Kleene algebra to compute with invariant sets of dynamical systems

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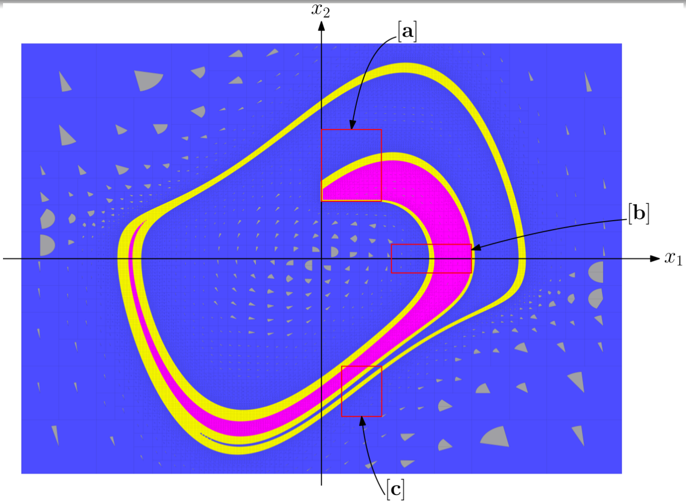
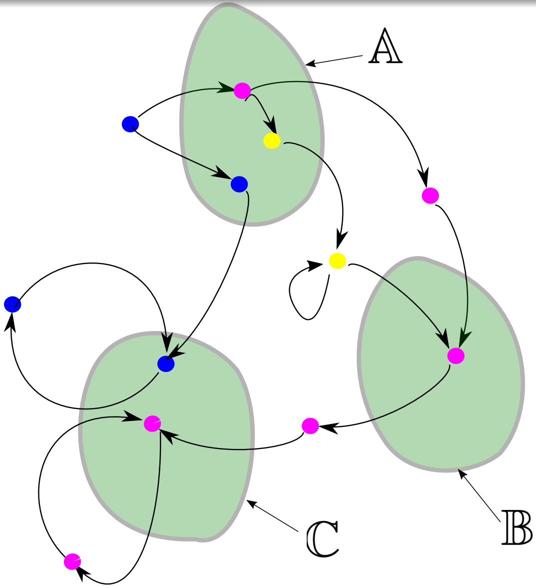
Motivation [5][4]


x_2
[a]
[b]
x_1
[c]


A
B
C

Consider the system

$$\mathscr{S} : \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$$

Denote by $\varphi(t, \mathbf{x})$ the flow map.

Positive invariant set

A set $\mathbb{A}$ is *positive invariant* [1] if

$$\mathbf{x} \in \mathbb{A}, t \geq 0 \implies \varphi(t, \mathbf{x}) \in \mathbb{A}.$$

Or equivalently

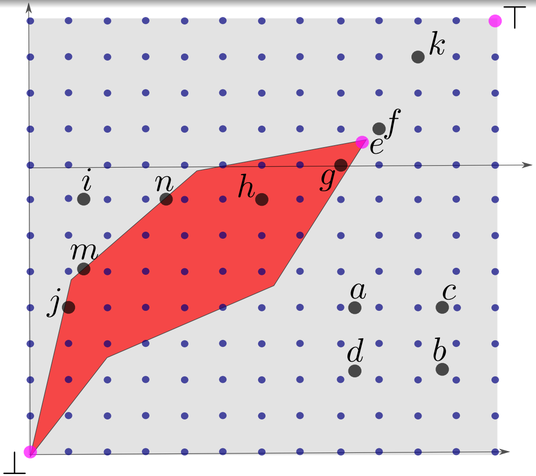
$$\varphi([0, \infty], \mathbb{A}) \subset \mathbb{A}.$$

The set of all positive invariant sets is a complete lattice.

Kleene algebra

Lattice

A *lattice* $(\mathscr{L}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds [2].
A *machine lattice* $(\mathscr{L}_M, \leq)$ of $\mathscr{L}$ is complete sublattice of $(\mathscr{L}, \leq)$ which is finite. Moreover both $\mathscr{L}$ and $\mathscr{L}_M$ have the same top and bottom.



Machine lattice

Kleene algebra

We consider a set $\mathscr{F}$ of automorphism $f:\mathscr{P}(\mathbb{X}) \to \mathscr{P}(\mathbb{X})$ such that

$$f(\mathbb{X}) = \mathbb{X}$$
$$f(\mathbb{A} \cap \mathbb{B}) = f(\mathbb{A}) \cap f(\mathbb{B})$$

Note that f is inclusion monotonic.

	Kleene$(+,\cdot,*)$	Kleene$(\cap,\circ,*)$
Addition	$a+b$	$f\cap g$
Product	$a\cdot b$	$f\circ g$
Associativity	$a+(b+c)=(a+b)+c$	$f\cap(g\cap h)=(f\cap g)\cap h$
	$a(bc)=(ab)c$	$f\circ(g\circ h)=(f\circ g)\circ h$
Commutativity	$a+b=b+a$	$f\cap g=g\cap f$
Distributivity	$a(b+c)=(ab)+(ac)$	$f\circ(g\cap h)=(f\circ g)\cap(f\circ h)$
	$(b+c)a=(ba)+(ca)$	$(g\cap h)\circ f=(g\circ f)\cap(h\circ f)$
zero	$a+0=a$	$f\cap\top=f$
One	$a1=1a=a$	$f\circ Id=Id\circ f=f$
Annihilation	$a0=0a=0$	$f\circ\top=\top$
Idempotence	$a+a=a$	$f\cap f=f$
Partial order	$a\leq b\Leftrightarrow a+b=b$	$f\supset g\Leftrightarrow f\cap g=g$
Kleene star	$a^*=1+a+aa+aaa+\ldots$	$f^*=Id\cap f\cap f^2\cap f^3\cap\ldots$

Reducers

To an automorphism $f \in \mathscr{F}$, we can associate the reducer $\mathscr{R} = Id \cap f$.

We have

$$\mathbb{A} \subset \mathbb{B} \Rightarrow \mathscr{R}(\mathbb{A}) \subset \mathscr{R}(\mathbb{B}) \qquad \text{monotonicity}$$

$$\mathscr{R}(\mathbb{A}) \subset \mathbb{A} \qquad \text{degrowth}$$

Theorem. We have

$$(Id \cap f)^{\infty} = f^{*}$$

Proof. Since f is such that $f(\mathbb{A} \cap \mathbb{B})) = f(\mathbb{A}) \cap f(\mathbb{B})$, we have

$$\begin{aligned} (Id \cap f)^{2}(\mathbb{A}) &= (Id \cap f)(\mathbb{A} \cap f(\mathbb{A})) = \\ &= \mathbb{A} \cap f(\mathbb{A}) \cap f(\mathbb{A} \cap f(\mathbb{A})) \\ &= \mathbb{A} \cap f(\mathbb{A}) \cap f^{2}(\mathbb{A}) \end{aligned}$$

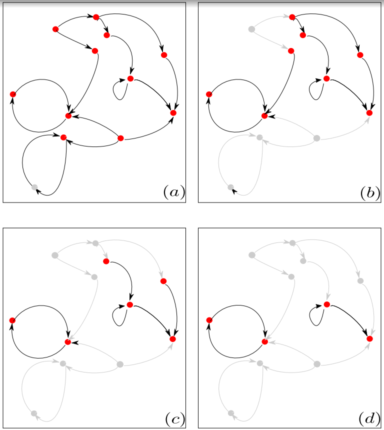
and

$$(Id \cap f)^{\infty}(\mathbb{A}) = \mathbb{A} \cap f(\mathbb{A}) \cap f^{2}(\mathbb{A}) \cap f^{3}(\mathbb{A}) \cap \cdots = f^{*}(\mathbb{A}).$$

We define

$$\mathrm{Fix}(f^*) = \{\mathbb{A} \,|\, f^*(\mathbb{A}) = \mathbb{A}\} = \mathrm{Fix}(Id \cap f)$$

From the Knaster–Tarski theorem, it is a complete sublattice of $\mathscr{L}$.



(a) : Red nodes : $\mathbb{A}$, (b):$\mathbb{A} \cap f(\mathbb{A})$, (c):$\mathbb{A} \cap f(\mathbb{A}) \cap f^2(\mathbb{A})$,
(d):$f^*(\mathbb{A})$.

Goal. Compute with closure sets $f_i^*, i \in \{1, 2, \dots\}$, *i.e.*, compute with expressions such as

$$f^*(\mathbb{A}) \cap (g^*(\mathbb{A}) \cup h^*(\mathbb{A}))^*$$

We want to factorize the fixed point operators as much as possible.

Factorization properties [3]

$$f^* \cap f^* = f^*$$
$$(f^*)^* = f^*$$
$$(f^* \cap g^*)^* = (f \cap g)^*$$
$$f^* \circ (f \circ g^*)^* = (f \cap g)^*$$

Dealing state equations

Define

$$\begin{aligned}\overrightarrow{f}(\mathbb{A}) &= \overline{\varphi([-1,0],\overline{\mathbb{A}})}\\ \overleftarrow{f}(\mathbb{A}) &= \overline{\varphi([0,1],\overline{\mathbb{A}})}\end{aligned}$$

We have

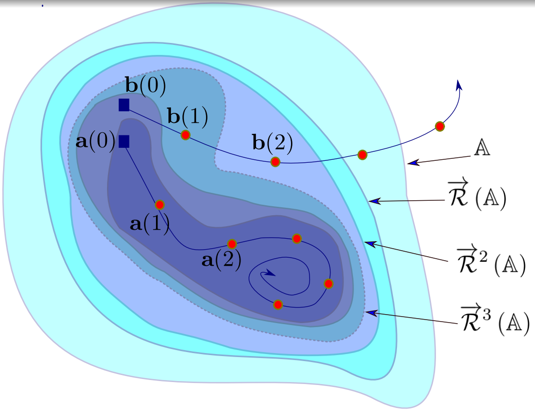
$$\begin{array}{ll}\overrightarrow{f}(\mathbb{R}^n)=\mathbb{R}^n & \overrightarrow{f}(\mathbb{A}\cap\mathbb{B})=\overrightarrow{f}(\mathbb{A})\cap\overrightarrow{f}(\mathbb{B})\\ \overleftarrow{f}(\mathbb{R}^n)=\mathbb{R}^n & \overleftarrow{f}(\mathbb{A}\cap\mathbb{B})=\overleftarrow{f}(\mathbb{A})\cap\overleftarrow{f}(\mathbb{B})\end{array}$$

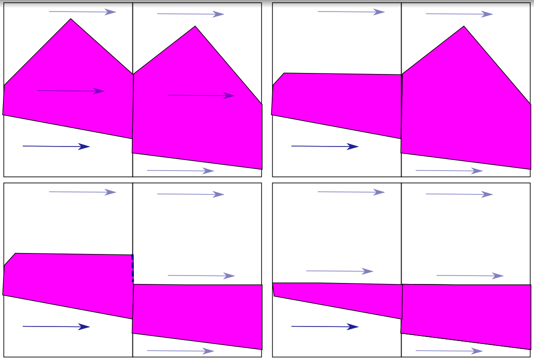
The sets $\overrightarrow{f}^*(\mathbb{A}), \overleftarrow{f}^*(\mathbb{A})$ correspond to the largest positive and negative invariant sets included in $\mathbb{A}$.

The largest invariant set included in $\mathbb{A}$ is

$$\left(\overrightarrow{f} \cap \overleftarrow{f}\right)^{*}(\mathbb{A})$$

Illustration


b(0)
b(1)
b(2)
a(0)
a(1)
a(2)
$\mathbb{A}$
$\overrightarrow{\mathcal{R}}(\mathbb{A})$
$\overrightarrow{\mathcal{R}}^2(\mathbb{A})$
$\overrightarrow{\mathcal{R}}^3(\mathbb{A})$



Kleene intervals

Given an automorphism f, we want to compute $f^*(a)$ where a is in $(\mathscr{L}, \leq)$ (for instance $(\mathbb{R}^n, \subset)$).
Machine sublattice $\mathscr{L}_M$ of $\mathscr{L}$ (maze for instance).

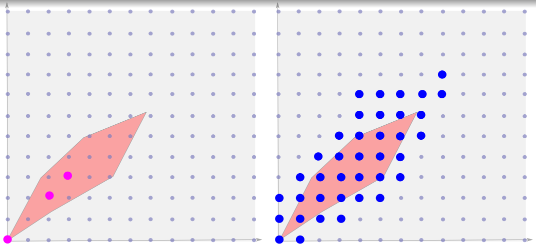
Interval automorphism

An interval automorphism $[f^-, f^+]$ containing f is a pair of two machine automorphism f^-, f^+ with such that

$$a \in \mathscr{L}_M \Rightarrow f^-(a) \leq f(a) \leq f^+(a).$$

Lemma. We have

$$\mathrm{Fix}\left((f^-)^*\right) \subset \mathscr{L}_M \cap \mathrm{Fix}(f^*) \subset \mathrm{Fix}\left((f^+)^*\right)$$

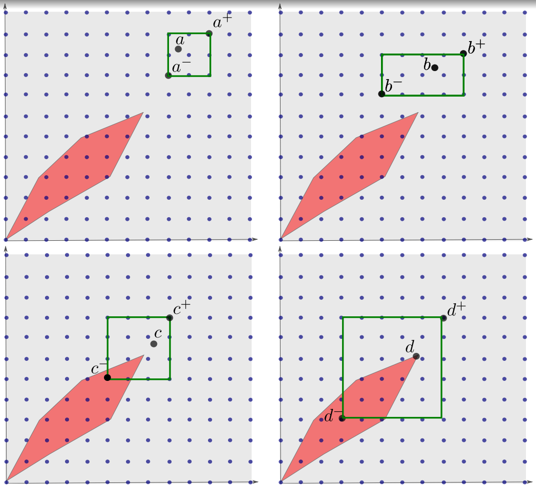


Fixed points $\mathrm{Fix}\left((f^-)^*\right)$ in magenta, $\mathrm{Fix}\left((f^+)^*\right)$ in blue

Theorem. If $a \in [a^-, a^+]$, where a^-, a^+ both belong to $\mathscr{L}_M$, then

$$
\begin{array}{ll}
(i) & f^*(a) \in \left[(f^-)^*(a^-), (f^+)^*(a^+) \right] \\
(ii) & f^* \circ (f^-)^*(a^-) = (f^-)^*(a^-) \\
(iii) & f^*(a) \leq (Id \cap f^+)^i (a^+), \ \forall i \geq 0
\end{array}
$$

Algorithm



Computation of $f^*(a), a \in [a]$

Boolean lattice

A Boolean lattice $\mathscr{L}$ is a complemented distributive lattice. Every element a has a unique complement $\bar{a}$, satisfying $a \vee \bar{a} = \top$ and $a \wedge \bar{a} = \bot$.

We have

$$\begin{aligned} a \leq b &\Leftrightarrow \overline{b} \leq \overline{a} \\ \overline{a \vee b} &= \overline{b} \wedge \overline{a} \qquad \text{(De Morgan's laws)} \\ \overline{a \wedge b} &= \overline{b} \vee \overline{a} \end{aligned}$$

Interval arithmetic

$$\begin{aligned}
\overline{[a^-, a^+]} &= \left[\overline{a^+}, \overline{a^-}\right] \\
f\left([a^-, a^+]\right) &= \left[f\left(a^-\right), f\left(a^+\right)\right] \\
[a^-, a^+] \wedge [b^-, b^+] &= [a^- \wedge b^-, a^+ \wedge b^+] \\
[a^-, a^+] \vee [b^-, b^+] &= [a^- \vee b^-, a^+ \vee b^+]
\end{aligned}$$

Monotonic case

Compute $x = f_1^*(a) \vee (f_2^*(b) \wedge f_3^*(c))$. We have

$$x \in \left[\begin{array}{c} f_1^*(a^-) \vee (f_2^*(b^-) \wedge f_3^*(c^-)) \\ \left(Id \cap f_1^i\right)(a^+) \vee \left(\left(Id \cap f_2^i\right)(b^+) \wedge \left(Id \cap f_3^i\right)(c^+)\right) \end{array} \right].$$

Non monotonic case

We want to compute $x = \overline{f_1^*(\overline{a})} \wedge f_2^*\left(\overline{b}\right)$. Applying interval arithmetic rules, we get

$$x \in \left[\overline{f_1^*\left(\overline{a^-}\right)} \wedge f_2^*\left(\overline{b^+}\right), \overline{f_1^*\left(\overline{a^+}\right)} \wedge f_2^*\left(\overline{b^-}\right)\right],$$

i.e., we need to go up to the fixed point for both bounds.

Forward reach set

Forward reach set of $\mathbb{A}$ defined by

$$\mathrm{Forw}(\mathbf{f},\mathbb{A}) = \{\mathbf{x} \mid \exists t \geq 0, \exists \mathbf{x}_0 \in \mathbb{A}, \varphi(t,\mathbf{x}_0) = \mathbf{x}\}.$$

We get

$$\mathrm{Forw}(\mathbf{f},\mathbb{A}) = \overline{\overleftarrow{f}^{\,*}(\overline{\mathbb{A}})}.$$

Monotonic path planning

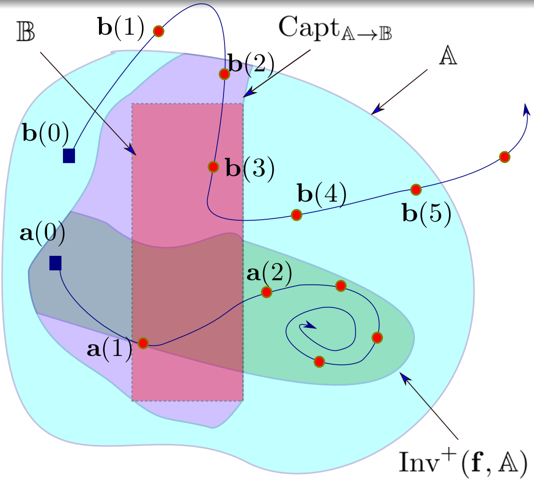
The set of paths that start in the set $\mathbb{A}$ and reach $\mathbb{B}$ is given by

$$\mathrm{Path}\left(\mathbb{A},\mathbb{B}\right)=\mathrm{Forw}\left(\mathbb{A}\right)\cap\mathrm{Back}\left(\mathbb{B}\right)=\overline{\overleftarrow{f}^{\,*}\left(\overline{\mathbb{A}}\right)}\cap\overline{\overrightarrow{f}^{\,*}\left(\overline{\mathbb{B}}\right)}.$$

A to B problem

Consider two sets $\mathbb{A},\mathbb{B}$ such that $\mathbb{B}\subset\mathbb{A}$. We want to compute the set

$$\mathbb{X}=\mathrm{Capt}_{\mathbb{A}\to\mathbb{B}}=\{\mathbf{x}\mid\exists t\geq 0,\varphi(t,\mathbf{x})\in\mathbb{B}\text{ and }\forall t_1\in[0,t],\varphi(t_1,\mathbf{x})\in\mathbb{A}\}.$$


$\mathbb{B}$
$\mathbf{b}(1)$
$\mathrm{Capt}_{\mathbb{A}\to\mathbb{B}}$
$\mathbf{b}(2)$
$\mathbb{A}$
$\mathbf{b}(0)$
$\mathbf{b}(3)$
$\mathbf{b}(4)$
$\mathbf{b}(5)$
$\mathbf{a}(0)$
$\mathbf{a}(2)$
$\mathbf{a}(1)$
$\mathrm{Inv}^{+}(\mathbf{f},\mathbb{A})$

Non monotonic path planning

Find the set $\mathbb{X}$ of all paths that start in $\mathbb{A}$, avoid $\mathbb{B}$ and reach $\mathbb{C}$.

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