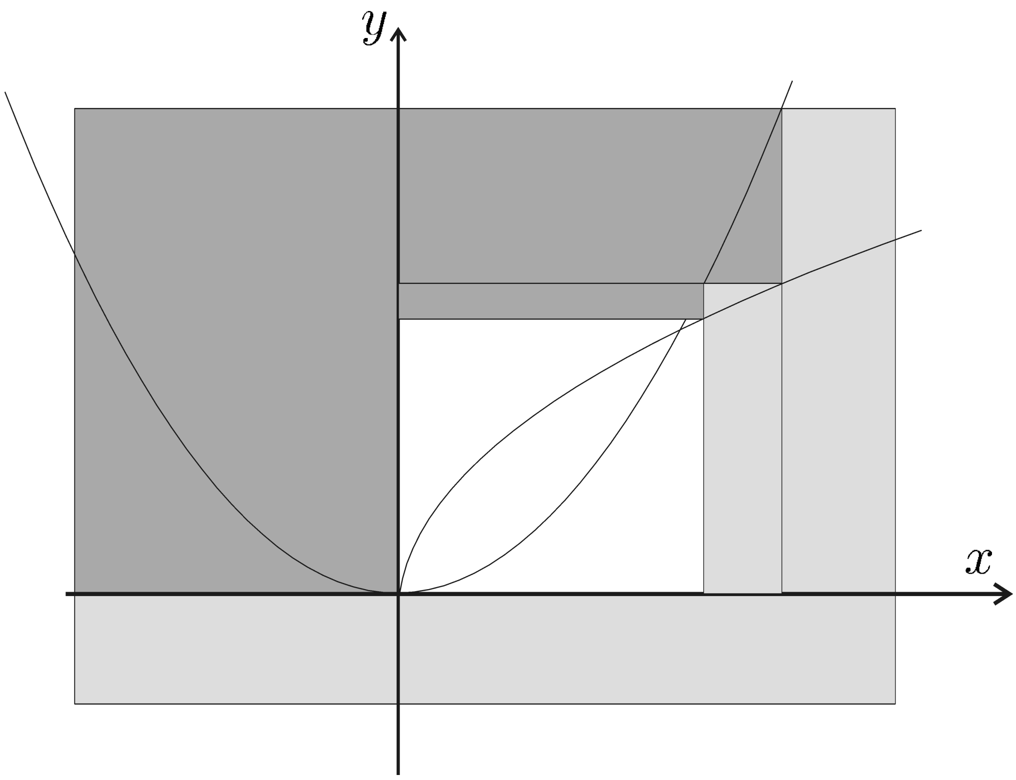


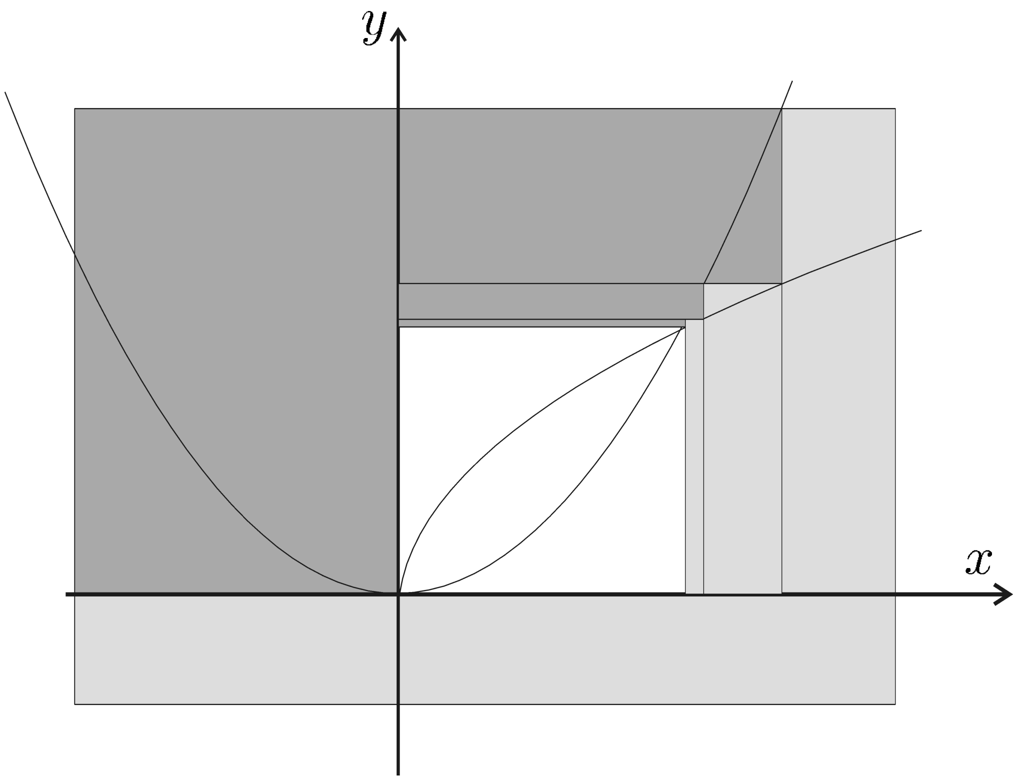
y
x

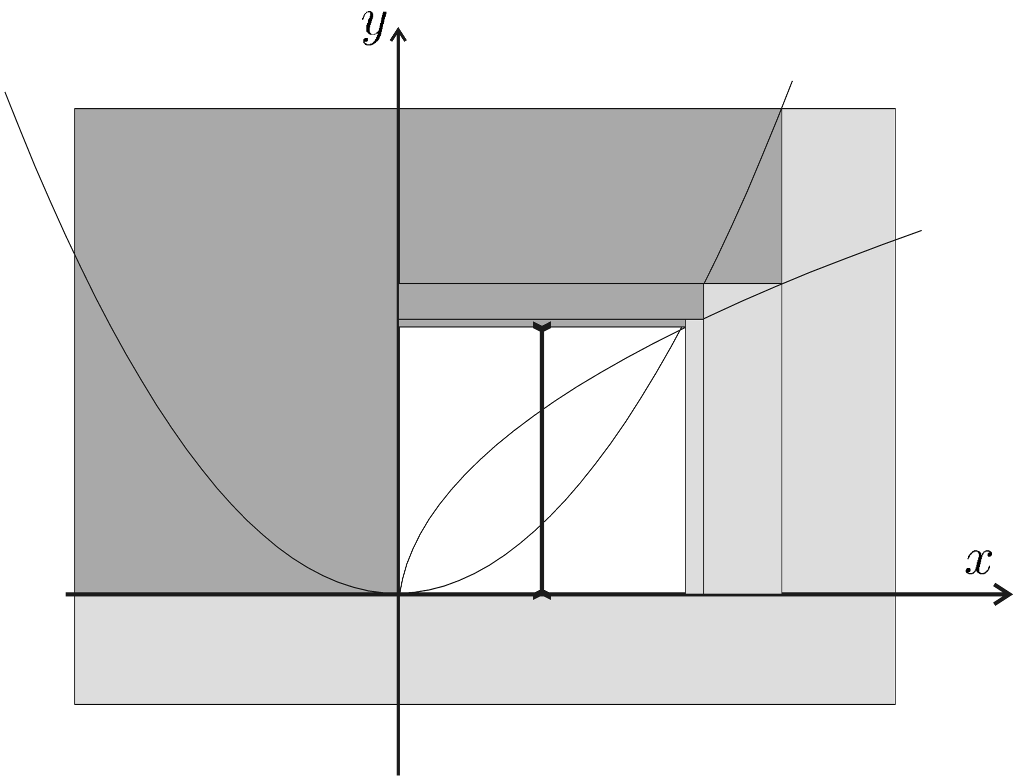

y
x



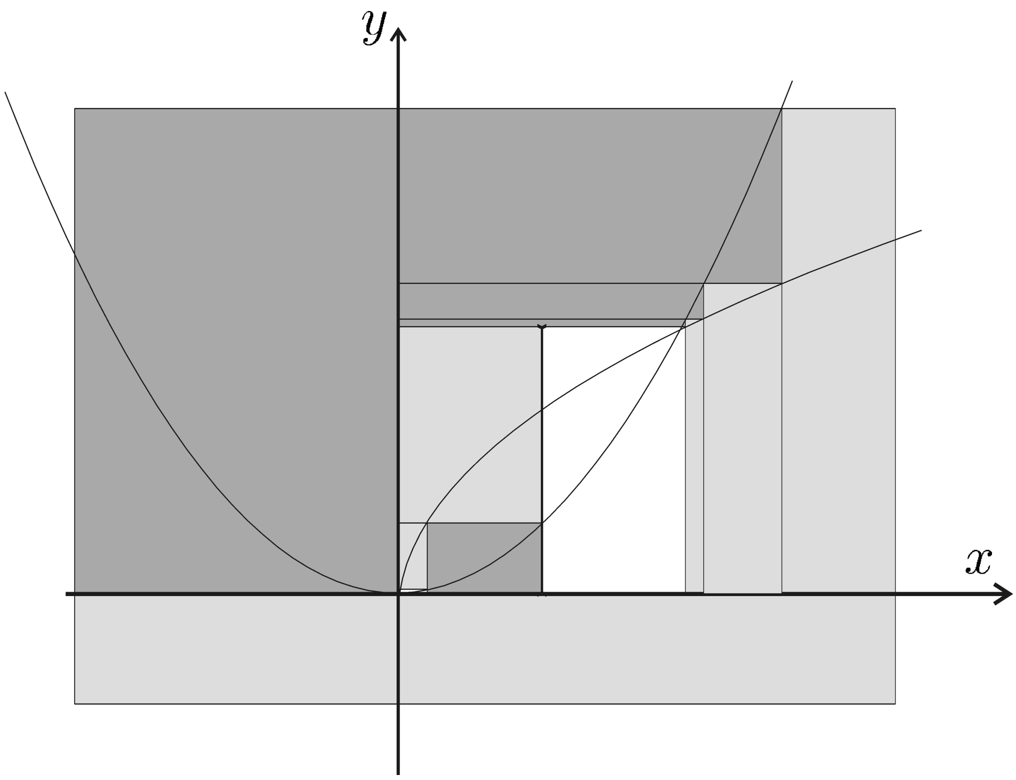
y
x


y
x

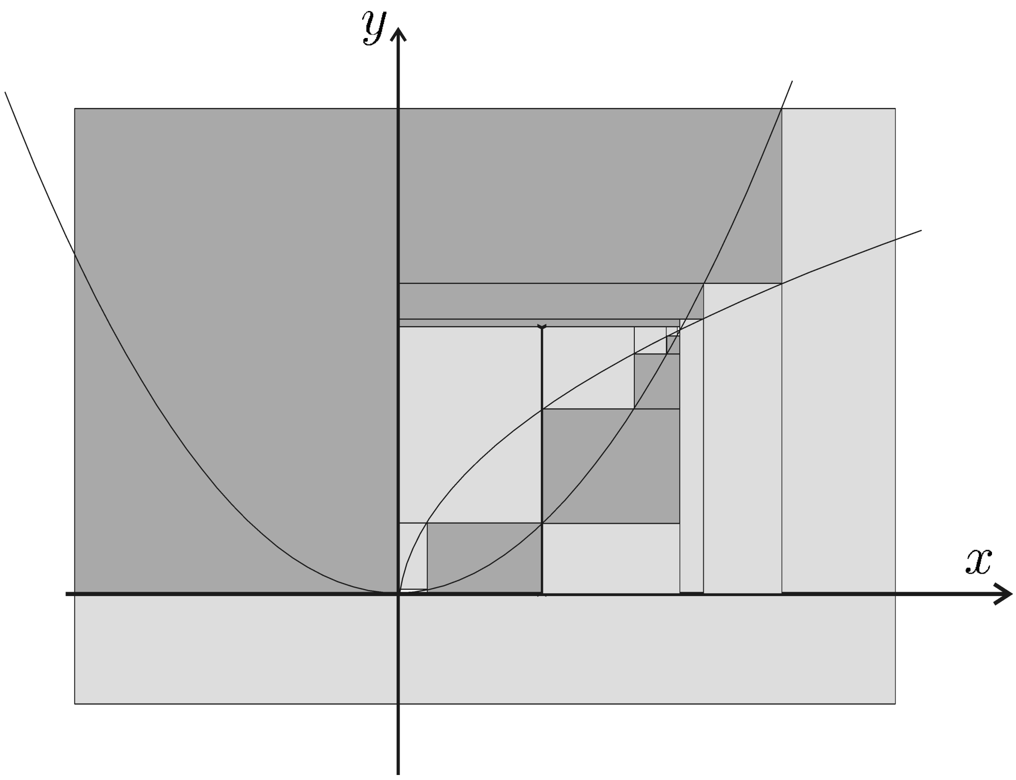

y
x



y
x

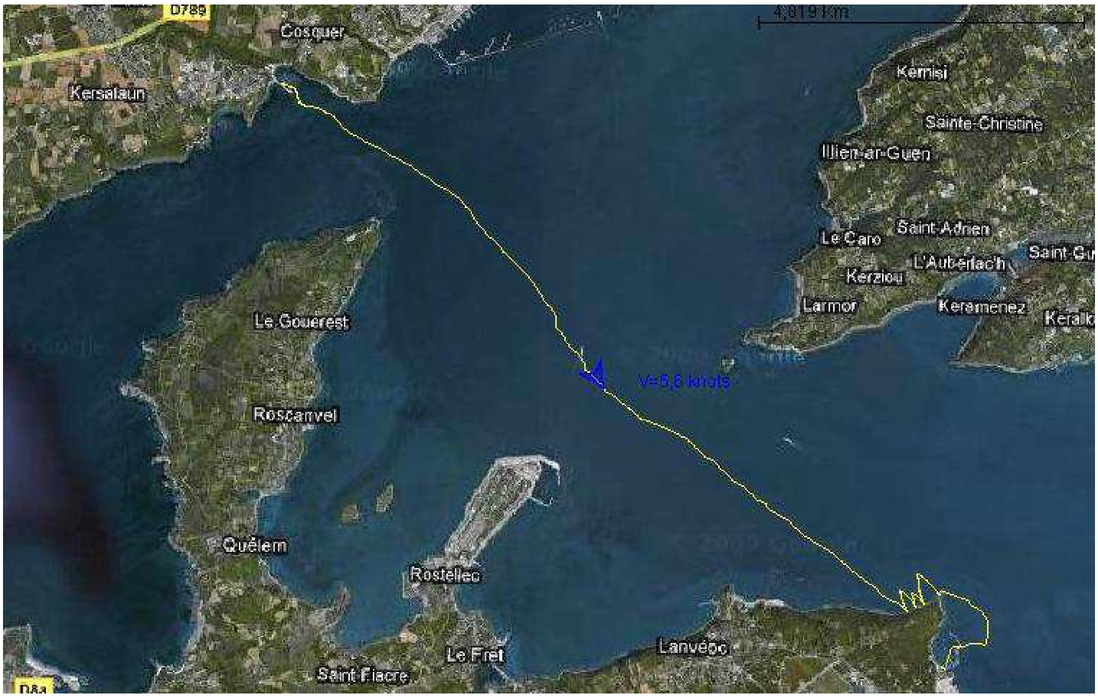


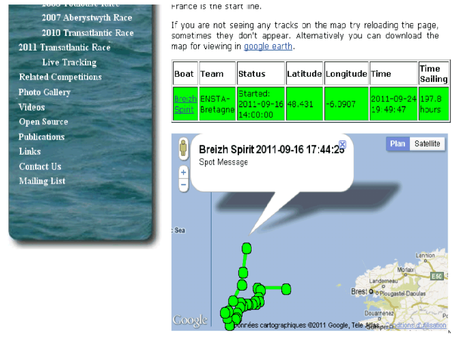
y
x


y
x

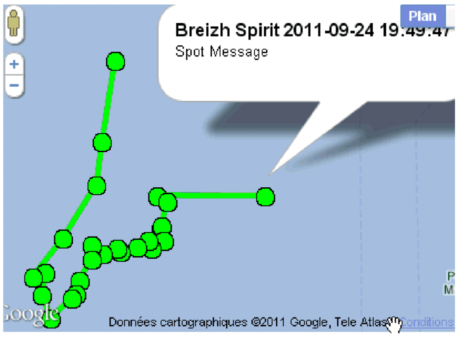
2 Sailboat robotics


BREIZH
ENSIETA
SPIRIT


D789
Cosquer
4,819 Km
Kersalaun
Kernisi
Sainte-Christine
Illien-ar-Guen
Le Caro
Saint-Adrien
L'Auberlac'h
Kerziou
Larmor
Keramenez
Le Gouerest
V=5,8 knots
Roscanvel
Quélern
Rostellec
Le Fret
Lanvéoc
Saint Fiacre



Boat	Team	Status	Latitude	Longitude	Time	Time Sailing
Breizh Spirit	ENSTA-Bretagne	Started: 2011-09-16 14:00:00	48.431	-6.0907	2011-09-24 19:49:47	197.8 hours



3 Vaimos

Collaboration ENSTA/IFREMER

Vaimos à la WRSC (ENSTA-IFREMER-Ecole Navale).

$$
\left\{
\begin{array}{lcl}
\dot{x} & = & v \cos\theta + p_1 a \cos\psi \\
\dot{y} & = & v \sin\theta + p_1 a \sin\psi \\
\dot{\theta} & = & \omega \\
\dot{v} & = & \frac{f_s \sin\delta_s - f_r \sin u_1 - p_2 v^2}{p_9} \\
\dot{\omega} & = & \frac{f_s(p_6 - p_7 \cos\delta_s) - p_8 f_r \cos u_1 - p_3 \omega}{p_{10}} \\
f_s & = & p_4 a \sin(\theta - \psi + \delta_s) \\
f_r & = & p_5 v \sin u_1 \\
\sigma & = & \cos(\theta - \psi) + \cos(u_2) \\
\delta_s & = & \left\{
\begin{array}{ll}
\pi - \theta + \psi & \text{si } \sigma \leq 0 \\
sign(\sin(\theta - \psi)) . u_2 & \text{sinon.}
\end{array}
\right.
\end{array}
\right.
$$

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right).$$

With the controller $\mathbf{u} = \mathbf{g}\left(\mathbf{x}\right)$, the robot satisfies

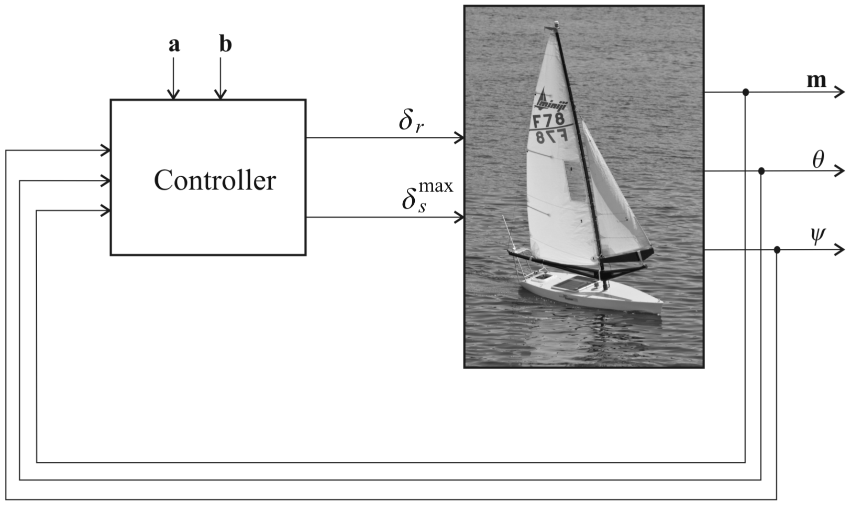
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right).$$

With all uncertainties, the robot satisfies.

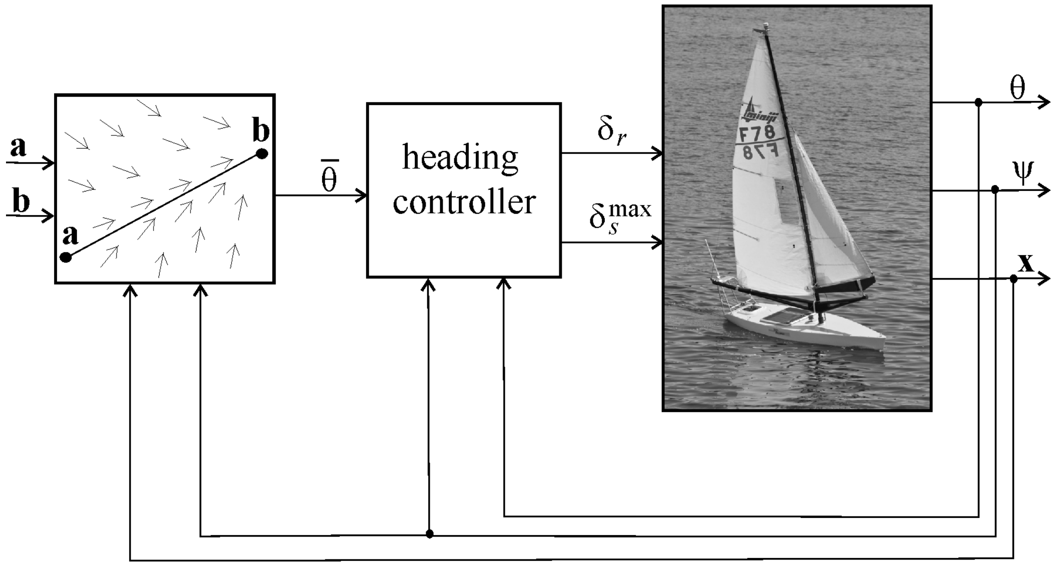
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

4 Line following



Controller of a sailboat robot



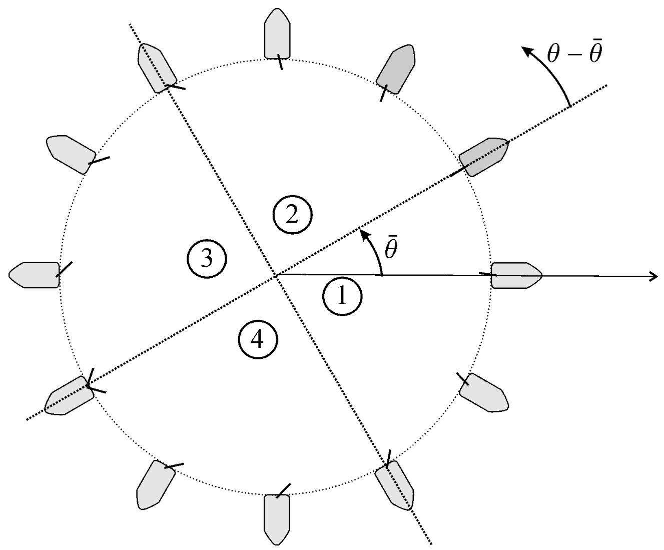
a
b
a
b
θ̄
heading controller
δ_r
δ_s^{max}
θ
ψ
x

Heading controller

$$\begin{cases} \delta_r & = & \frac{\delta_r^{\mathsf{max}}}{\pi}.\mathsf{atan}(\tan\frac{\theta-\bar{\theta}}{2}) \\ \delta_s^{\mathsf{max}} & = & \frac{\pi}{2}.\left(\frac{\cos(\psi-\bar{\theta})+1}{2}\right). \end{cases}$$

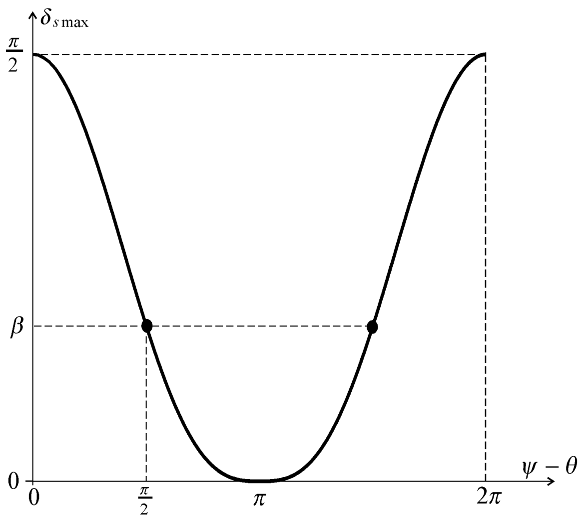
Rudder

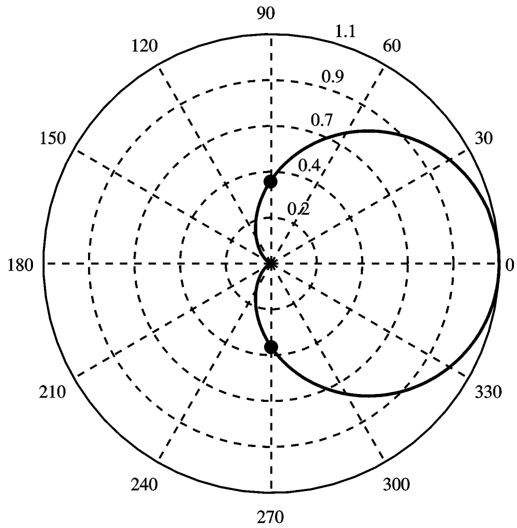
$$\left\{ \delta_r = \frac{\delta_r^{\max}}{\pi}.\text{atan}(\tan\frac{\theta-\bar{\theta}}{2}) \right.$$



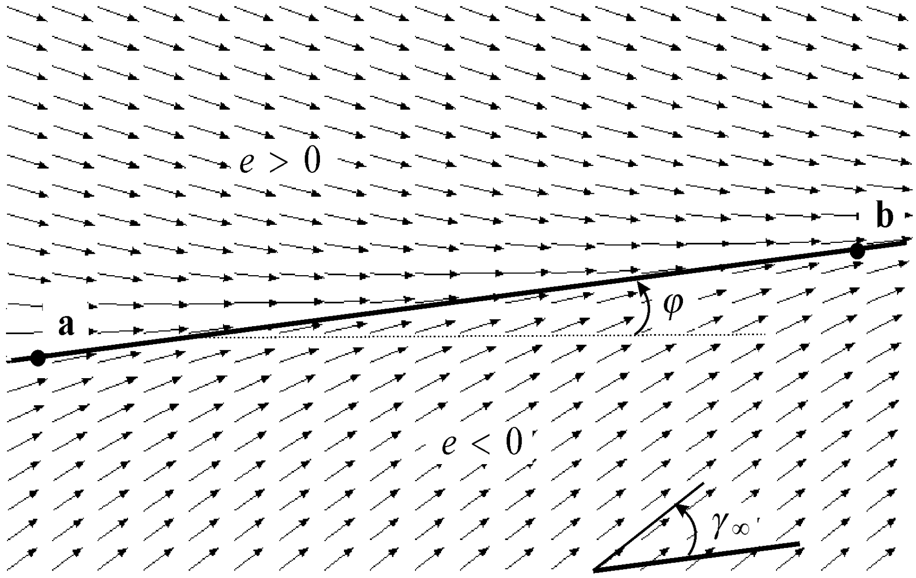
Sail

$$\delta_s^{\max} = \frac{\pi}{2} \cdot \left(\frac{\cos\left(\psi - \bar{\theta}\right) + 1}{2} \right)$$

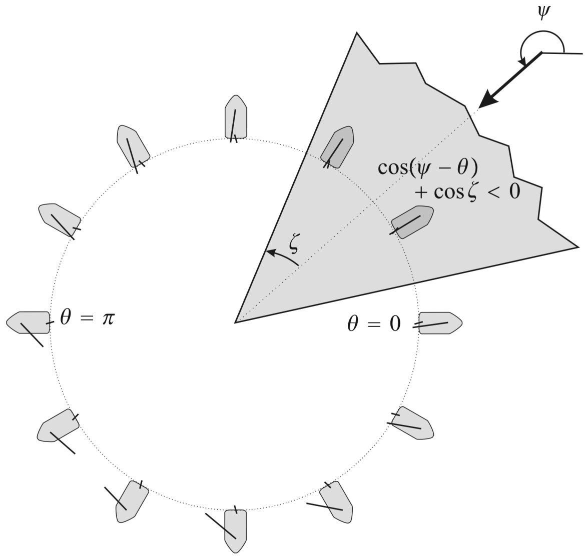




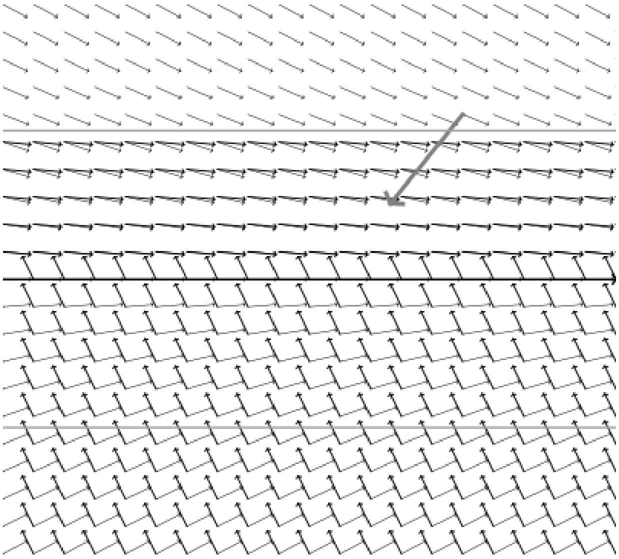
4.1 Vector field



Nominal vector field: $\theta^* = \varphi - \frac{1}{2}.\text{atan}\left(\frac{e}{r}\right)$



A course θ^* may be unfeasible



Keep close hauled strategy.

4.2 Controller

Controlleur : in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$;   out: $\delta_r, \delta_s^{\mathsf{max}}$;   inout: q

$e = \frac{\det(\mathbf{b}-\mathbf{a},\mathbf{m}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|}$

if $|e| > \frac{r}{2}$ then $q = \text{sign}(e)$

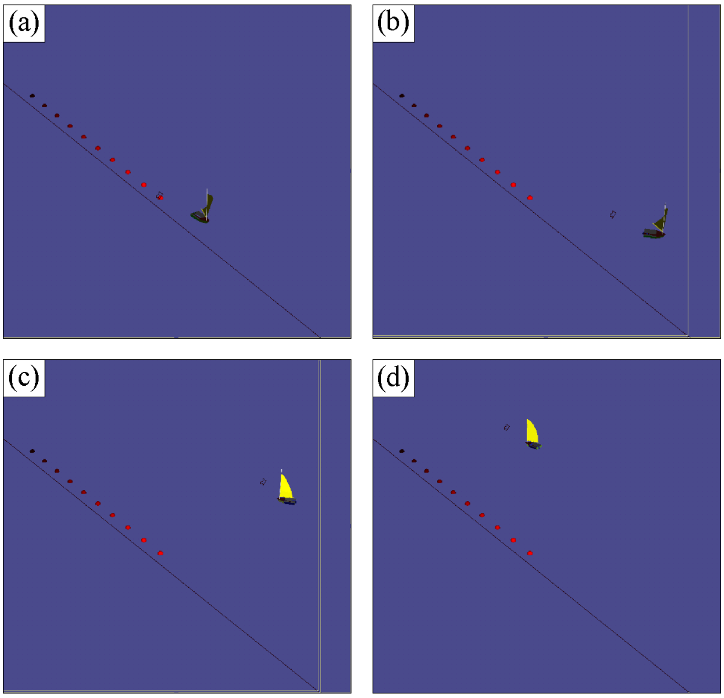
$\bar{\theta} = \text{atan2}(\mathbf{b}-\mathbf{a}) - \frac{1}{2}.\text{atan}\left(\frac{e}{r}\right)$

if $\cos\left(\psi - \bar{\theta}\right) + \cos\zeta < 0$ then $\bar{\theta} = \pi + \psi - q.\zeta.$

$\delta_r = \frac{\delta_r^{\mathsf{max}}}{\pi}.\text{atan}(\tan\frac{\theta-\bar{\theta}}{2})$

$\delta_s^{\mathsf{max}} = \frac{\pi}{2}.\left(\frac{\cos(\psi-\bar{\theta})+1}{2}\right).$

5 Validation by simulation


(a)
(b)
(c)
(d)

6 Theoretical validation

*Jaulin, Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE TRO.

When the wind is known, the sailboat with the heading controller is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

The system

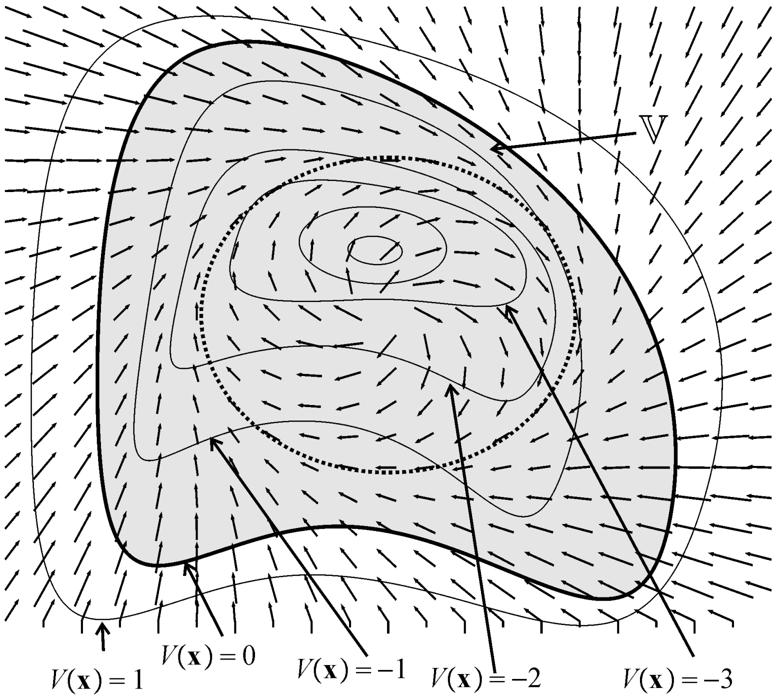
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) is there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned} \dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}, \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0}. \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable if

$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) \leq \varepsilon < 0\right).$$



$\mathbb{V}$
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$
\begin{aligned}
& \Big(V\left(\mathbf{x}\right) \geq 0 \;\Rightarrow\; \dot{V}\left(\mathbf{x}\right) < 0\Big) \\
\Leftrightarrow\; & \Big(V\left(\mathbf{x}\right) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) < 0\Big) \\
\Leftrightarrow\; & \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow\; & \neg\Big(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{f}\left(\mathbf{x}\right) \geq 0 \text{ and } V(\mathbf{x}) \geq 0\Big)
\end{aligned}
$$

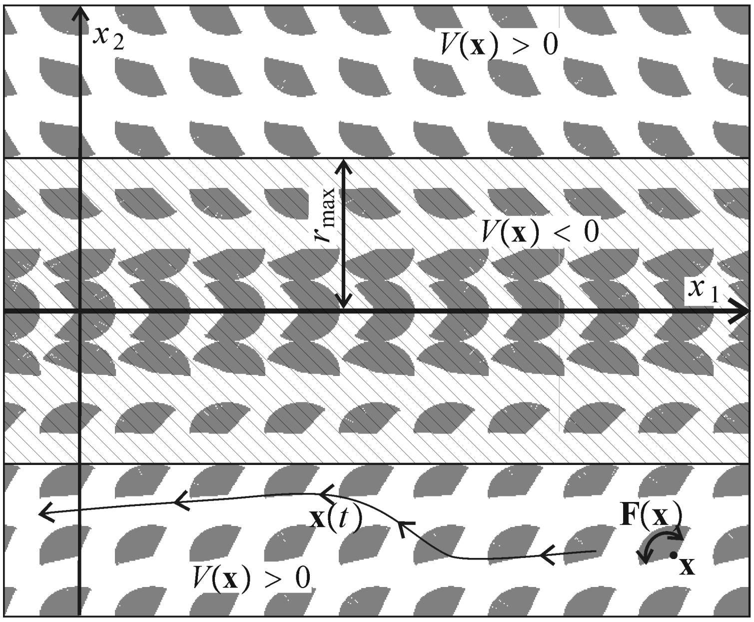
Theorem. We have

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{f}(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V-stability.

Theorem. We have

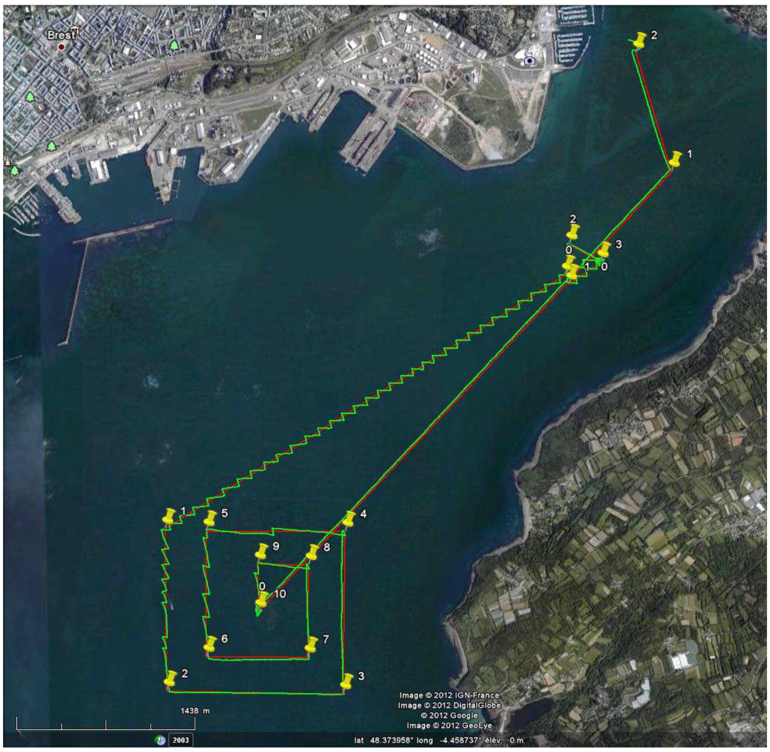
$$\left\{\begin{array}{c}\frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{x}\right).\mathbf{a}\geq 0\\ \mathbf{a}\in\mathbf{F}\left(\mathbf{x}\right)\\ V(\mathbf{x})\geq 0\end{array}\right. \quad \text{inconsistent} \quad \Leftrightarrow \quad \dot{\mathbf{x}}\in\mathbf{F}\left(\mathbf{x}\right) \text{ is } V\text{-stable}$$



Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\max}^2.$$

7 Experimental validation

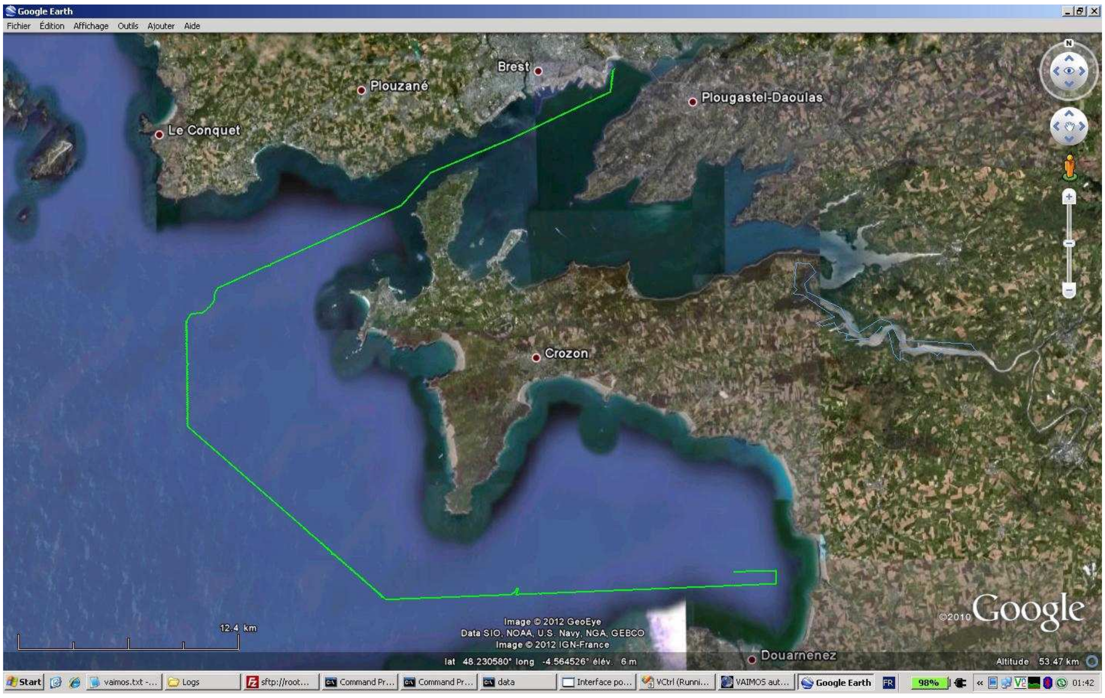


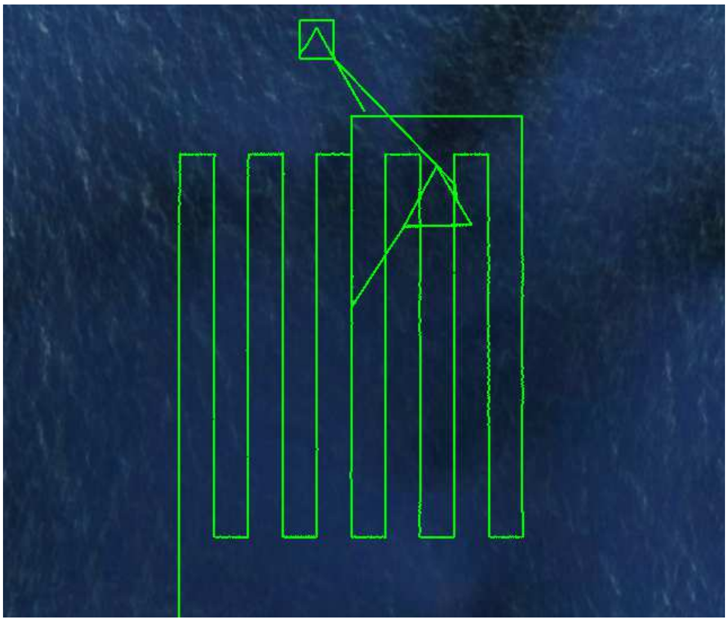
Rade de Brest

Brest-Douarnenez. January 17, 2012, 8am


Brest
PRO
BR D98 303
YAMAHA
Ifremer


F78


Google Earth
Fichier Édition Affichage Outils Ajouter Aide
Brest
Plouzané
Le Conquet
Plougastel-Daoulas
Crozon
Douarnenez
12.4 km
Image © 2012 GeoEye
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image © 2012 IGN-France
lat 48.230580° long -4.564526° élév 6 m
Altitude 53.47 km
©2010 Google
Start
98%
01:42



Middle of Atlantic ocean, 350 km made by Vaimos in 53h, September 6-9, 2012.

Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.